Small Scale Equidistribution of Lattice Points on the Sphere

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Sums of Three Squares

Question

When can a positive integer n be written as the sum of three squares?

Sums of Three Squares

Example

$x_1^2 + x_2^2 + x_3^2 = 11$ with $(x_1, x_2) \in \{(3,1,1),(1,3,1),\ldots\}$.

Example

$x_1^2 + x_2^2 + x_3^2 = 7$ has no integral solutions.

Sums of Three Squares: Classification

If $x \in \mathbb{Z}$, then $x^2 \equiv 0, 1, \text{ or } 4 \pmod 8$. So if $(x_1, x_2, x_3) \in \mathbb{Z}^3$, then $x_1^2 + x_2^2 + x_3^2 \equiv 0, 1, 2, 3, 4, 5, \text{ or } 6 \pmod 8$.

Theorem (Legendre (1798))

Any positive integer n can be written as a sum of three squares if and only if n is not of the form $n = 4^a(8b + 7)$ for some nonnegative integers a, b.

Sums of Three Squares: Number of Solutions

Question

Given a positive integer n, how many ways are there to write n as the sum of three squares?

Given n, want to find all $(x_1, x_2, x_3) \in \mathbb{Z}^3$ for which $x_1^2 + x_2^2 + x_3^2 = n$.

Sums of Three Squares: Number of Solutions

Interesting case is n squarefree.

Theorem (Gauss (1801))

For odd squarefree $n \not\equiv 7 \pmod 8$,

$$r_3(n) = \begin{cases} 12h(D) & \text{for } n \equiv 1, 2 \pmod 4 \text{ with } D = -4n, \\ 24h(D) & \text{for } n \equiv 3 \pmod 8 \text{ with } D = -n. \end{cases}$$

$h(D)$ is the class number of the imaginary quadratic number field $\mathbb{Q}(\sqrt{D})$.

Sums of Three Squares: Number of Solutions

Corollary

For all $\varepsilon > 0$, we have that

$$n^{\frac{1}{2}-\varepsilon} \ll_{\varepsilon} r_3(n) \ll \sqrt{n} \log n.$$

Upper bound is easy; not hard to show that $h(D) \ll \sqrt{|D|} \log |D|$.

Lower bound is nontrivial; Dirichlet class number formula (1839) plus Siegel ineffective bound (1935)

$$L(1, \chi_D) \gg_{\varepsilon} |D|^{-\varepsilon},$$

where

$$L(s, \chi_D) := \sum_{m=1}^{\infty} \frac{\chi_D(m)}{m^s} = \prod_p \frac{1}{1 - \chi_D(p) p^{-s}}$$

is the Dirichlet L-function associated to χ_D, the quadratic Dirichlet character modulo $-D$.

Sums of Three Squares: Limiting Behaviour

For n not of the form $4^a(8b+7)$, let

$$\mathcal{E}(n) := \left\{(x_1, x_2, x_3) \in \mathbb{Z}^3 : x_1^2 + x_2^2 + x_3^2 = n\right\}.$$

Geometric viewpoint: $\mathcal{E}(n)$ is the set of points on the lattice $\mathbb{Z}^3$ in $\mathbb{R}^3$ that lie on the sphere centred at the origin of radius $\sqrt{n}$.

Sums of Three Squares: Limiting Behaviour

Let

$$\widehat{\mathcal{E}}(n) := \left\{ \left(\frac{x_1}{\sqrt{n}}, \frac{x_2}{\sqrt{n}}, \frac{x_3}{\sqrt{n}} \right) \in \mathbb{R}^3 : (x_1, x_2, x_3) \in \mathcal{E}(n) \right\}$$

denote the projection of $\mathcal{E}(n)$ onto the unit sphere

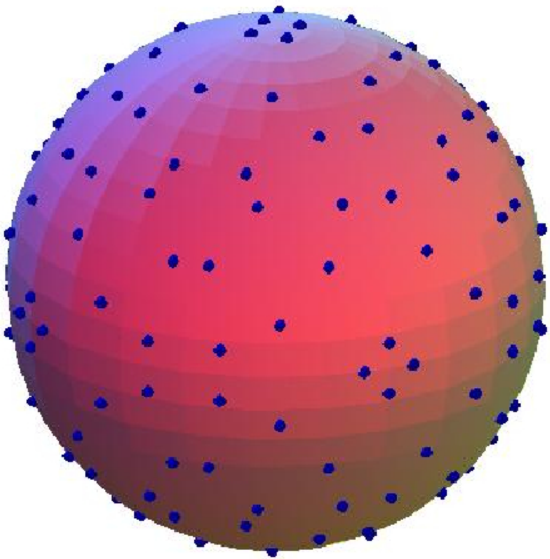
$$S^2 := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1 \right\}.$$

Question

What are the limiting statistical properties of $\widehat{\mathcal{E}}(n) \subset S^2$ as $n \to \infty$?

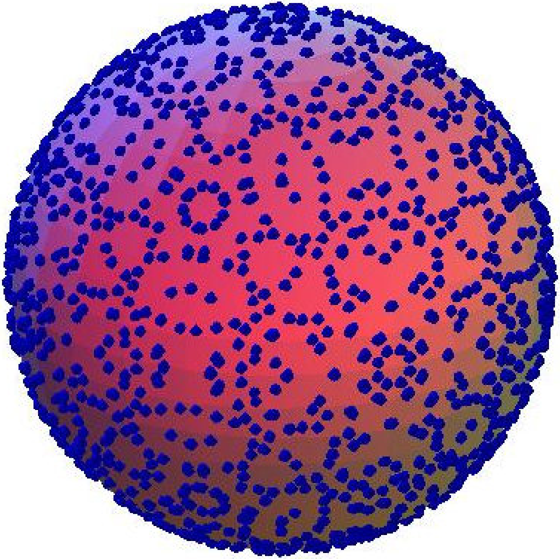
Examples: $n = 101$

(Image: Ellenberg–Michel–Venkatesh)



Examples: $n = 104851$

(Image: Ellenberg–Michel–Venkatesh)



Duke's Theorem

Theorem (Duke (1988), Duke–Schulze-Pillot (1990), Golubeva–Fomenko (1990))

As $n \to \infty$ along squarefree integers with $n \not\equiv 7 \pmod 8$, the lattice points on the sphere $\widehat{\mathcal{E}}(n)$ equidistribute on S^2.

Informally, the points $\widehat{\mathcal{E}}(n)$ spread out randomly on S^2.

Proved earlier by Linnik (1968) for certain subsequences of n.

Equidistribution

Let M be a topological space and μ a probability measure on M.
Let μ_n be a sequence of probability measures on M.

Definition

The probability measures μ_n equidistribute on M w.r.t. μ if

$$\lim_{n\to\infty} \mu_n(B) = \mu(B)$$

for every continuity set $B \subset M$ (boundary has μ-measure zero).

Equidistribution

Let M be a topological space and μ a probability measure on M. Let μ_n be a sequence of probability measures on M.

Definition

The probability measures μ_n equidistribute on M w.r.t. μ if

$$\lim_{n\to\infty} \int_M f(x)\, d\mu_n(x) = \int_M f(x)\, d\mu(x)$$

for all $f \in C_b(M)$ (continuous bounded).

Equidistribution

Define the probability measure μ_n on S^2 by

$$\mu_n := \frac{1}{\#\widehat{\mathcal{E}}(n)} \sum_{x \in \widehat{\mathcal{E}}(n)} \delta_x,$$

so that for $B \subset S^2$ and $f : S^2 \to \mathbb{C}$,

$$\mu_n(B) := \frac{\#(\widehat{\mathcal{E}}(n) \cap B)}{\#\widehat{\mathcal{E}}(n)},$$

$$\int_{S^2} f(y)\, d\mu_n(y) := \frac{1}{\#\widehat{\mathcal{E}}(n)} \sum_{x \in \widehat{\mathcal{E}}(n)} f(x).$$

Duke's Theorem

Theorem (Duke (1988), Duke–Schulze-Pillot (1990), Golubeva–Fomenko (1990))

As $n \to \infty$ along squarefree integers with $n \not\equiv 7 \pmod 8$, the probability measures μ_n equidistribute on S^2 with respect to the normalised surface measure on S^2.

Duke's Theorem

Theorem (Duke (1988), Duke–Schulze-Pillot (1990), Golubeva–Fomenko (1990))

As $n \to \infty$ along squarefree integers with $n \not\equiv 7 \pmod 8$,

$$\frac{\#(\widehat{\mathcal{E}}(n) \cap B)}{\#\widehat{\mathcal{E}}(n)} \to \mathrm{vol}(B)$$

for every continuity set $B \subset S^2$.

Duke's Theorem

Theorem (Duke (1988), Duke–Schulze-Pillot (1990), Golubeva–Fomenko (1990))

As $n \to \infty$ along squarefree integers with $n \not\equiv 7 \pmod 8$,

$$\frac{1}{\#\widehat{\mathcal{E}}(n)} \sum_{x \in \widehat{\mathcal{E}}(n)} f(x) \to \int_{S^2} f(y)\, dy$$

for every continuous function f on S^2.

Proof of Duke's Theorem

Idea of proof: approximate $f \in C(S^2)$ by spherical harmonics.

Reduces problem to showing

$$\frac{1}{\#\widehat{\mathcal{E}}(n)} \sum_{x \in \widehat{\mathcal{E}}(n)} \phi(x) \to \int_{S^2} \phi(y)\, dy$$

for a spherical harmonic ϕ. Trivial if ϕ is constant. RHS is zero if ϕ is nonconstant.

Since $\#\widehat{\mathcal{E}}(n) \gg_\varepsilon n^{1/2-\varepsilon}$, suffices to show that there exists $\delta > 0$ such that

$$\sum_{x \in \widehat{\mathcal{E}}(n)} \phi(x) \ll_\phi n^{\frac{1}{2}-\delta}.$$

Waldspurger's Identity

Theorem (Waldspurger (1981))

Given a spherical harmonic of degree $m_\phi \geq 1$, there exists a modular form f of weight $2 + 2m_\phi$ such that

$$\left| \sum_{x \in \widehat{\mathcal{E}}(n)} \phi(x) \right|^2 \approx \sqrt{n} L\left(\frac{1}{2}, f\right) L\left(\frac{1}{2}, f \otimes \chi_{-n}\right).$$

L-Functions of Modular Forms

The L-function $L(s, f)$ of a modular form f is given by

$$L(s, f) = \sum_{m=1}^{\infty} \frac{\lambda_f(m)}{m^s} = \prod_p \frac{1}{1 - \lambda_f(p)p^{-s} + p^{-2s}},$$

where $\lambda_f(p) \in [-2, 2]$ are the Hecke eigenvalues of f.

The L-function $L(s, f \otimes \chi)$ of the twist of f by a Dirichlet character χ is given by

$$L(s, f \otimes \chi) = \sum_{m=1}^{\infty} \frac{\lambda_f(m)\chi(m)}{m^s} = \prod_p \frac{1}{1 - \lambda_f(p)\chi(p)p^{-s} + \chi^2(p)p^{-2s}}.$$

Iwaniec's Subconvex Bound

Theorem (Iwaniec (1987))

There exists $\delta > 0$ such that

$$L\left(\frac{1}{2}, f \otimes \chi_{-n}\right) \ll_f n^{\frac{1}{2}-\delta}.$$

This is a case of *subconvexity*. Trivial bound is

$$L\left(\frac{1}{2}, f \otimes \chi_{-n}\right) \ll_{f,\varepsilon} n^{\frac{1}{2}+\varepsilon}.$$

Consequence of the Phragmén–Lindelöf convexity principle.
Generalisation of the bound $\zeta(1/2 + it) \ll_\varepsilon (|t| + 1)^{\frac{1}{4}+\varepsilon}$.

Rate of Equidistribution: Decay of Error Term

Suppose that μ_n equidistributes on S^2 w.r.t. μ. At what rate?

Goal

Find the most rapidly decreasing function $\alpha(n)$ for which

$$\mu_n(B) = \frac{\#(\widehat{\mathcal{E}}(n) \cap B)}{\#\widehat{\mathcal{E}}(n)}$$

is equal to

$$\mathrm{vol}(B) + O_B(\alpha(n))$$

for a fixed continuity set $B \subset S^2$.

Informally, determine how quickly the points $\widehat{\mathcal{E}}(n)$ spread out randomly on S^2.

Rate of Equidistribution: Decay of Error Term

Heuristic

Like *random* points, we should expect square-root cancellation: since $\#\widehat{\mathcal{E}}(n) \approx \sqrt{n}$, we should hope for $\alpha(n) \approx n^{-1/4}$.

Rate of Equidistribution: Decay of Error Term

Theorem (Conrey–Iwaniec (2000))

For a fixed continuity set $B \subset S^2$,

$$\frac{\#(\widehat{\mathcal{E}}(n) \cap B)}{\#\widehat{\mathcal{E}}(n)} = \mathrm{vol}(B) + O_{B,\varepsilon}\left(n^{-\frac{1}{12}+\varepsilon}\right)$$

for all $\varepsilon > 0$.

Follows from the *Weyl-strength* subconvex bound $L(1/2, f \otimes \chi_D) \ll_{f,\varepsilon} |D|^{1/3+\varepsilon}$.

Assuming the generalised Lindelöf hypothesis,

$$\frac{\#(\widehat{\mathcal{E}}(n) \cap B)}{\#\widehat{\mathcal{E}}(n)} = \mathrm{vol}(B) + O_{B,\varepsilon}\left(n^{-\frac{1}{4}+\varepsilon}\right).$$

Optimal.

Rate of Equidistribution: Small Scale Equidistribution

Suppose that μ_n equidistributes on S^2 w.r.t. μ. At what rate?

Goal

Find the most rapidly decreasing function $\alpha(n)$ for which

$$\lim_{n \to \infty} \frac{1}{\mathrm{vol}(B_n)} \frac{\#(\widehat{\mathcal{E}}(n) \cap B_n)}{\#\widehat{\mathcal{E}}(n)} = 1$$

for a family of sets $B = B_n$ with $0 < \mathrm{vol}(B_n) \leq \alpha(n)$.

Informally, determine the scale at which the points $\widehat{\mathcal{E}}(n)$ no longer look random.

Rate of Equidistribution: Small Scale Equidistribution

Heuristic

Like *random* points, we should expect small scale equidistribution provided we are at a scale for which $\#(\widehat{\mathcal{E}}(n) \cap B_n) \to \infty$. Since $\widehat{\mathcal{E}}(n) \approx \sqrt{n}$, the optimal scale should be $\alpha(n) \approx n^{-1/2}$.

Optimal Small Scale Equidistribution

Proposition

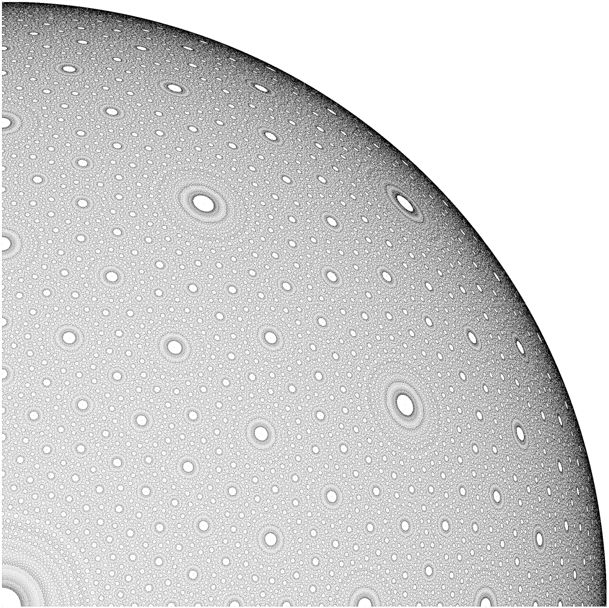
Generically, $\widehat{\mathcal{E}}(n)$ cannot equidistribute on shrinking sets B_n for which $\mathrm{vol}(B_n) \leq n^{-1/2-\delta}$ for some $\delta > 0$.

Sketch of Proof.

There are $\approx \sqrt{n}$ lattice points in $\widehat{\mathcal{E}}(n)$, so if $\mathrm{vol}(B_n) \leq n^{-1/2-\delta}$, then generically $\widehat{\mathcal{E}}(n) \cap B_n = \emptyset$. $\square$

Example: $n \leq 2048$

(Image: Stefan Kohl)



Optimal Small Scale Equidistribution

Conjecture

Lattice points $\widehat{\mathcal{E}}(n)$ equidistribute on shrinking sets B_n for which $\mathrm{vol}(B_n) \gg n^{-1/2+\delta}$ for some $\delta > 0$.

Optimal scale.

Linnik's conjecture

Conjecture (Linnik (1968))

Fix $\delta > 0$. *For all sufficiently large squarefree* $n \not\equiv 7 \pmod 8$,

$$x_1^2 + x_2^2 + x_3^2 = n$$

has an integral solution $(x_1, x_2, x_3) \in \mathbb{Z}^3$ *with* $|x_3| < n^\delta$.

Special case of optimal small scale equidistribution:
B_n the annulus (belt about the equator) of optimally shrinking width.

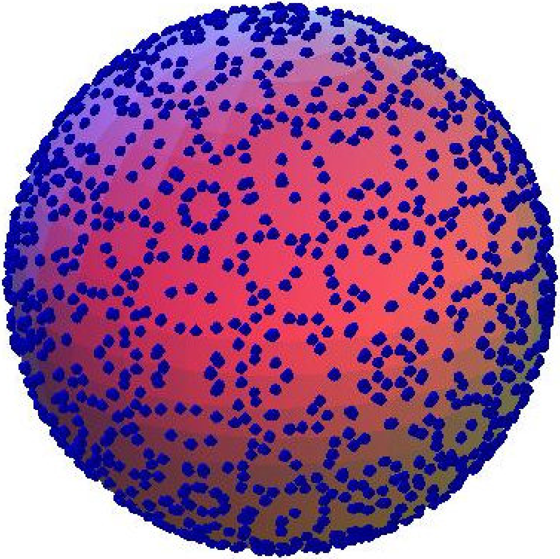
Conjecture (Linnik (1968))

Fix $\delta > 0$. For all sufficiently large squarefree $n \not\equiv 7 \pmod 8$, there exists $(x_1, x_2, x_3) \in \widehat{\mathcal{E}}(n)$ with $|x_3| < n^{-\frac{1}{2}+\delta}$.

Special case of optimal small scale equidistribution:
B_n the annulus (belt about the equator) of optimally shrinking width.

Example: $n = 104851$

(Image: Ellenberg–Michel–Venkatesh)



Linnik's Conjecture

Theorem (H.–Radziwiłł (2019))

Fix $\delta > 0$. For all sufficiently large squarefree $n \not\equiv 7 \pmod 8$,

$$x_1^2 + x_2^2 + x_3^2 = n$$

has an integral solution $(x_1, x_2, x_3) \in \mathbb{Z}^3$ with $|x_3| < n^{\frac{4}{9}+\delta}$.

Assuming the generalised Lindelöf hypothesis, the same result is true with $|x_3| < n^{\frac{1}{4}+\delta}$.

Still fall well short of Linnik's conjecture $|x_3| < n^{\delta}$.

Proof shows small scale equidistribution when $\mathrm{vol}(B_n) \gg n^{-\frac{1}{18}+\delta}$.

Rotated Linnik's Conjecture

Linnik's conjecture is small scale equidistribution on thin annuli around the equator, with respect to the north pole $(0, 0, 1) \in S^2$.

Nothing special about this choice of north pole; could also choose any other equator with respect to a point $w = (w_1, w_2, w_3) \in S^2$.

Conjecture (Rotated Linnik's Conjecture)

Fix $\delta > 0$. For all sufficiently large squarefree $n \not\equiv 7 \pmod 8$,

$$x_1^2 + x_2^2 + x_3^2 = n$$

has an integral solution $x = (x_1, x_2, x_3) \in \mathbb{Z}^3$ with $|x \cdot w| < n^\delta$.

Averaged Rotated Linnik's Conjecture

Theorem (H.–Radziwiłł (2019))

Fix $\delta > 0$. *For squarefree* $n \not\equiv 7 \pmod 8$, *the volume of the set of* $w \in S^2$ *for which*

$$x_1^2 + x_2^2 + x_3^2 = n$$

has no integral solutions $x = (x_1, x_2, x_3) \in \mathbb{Z}^3$ *with* $|x \cdot w| < n^\delta$ *is* $O(n^{-\delta})$ *as* $n \to \infty$.

Unconditionally resolves the rotated Linnik's conjecture for almost every pole $w \in S^2$.

Optimal. Fails if instead one demands $|x \cdot w| < 1000$.

Optimal Small Scale Equidistribution on Annuli

Proof follows from a stronger result: the volume of

$$\left\{ w \in S^2 : \left| \frac{1}{\mathrm{vol}(B_n)} \frac{\#(\widehat{\mathcal{E}}(n) \cap B_n(w))}{\#\mathcal{E}(n)} - 1 \right| > c \right\}$$

is $O(n^{-\delta})$ for any fixed $c > 0$.

$B_n = B_n(w)$ denotes the annulus around the equator with respect to the north pole $w = (w_1, w_2, w_3) \in S^2$ of volume $n^{-\frac{1}{2}+\delta}$.

This implies the equidistribution of $\widehat{\mathcal{E}}(n)$ on the shrinking annulus $B_n(w)$ of volume $n^{-\frac{1}{2}+\delta}$ for almost every $w \in S^2$.

Rate of shrinking is **optimal**.

By Chebyshev's inequality, this result follows upon showing that

$$\mathrm{Var}(\widehat{\mathcal{E}}(n); B_n) := \int_{S^2} \left(\frac{1}{\mathrm{vol}(B_n)} \frac{\#(\widehat{\mathcal{E}}(n) \cap B_n(w))}{\#\mathcal{E}(n)} - 1 \right)^2 dw$$

is $O(n^{-\delta})$ as $n \to \infty$.

Can ask for more refined results about this variance other than just tending to zero as $n \to \infty$.

Bourgain–Rudnick–Sarnak Conjecture

Conjecture (Bourgain–Rudnick–Sarnak (2017))

Let $B_n(w)$ be a sequence of balls (spherical caps) or annuli on S^2 of decreasing volume. If $\mathrm{vol}(B_n) \ll n^{-\delta}$ for some $\delta > 0$,

$$\mathrm{Var}(\widehat{\mathcal{E}}(n); B_n) \sim \frac{1}{\mathrm{vol}(B_n)\#\widehat{\mathcal{E}}(n)}.$$

After a renormalisation, this states that the variance is asymptotic to the expectation.

Motivation

Such an asymptotic holds for *random* points.

Bourgain–Rudnick–Sarnak Conjecture

Theorem (H.–Radziwiłł (2019))

Let $B_n(w)$ be a sequence of annuli on S^2 with fixed inner radius for which $\mathrm{vol}(B_n) \ll n^{-\frac{5}{12}-\delta}$ for some $\delta > 0$. Then

$$\mathrm{Var}(\widehat{\mathcal{E}}(n); B_n) \sim \frac{1}{\mathrm{vol}(B_n)\#\widehat{\mathcal{E}}(n)}.$$

Resolves the Bourgain–Rudnick–Sarnak conjecture for small annuli.

Idea of Proof

Method of proof to bound the variance: spectral expansion on $L^2(S^2)$ plus Waldspurger's formula.

Lemma

We have that

$$\mathrm{Var}(\widehat{\mathcal{E}}(n); B_n) \approx \frac{1}{\sqrt{n}} \sum_f L\left(\frac{1}{2}, f\right) L\left(\frac{1}{2}, f \otimes \chi_{-n}\right) |h(k_f)|^2$$

where the sum is over modular forms of even weight $k_f \in 2\mathbb{N}$, and

$$h(k) \ll \begin{cases} \dfrac{1}{\sqrt{k}} & \text{for } k \leq \dfrac{1}{\mathrm{vol}(B_n)}, \\ \dfrac{1}{\mathrm{vol}(B_n) k^{3/2}} & \text{for } k \geq \dfrac{1}{\mathrm{vol}(B_n)}. \end{cases}$$

Reduction to Bounds for Moments of L-Functions

Break up sum into dyadic ranges; reduces problem to bounding moments of L-functions.

Corollary

Good bounds for $\mathrm{Var}(\widehat{\mathcal{E}}(n); B_n)$ *follow from good bounds for the moment of L-functions*

$$\sum_{T \leq k_f \leq 2T} L\left(\frac{1}{2}, f\right) L\left(\frac{1}{2}, f \otimes \chi_{-n}\right)$$

associated to modular forms f *of even weight* $k_f \in [T, 2T] \cap 2\mathbb{N}$.

Bounds for Moments of L-Functions

Assuming the generalised Lindelöf hypothesis,

$$\sum_{T \leq k_f \leq 2T} L\left(\frac{1}{2}, f\right) L\left(\frac{1}{2}, f \otimes \chi_{-n}\right) \ll_\varepsilon n^\varepsilon T^{2+\varepsilon}.$$

Would like results of this strength *unconditionally*.

Lemma

Unconditionally, the moment above is

$$\ll_\varepsilon \begin{cases} n^{\frac{1}{3}+\varepsilon} T^{2+\varepsilon} & \text{for } T \ll n^{\frac{1}{12}}, \\ n^{\frac{1}{2}+\varepsilon} & \text{for } n^{\frac{1}{12}} \ll T \ll n^{\frac{1}{4}}, \\ n^\varepsilon T^{2+\varepsilon} & \text{for } T \gg n^{\frac{1}{4}}. \end{cases}$$

"Lindelöf on average" for T sufficiently large.

Asymptotics for Moments of L-Functions

For the Bourgain–Rudnick–Sarnak conjecture, we need *asymptotics* instead of upper bounds for the moment

$$\sum_{T \leq k_f \leq 2T} L\left(\frac{1}{2}, f\right) L\left(\frac{1}{2}, f \otimes \chi_{-n}\right).$$

Lemma

When $T \gg n^{\frac{1}{4}}$, we have asymptotics with a main term

$$L(1, \chi_{-n}) T^2.$$

Sketch of Proof

Sketch of proof:

- Petersson trace formula for modular forms,
- Poisson summation formula,
- Stationary phase to bound complicated integrals involving Bessel functions.

Problem is reduced to analysis; difficulties due to uniformly bounding integrals and sums involving many variables.

Major issue: special functions behave differently in various regimes, so many separate cases to deal with.

Sketch of Proof

End up with

$$\sum_{T \leq k_f \leq 2T} L\left(\frac{1}{2}, f\right) L\left(\frac{1}{2}, f \otimes \chi_{-n}\right) \approx L(1, \chi_{-n}) T^2 + O_\varepsilon\left(n^{\frac{1}{2}+\varepsilon}\right).$$

Main term dominates when $T \gg n^{\frac{1}{4}}$.

Alternative strategy is to use Hölder's inequality and bounds for cubic moments of L-functions (Conrey–Iwaniec, Young, Petrow–Young) to get

$$\sum_{T \leq k_f \leq 2T} L\left(\frac{1}{2}, f\right) L\left(\frac{1}{2}, f \otimes \chi_{-n}\right) \ll_\varepsilon n^{\frac{1}{3}+\varepsilon} T^{2+\varepsilon}.$$

Better when $T \ll n^{\frac{1}{12}}$.

Optimal Small Scale Equidistribution on Balls

Want to show optimal small scale equidistribution on almost every shrinking ball (spherical cap) B_n on S^2. Implied by

$$\mathrm{Var}(\widehat{\mathcal{E}}(n); B_n) = o(1)$$

for $\mathrm{vol}(B_n) \gg n^{-\frac{1}{2}+\delta}$. Need the bound

$$\sum_{T \leq k_f \leq 2T} L\left(\frac{1}{2}, f\right) L\left(\frac{1}{2}, f \otimes \chi_{-n}\right) = o(\sqrt{n})$$

for $T = o\left(n^{\frac{1}{4}}\right)$.

Unfortunately, can only prove $O_\varepsilon\left(n^{\frac{1}{2}+\varepsilon}\right)$ for $n^{\frac{1}{12}} \ll T \ll n^{\frac{1}{4}}$. Need to find additional cancellation from error term (shifted convolution sum).

Related Problems

Method also works for ternary quadratic forms other than just

$$x_1^2 + x_2^2 + x_3^2 = n,$$

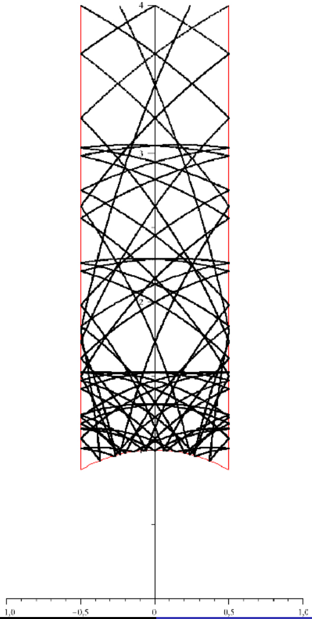
namely

$$x_2^2 - 4x_1x_3 = D.$$

- For $D > 0$, we get closed geodesics on the modular surface $\Gamma\backslash\mathbb{H}$ instead of lattice points on the sphere S^2.
- For $D < 0$, we get Heegner points on $\Gamma\backslash\mathbb{H}$.

Example: $D = 377$

(Image: Einsiedler–Lindenstrauss–Michel–Venkatesh)



Theorem (H.–Radziwiłł (2019))

(1) *Exact same results hold for Heegner points as for lattice points on the sphere.*

(2) *For closed geodesics, we obtain stronger results: small scale equidistribution on almost every shrinking* ball *down to the optimal scale.*

Chief difference for closed geodesics versus Heegner points and lattice points on the sphere: codimension 1 instead of 2.

Method is essentially the same; spectral expansion of the variance involves Maaß forms instead of modular forms, so we use the Kuznetsov formula instead of the Petersson formula.

Thank you!