Chain conditions, unbounded colorings and the C-sequence spectrum

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Bibliography

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3. **Knaster and friends III: Subadditive colorings and stationarily layered posets**, *in preparation*.

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- For a, b, nonempty sets of ordinals,
  $a < b$ means that $\sup(a) < \min(b)$.

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  $\forall A \in [P]^\kappa\ \exists B \in [A]^\kappa\ \forall X \in [B]^{<\omega}\ \bigwedge X \neq \emptyset$.
- $\mathbb{P}$ is κ-stationarily layered iff the following set is stationary:
  $\{Q \in [P]^{<\kappa} \mid \langle Q, \leq \rangle$ is a regular suborder of $\mathbb{P}\}$.

The product order (aka, coordinatewise order)

Given posets $\langle P_1, \leq_1 \rangle, \langle P_2, \leq_2 \rangle$, consider their product $\langle P_1 \times P_2, \trianglelefteq \rangle$, where $(x, y) \trianglelefteq (x', y')$ iff $x \leq_1 x'$ and $y \leq_2 y'$.

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Note: It suffices to consider squares

$\mathcal{C}_\kappa$ iff $\mathbb{P}^2$ is κ-cc for every κ-cc poset $\mathbb{P}$.

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- Kurepa (1963): If $\mathbb{P}$ satisfies the λ^+-cc, then $\mathbb{P}^2$ satisfies the $(2^\lambda)^+$-cc. $\square$

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- (Galvin, 1980) after (Laver, unpublished): $\mathfrak{c} = \aleph_1$ refutes $\mathcal{C}_{\aleph_1}$.
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Open problem

Is $\mathrm{MA}_{\aleph_1}$ equivalent to $\mathcal{C}_{\aleph_1}$?

The case $\kappa > \aleph_1$. Counterexamples in ZFC

Theorem (Todorcevic, 1985)

$\mathcal{C}_{\mathrm{cf}(\beth_{\alpha+1})}$ fails for every limit ordinal α.
Moreover, if λ is a cardinal for which there exists a linear order of size 2^λ with a dense subset of size λ, then $\mathcal{C}_\kappa$ fails, for $\kappa = \mathrm{cf}(2^\lambda)$.

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More counterexamples in ZFC

Theorem (Shelah, 1990–1997)

$\mathcal{C}_{\lambda^+}$ fails whenever λ is a regular cardinal $\geq \aleph_1$. Specifically:

- [Sh:280]: $\lambda > 2^{\aleph_0}$;
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Conjecture (Todorcevic, 1980's)

For all regular cardinal $\kappa > \aleph_1$, $\mathcal{C}_\kappa$ iff κ is weakly compact.

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In fact, $\mathcal{C}_\kappa$ entails $\neg\Box(\kappa)$ & every stationary subset of κ reflects.

Longer products and stronger chain conditions

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Characterization theorem (Cox and Lücke, 2016)

For every regular uncountable cardinal κ*:*
κ *is weakly compact iff every* κ*-cc poset is* κ*-stationarily layered.*

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Theorem (Lambie-Hanson and Lücke, 2018)

Suppose $\theta < \kappa$ are infinite and regular.
If the class of κ-Knaster posets is closed under θ-support products, then $\neg\square(\kappa)$, so that (κ is weakly comapct)^L.

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Galvin (1980) gave a consistent construction of an anti-Ramsey coloring $c : [\kappa]^2 \to 2$ from which he derived a κ-cc poset whose square is not κ-cc.

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Galvin (1980) gave a consistent construction of an anti-Ramsey coloring $c : [\kappa]^2 \to 2$ from which he derived a κ-cc poset whose square is not κ-cc.
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We would like to obtain the conclusions of Lambie-Hanson and Lücke from ZFC, e.g., getting a ZFC example of an $\aleph_2$-Knaster poset whose ω^{th}-power is not $\aleph_2$-cc.

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For this, let us revisit Galvin's approach.

Colorings FTW

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From a coloring $c : [\kappa]^2 \to \theta$ with $\theta \in \mathrm{Reg}(\kappa)$, we derive posets:

- $\mathbb{P} := \{(x, i) \mid x \in [\kappa]^{<\omega}, c``[x]^2 \subseteq \{i\}\}$;

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Key feature

- $\mathbb{P}^2$ fails to have the κ-cc;
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Key feature

- $\mathbb{P}^2$ fails to have the κ-cc, e.g., $\{\langle(\{\alpha\}, 0), (\{\alpha\}, 1)\rangle \mid \alpha < \kappa\}$.
- $\mathbb{Q}^\theta$ fails to have the κ-cc.

About $\mathbb{P}^2$.

For $\alpha < \beta < \kappa$ and $i := c(\alpha, \beta)$, $(\{\alpha\}, 1 - i)$ and $(\{\beta\}, 1 - i)$ are incompatible in $\mathbb{P}$. □

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About $\mathbb{Q}^\theta$.

For $\alpha < \beta < \kappa$ and $i := c(\alpha, \beta)$, $(\{\alpha\}, i + 1)$ and $(\{\beta\}, i + 1)$ are incompatible in $\mathbb{Q}$. □

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The heart of the matter is to construct c for which the corresponding $\mathbb{P}$ be κ-cc, or $\mathbb{Q}^\tau$ be κ-Knaster for all $\tau < \theta$.

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The poset $\mathbb{P}$ was analyzed by Galvin, giving birth to $\mathrm{Pr}_1(\ldots)$.
Today, we shall focus on the poset $\mathbb{Q}$.

Unbounded functions

Suppose $\mathbb{Q} := \{(x, i) \mid x \in [\kappa]^{<\omega}, c``[x]^2 \cap i = \emptyset\}$ is derived from $c : [\kappa]^2 \to \theta$. Assuming $\theta \in \mathrm{Reg}(\kappa)$, $\mathbb{Q}$ is κ-Knaster iff it has precaliber κ iff c witnesses $\mathsf{U}(\kappa, \theta)$:

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$\mathrm{U}(\kappa, \theta)$ asserts that there exists a coloring $c : [\kappa]^2 \to \theta$ such that for every family $\mathcal{A} \subseteq [\kappa]^{<\omega}$ consisting of κ-many pairwise disjoint sets, and every $i < \theta$, there is $\mathcal{B} \in [\mathcal{A}]^\kappa$ such that $\min(c[a \times b]) \geq i$ for every pair $a < b$ from $\mathcal{B}$.

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$\mathrm{U}(\kappa, \mu, \theta, \chi)$ asserts there is a coloring $c : [\kappa]^2 \to \theta$ such that for every $\chi' < \chi$, every family $\mathcal{A} \subseteq [\kappa]^{\chi'}$ consisting of κ-many pairwise disjoint sets, and every $i < \theta$, there is $\mathcal{B} \in [\mathcal{A}]^\mu$ such that $\min(c[a \times b]) \geq i$ for every pair $a < b$ from $\mathcal{B}$.

Note that $\mathrm{Pr}_1(\kappa, \kappa, \theta, \chi)$ entails $\mathrm{U}(\kappa, 2, \theta, \chi)$.

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Suppose $\chi, \theta \in \mathrm{Reg}(\kappa)$ and that κ is $({<}\chi)$-inaccessible.
For every coloring $c : [\kappa]^2 \to \theta$ witnessing $\mathsf{U}(\kappa, \mu, \theta, \chi)$,
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- *$\mathbb{Q}$ is well-met and χ-directed-closed with greatest lower bounds.*

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Conjecture

For κ regular uncountable, κ is weakly compact iff $\neg\,\mathsf{U}(\kappa, 2, \omega, 2)$.

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Put differently, we ask whether the existence of a κ-Aronszajn tree gives rise to a coloring $c : [\kappa]^2 \to \omega$ with the property that $\sup(c``[A]^2) = \omega$ for every $A \in [\kappa]^\kappa$.

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Partial answer 1

The existence of a κ-Aronszajn tree with an ω-ascent path entails $\mathsf{U}(\kappa,2,\omega,\omega)$.

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Partial answer 1

The existence of a κ-Aronszajn tree with an ω-ascent path entails $\mathrm{U}(\kappa, 2, \omega, \omega)$.

Partial answer 2 (with Todorcevic)

The existence of a coherent κ-Aronszajn tree entails $\mathrm{U}(\kappa, 2, \omega, \omega)$ but not $\mathrm{U}(\kappa, \kappa, \omega, \omega)$.

Inspecting the parameters

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- $\mathsf{U}(\kappa, 2, \theta, \chi)$ iff $\mathsf{U}(\kappa, \omega, \theta, \chi)$;
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Definition

$c : [\kappa]^2 \to \theta$ is closed iff $\{\alpha < \beta \mid c(\alpha, \beta) \leq i\}$ is closed below β for all $\beta < \kappa$, $i < \theta$.

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- $\mathrm{U}(\kappa, \kappa, \kappa, \kappa)$ holds;
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$\mathsf{U}(\kappa,\mu,\theta,\chi)$ asserts there is a coloring $c:[\kappa]^2\rightarrow\theta$ such that for every $\chi'<\chi$, every family $\mathcal{A}\subseteq[\kappa]^{\chi'}$ consisting of κ-many pairwise disjoint sets, and every $i<\theta$, there is $\mathcal{B}\in[\mathcal{A}]^{\mu}$ such that $\min(c[a\times b])\geq i$ for every pair $a<b$ from $\mathcal{B}$.

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- $\mathsf{U}(\kappa,\kappa,\kappa,\kappa)$ holds;
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Therefore, hereafter, we shall focus on $\theta\in\mathrm{Reg}(\kappa)$.

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- $\mathsf{U}(\kappa, \kappa, \kappa, \kappa)$ holds;
- $\mathsf{U}(\kappa, \mu, \theta, \chi)$ iff $\mathsf{U}(\kappa, \mu, \mathrm{cf}(\theta), \chi)$;
- Lack of monotonicity: If λ is the singular limit of strongly compact cardinals, then, for every $\theta \leq \lambda$, $\mathsf{U}(\lambda^+, \lambda^+, \theta, \lambda)$ iff $\mathrm{cf}(\theta) = \mathrm{cf}(\lambda)$.

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- $\mathsf{U}(\kappa, \kappa, \theta, 3)$ iff $\mathsf{U}(\kappa, \kappa, \theta, \omega)$;
- $\mathsf{U}(\lambda^+, 2, \theta, 2)$ iff $\mathsf{U}(\lambda^+, 2, \theta, \mathrm{cf}(\lambda))$;

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$\mathsf{U}(\kappa, \mu, \theta, \chi)$ asserts there is a coloring $c : [\kappa]^2 \to \theta$ such that for every $\chi' < \chi$, every family $\mathcal{A} \subseteq [\kappa]^{\chi'}$ consisting of κ-many pairwise disjoint sets, and every $i < \theta$, there is $\mathcal{B} \in [\mathcal{A}]^{\mu}$ such that $\min(c[a \times b]) \geq i$ for every pair $a < b$ from $\mathcal{B}$.

About the fourth parameter

- $\mathsf{U}(\kappa, \kappa, \theta, 3)$ iff $\mathsf{U}(\kappa, \kappa, \theta, \omega)$;
- $\mathsf{U}(\lambda^+, 2, \theta, 2)$ iff $\mathsf{U}(\lambda^+, 2, \theta, \mathrm{cf}(\lambda))$;

The above is optimal: If λ is the limit of strongly compact cardinals, $\theta \in \mathrm{Reg}(\lambda)$ with $\theta \neq \mathrm{cf}(\lambda)$, then $\mathsf{U}(\lambda^+, 2, \theta, \chi)$ holds for $\chi := \mathrm{cf}(\lambda)$, but fails for $\chi := \mathrm{cf}(\lambda)^+$.

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- There are κ, θ and colorings c, $c \models \mathrm{U}(\kappa, \kappa, \theta, 2)$, but $c \not\models \mathrm{U}(\kappa, 2, \theta, 3)$;
- If there is a closed witness to $\mathrm{U}(\lambda^+, \lambda^+, \theta, 2)$, then there is one for $\mathrm{U}(\lambda^+, \lambda^+, \theta, \mathrm{cf}(\lambda))$.

Further findings

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Theorem

For every regular λ *and* $\theta \in \mathrm{Reg}(\lambda^+)$*, there is* $c : [\lambda^+]^2 \to \theta$ *witnessing* $\mathsf{U}(\lambda^+, \lambda^+, \theta, \lambda)$ *which is moreover closed.*

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In case you wondered

The corresponding tree $\mathcal{T}(c) := \{c(\cdot, \gamma) \restriction \beta \mid \beta \leq \gamma < \kappa\}$ may consistently be a special κ-Aronszajn tree, as well as an almost Souslin κ-Aronszajn tree.

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There exists an $\aleph_2$*-Knaster poset whose* ω^{th}*-power is not* $\aleph_2$*-cc.*

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More generally

Suppose that $\theta \leq \chi \leq \lambda$ *are regular, with* $\lambda^{<\chi} = \lambda$*.*
Then there exists a χ*-directed-closed poset* $\mathbb{Q}$ *such that:*

- $\mathbb{Q}^\tau$ *has precaliber* λ^+ *for all* $\tau < \theta$*;*
- $\mathbb{Q}^\theta$ *is not* λ^+*-cc.*

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CH $\Rightarrow \exists\sigma$*-closed* $\aleph_2$*-Knaster poset whose* ω^{th}*-power is not* $\aleph_2$*-cc.*

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Open problem

Does CH entail a σ-closed $\aleph_2$-cc poset whose square is not $\aleph_2$-cc?

Further findings (cont.)

Theorem

For every singular λ *and* $\theta \in \mathrm{Reg}(\lambda)$, *any of the following entail the existence of a closed witness to* $\mathrm{U}(\lambda^+, \lambda^+, \theta, \mathrm{cf}(\lambda))$*:*

- $2^\lambda = \lambda^+$;
- $\mathrm{Refl}(<\mathrm{cf}(\lambda), \lambda^+)$ *fails;*
- $\theta = \omega$ *or* $\theta = \mathrm{cf}(\lambda)$;
- $\theta < \nu < \nu^+ = \mathrm{cf}(\lambda)$;
- $\theta < \mathrm{cf}(\lambda)$ *and* $\mathrm{cf}(\mathrm{NS}_{\mathrm{cf}(\lambda)}, \subseteq) < \lambda$.

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Corollary

If the class of κ*-Knaster posets is closed under* ω *powers, then* κ *is inaccessible.*

Further findings (cont.)

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Theorem

For every $\theta, \chi \in \mathrm{Reg}(\kappa)$, any of the following entails the existence of a closed witness to $\mathrm{U}(\kappa, \kappa, \theta, \chi)$:

- $\square(\kappa, {<}\omega)$ *or* $\square^{\mathrm{ind}}(\kappa, \theta)$;
- $\exists$*stationary* $S \subseteq E^\kappa_{\geq\chi}$ *with* $S \cap \alpha$ *nonstationary for* $\alpha \in E^\kappa_{>\omega}$;
- $\exists$*stationary* $S \subseteq E^\kappa_{\geq\chi}$ *with* $S \cap \alpha$ *nonstationary for all* $\alpha \in \mathrm{Reg}(\kappa)$, *and* κ *is inacc.*

A new cardinal invariant

The C-sequence number

Theorem (Todorcevic, 1987)

For every strongly inaccessible cardinal κ, the following are equivalent:

1. κ is weakly compact;
2. For every C-sequence $\langle C_\beta \mid \beta < \kappa \rangle$, there exist $\Delta \in [\kappa]^\kappa$ and $b : \kappa \to \kappa$ such that $\Delta \cap \alpha = C_{b(\alpha)} \cap \alpha$ for every $\alpha < \kappa$.

Recall

$\langle C_\beta \mid \beta < \kappa \rangle$ is a C-sequence iff each C_β is closed subset of β with $\sup(C_\beta) = \sup(\beta)$.

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The cardinal invariant that we introduce suggests a way to measure how far an inaccessible cardinal κ is from being weakly compact, though, will see it is of interest for successor cardinals as well.

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If κ is weakly compact, then let $\chi(\kappa) := 0$.

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Definition (The C-sequence number of κ)

If κ is weakly compact, then let $\chi(\kappa) := 0$.
Otherwise, let $\chi(\kappa)$ denote the least $\chi \leq \kappa$ such that, for every C-sequence $\langle C_\beta \mid \beta < \kappa \rangle$, there exist $\Delta \in [\kappa]^\kappa$ and $b : \kappa \to [\kappa]^\chi$ with $\Delta \cap \alpha \subseteq \bigcup_{\beta \in b(\alpha)} C_\beta$ for all $\alpha < \kappa$.

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Note that $\chi(\kappa)$ is well-defined. In fact, $\chi(\kappa) \leq \sup(\mathrm{Reg}(\kappa))$.

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Todorcevic's analysis of *the number of steps* function readily establishes the following.

The C-sequence number and yoU

$\mathsf{U}(\kappa, \kappa, \omega, \chi(\kappa))$ *holds, as witnessed by the closed function* ρ_2*.*

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If the class of κ-Knaster posets is closed under taking ω powers, then $\chi(\kappa) < \omega$.

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Questions

- Is "$\chi(\kappa) < \omega$" a large cardinal property?
- How about "$\chi(\kappa) < \sup(\mathrm{Reg}(\kappa))$"?
- Could $\chi(\kappa)$ be singular?

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Increasing the C-sequence number

Kunen (1978) showed that by forcing over a model with a weakly compact cardinal κ, one obtains a model V having a κ-Souslin tree $\mathbb{S}$ such that $V^{\mathbb{S}} \models \kappa$ is weakly compact.

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As $\mathbb{S}$ is κ-cc, there is a club $D \subseteq \kappa$ in V, with $D \subseteq \Delta$.

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As $\mathbb{S}$ is κ-cc, there is a club $D \subseteq \kappa$ in V, with $D \subseteq \Delta$.
Then $D \cap \alpha \subseteq C_{b(\alpha)} \cap \alpha$ for each $\alpha < \kappa$. $\square$

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Then $D \cap \alpha \subseteq C_{b(\alpha)} \cap \alpha$ for each $\alpha < \kappa$. $\square$

Theorem

Suppose $\chi(\kappa) = 0$. For every $\theta \in \mathrm{Reg}(\kappa^+)$, there is a cofinality-preserving forcing extension in which κ remains strongly inaccessible, and $\chi(\kappa) = \theta$.

Increasing the C-sequence number (cont.)

Observation

$\mathrm{cf}(\lambda) \leq \chi(\lambda^+) \leq \lambda$.

[1] The latter assumes the consistency of a supercompact.

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If λ is a singular limit of supercompacts, then $\chi(\lambda^+) = \mathrm{cf}(\lambda)$.

Theorem

If λ is a singular limit of supercompacts, and $\theta \in \mathrm{Reg}(\lambda) \setminus \mathrm{cf}(\lambda)$, then, in some cofinality-preserving forcing extension, $\chi(\lambda^+) = \theta$.

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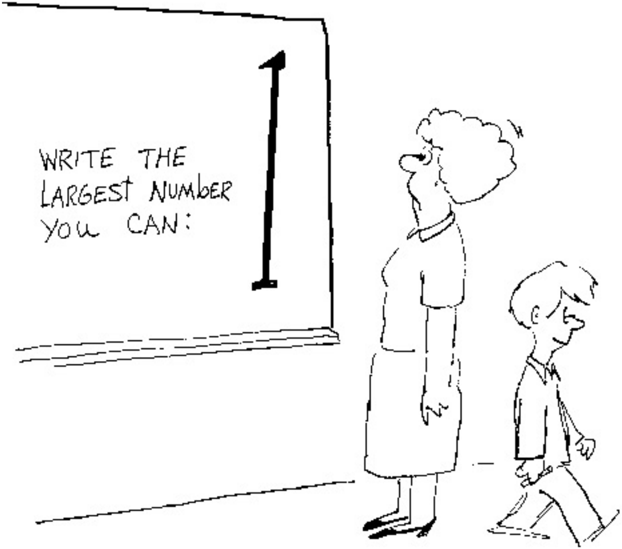
Theorem

$\chi(\aleph_{\omega+1}) = \aleph_\omega$ is consistent, and so is $\chi(\aleph_{\omega+1}) = \omega$.[1]

[1] The latter assumes the consistency of a supercompact.

How large



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Corollary

- *In L, either* $\chi(\kappa) = 0$ *or* $\chi(\kappa) = \sup(\mathrm{Reg}(\kappa))$;
- $\square(\kappa, {<}\omega)$ *entails* $\chi(\kappa) = \sup(\mathrm{Reg}(\kappa))$;
- *If* $\chi(\kappa) = 1$, *then* κ *is greatly Mahlo.*

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Theorem

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Corollary

If the class of κ*-Knaster posets is closed under* ω *powers, then* κ *is greatly Mahlo.*

The C-sequence spectrum

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Definition

For a C-sequence $\vec{C} = \langle C_\beta \mid \beta < \kappa \rangle$, let $\chi(\vec{C})$ denote the least cardinal $\chi \leq \kappa$ such that there exist $\Delta \in [\kappa]^\kappa$ and $b : \kappa \to [\kappa]^\chi$ with $\Delta \cap \alpha \subseteq \bigcup_{\beta \in b(\alpha)} C_\beta$ for every $\alpha < \kappa$.

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Definition

$\mathrm{Cspec}(\kappa) := \{\chi(\vec{C}) \mid \vec{C} \text{ is a } C\text{-sequence over } \kappa\} \setminus \omega$.

The C-sequence spectrum

Definition

For a C-sequence $\vec{C} = \langle C_\beta \mid \beta < \kappa \rangle$, let $\chi(\vec{C})$ denote the least cardinal $\chi \leq \kappa$ such that there exist $\Delta \in [\kappa]^\kappa$ and $b : \kappa \to [\kappa]^\chi$ with $\Delta \cap \alpha \subseteq \bigcup_{\beta \in b(\alpha)} C_\beta$ for every $\alpha < \kappa$.

Definition

$\mathrm{Cspec}(\kappa) := \{\chi(\vec{C}) \mid \vec{C} \text{ is a } C\text{-sequence over } \kappa\} \setminus \omega$.

Theorem

1. *If* $\mathrm{Cspec}(\kappa) \neq \emptyset$, *then* $\chi(\kappa) = \max(\mathrm{Cspec}(\kappa))$;

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Open problem

Is $\mathrm{Cspec}(\kappa)$ an interval? Is it a closed set?

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Is $\mathrm{Cspec}(\kappa)$ an interval? Is it a closed set?
Is every limit uncountable cardinal in $\mathrm{Cspec}(\kappa)$ an accumulation point of $\mathrm{Cspec}(\kappa)$?

Unexpected equivalency

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Theorem

For every $\theta \in \mathsf{Reg}(\kappa)$, the following are equivalent:

- $\theta \in \mathsf{Cspec}(\kappa)$;
- *There exists a closed witness to* $\mathsf{U}(\kappa, \kappa, \theta, \theta)$.

The forward implication works for θ singular; the backward does not.

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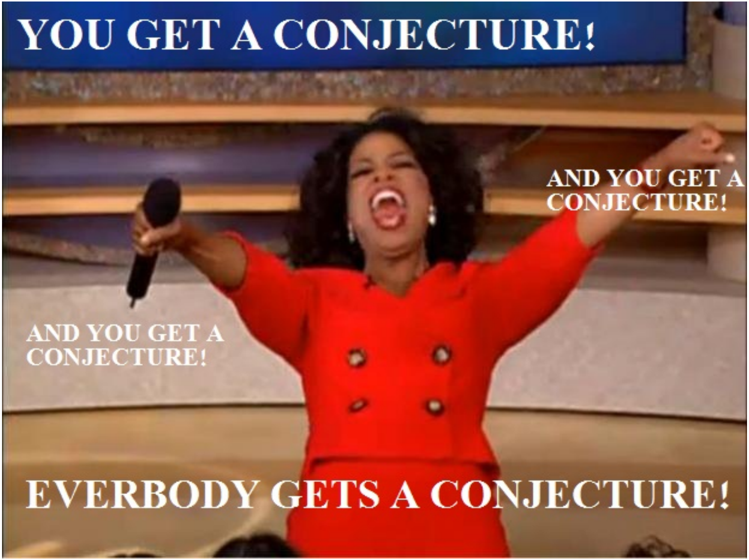
Corollary

- *If* κ *is a successor of a regular, then* $\mathrm{Reg}(\kappa) \subseteq \mathrm{Cspec}(\kappa)$;
- *If* κ *is a non-Mahlo inaccessible, then* $\mathrm{Reg}(\kappa) \subseteq \mathrm{Cspec}(\kappa)$;
- *If* $\square(\kappa, {<}\omega)$ *holds, then* $\mathrm{Reg}(\kappa) \subseteq \mathrm{Cspec}(\kappa)$;
- *If* $E^{\kappa}_{\geq\chi}$ *admits a non-reflecting stationary subset, then* $\mathrm{Reg}(\chi^{+}) \subseteq \mathrm{Cspec}(\kappa)$.

Conjectures



Conjectures

1. If $\chi(\kappa) = 1$, then, in some set-forcing extension, $\chi(\kappa) = 0$.
2. If $\chi(\kappa) = 1$, then, there exists a coherent κ-Aronszajn tree.
3. If κ is inaccessible and $1 < \chi(\kappa) < \kappa$, then there exists a κ-Aronszajn tree with a $\chi(\kappa)$-ascent path.
4. Any $\mathsf{U}(\kappa, \kappa, \ldots)$ may be witnessed by a closed coloring.
5. If $\chi(\kappa)$ is singular, then $\mathrm{cf}(\chi(\kappa)) = \mathrm{cf}(\sup(\mathrm{Reg}(\kappa)))$.
6. $\mathrm{Reg}(\mathrm{cf}(\lambda)^+) \subseteq \mathrm{Cspec}(\lambda^+)$ for every singular λ.
7. For all $\theta, \chi \in \mathrm{Cspec}(\kappa)$, $\mathsf{U}(\kappa, \kappa, \theta, \chi)$ holds.

Thank you for your attention!