M_1 and its strategy extensions (and beyond)

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1. How much of its own iteration strategy can be added to M_1 without destroying the Woodinness of δ?
2. Given an M_1-cardinal $\kappa > \delta$, what is the κ-mantle of M_1?

Background...

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Set $M_0 = M$ and $M_1 = U$...

Iteration trees $\mathcal{T}$ on M:

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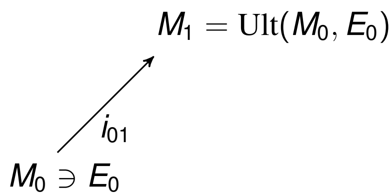

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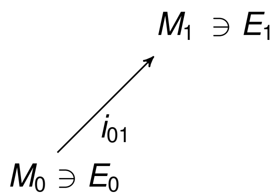

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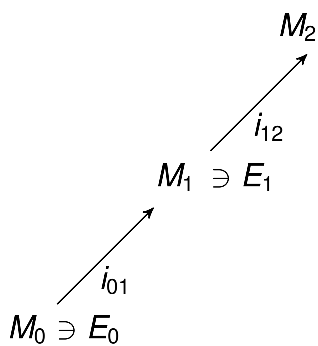

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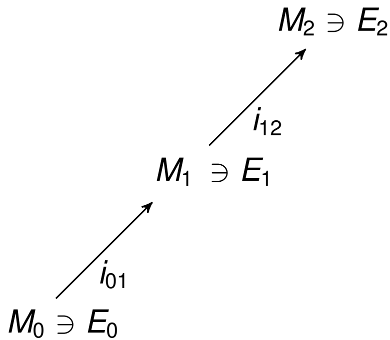

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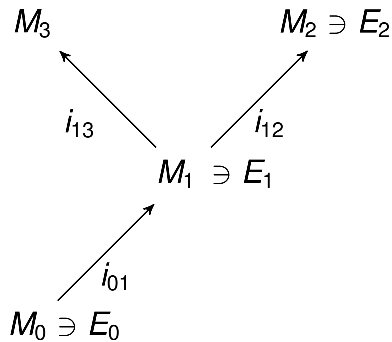

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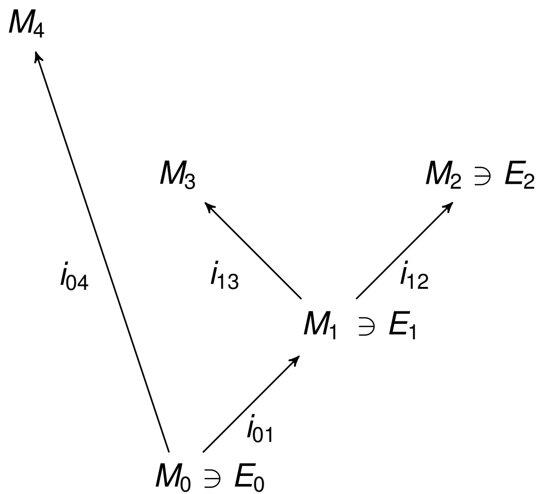

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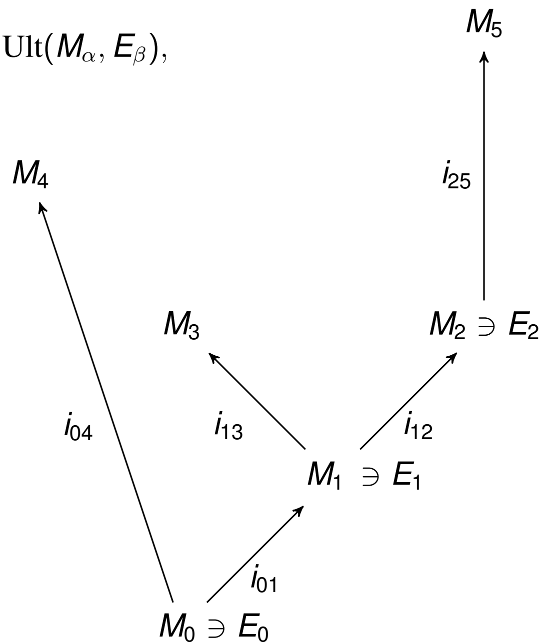

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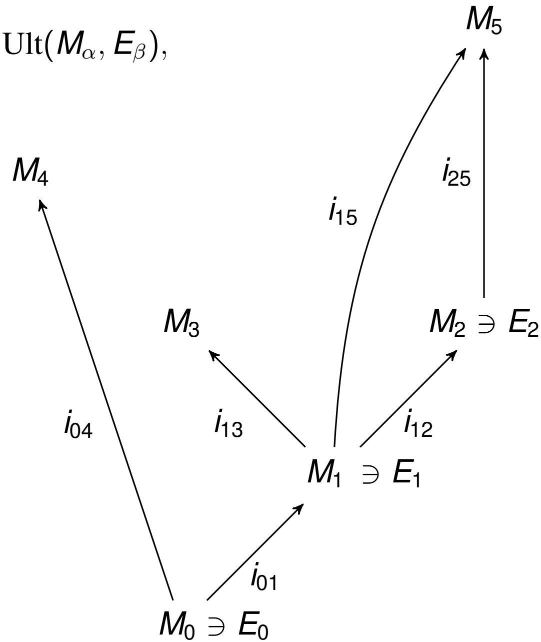

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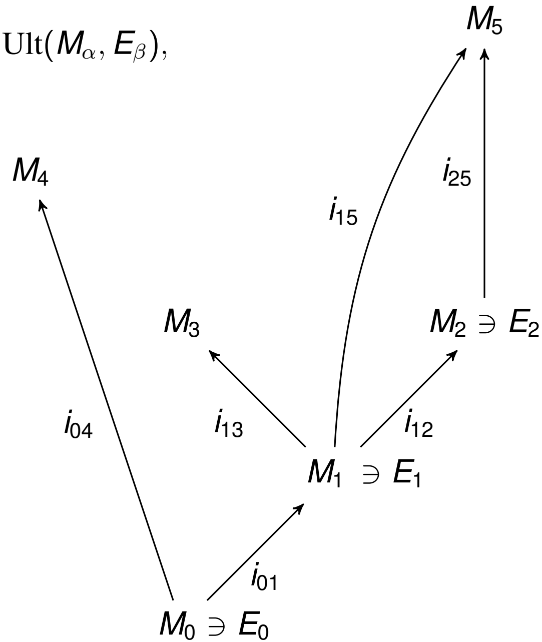
– Write $M^{\mathcal{T}}_\alpha = M_\alpha$, $E^{\mathcal{T}}_\alpha = E_\alpha$, etc.



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$$M_\lambda = M_b = \text{ direct limit of models along } b.$$

- Iteration strategy Σ chooses branches *b*, guarantees wellfounded models.
- *M* is (fully) iterable if such a Σ exists.

Definition 1.2.

An iteration tree $\mathcal{T}$ is normal iff

$$\alpha < \beta \implies \mathrm{lh}(E^{\mathcal{T}}_{\alpha}) \leq \mathrm{lh}(E^{\mathcal{T}}_{\beta}),$$

and all extenders apply to the earliest and largest model possible.
We write $\mathrm{lh}(\mathcal{T})$ for the length of $\mathcal{T}$.
$\mathcal{T}$ is on $M^{\mathcal{T}}_0$.
If $\mathcal{T}$ has successor length then $M^{\mathcal{T}}_{\infty} =$ last model of $\mathcal{T}$.
If b is a branch through $\mathcal{T}$ then $M^{\mathcal{T}}_b =$ direct limit model along b, and

$$i^{\mathcal{T}}_b : M^{\mathcal{T}}_0 \to M^{\mathcal{T}}_b$$

is the direct limit map.
Let $\mathcal{T}$ be a normal iteration tree, of limit length. Then (Coherence)

$$\alpha < \beta \implies M^{\mathcal{T}}_{\alpha}||\mathrm{lh}(E^{\mathcal{T}}_{\alpha}) = M^{\mathcal{T}}_{\beta}||\mathrm{lh}(E^{\mathcal{T}}_{\alpha}).$$

Write:

- $\delta(\mathcal{T}) = \sup_{\alpha<\mathrm{lh}(\mathcal{T})} \mathrm{lh}(E^{\mathcal{T}}_{\alpha})$.
- $M(\mathcal{T}) =$ eventual model of agreement of height $\delta(\mathcal{T})$,

$$M(\mathcal{T}) = \bigcup_{\alpha<\mathrm{lh}(\mathcal{T})} M^{\mathcal{T}}_{\alpha}||\mathrm{lh}(E^{\mathcal{T}}_{\alpha}).$$

Definition 1.3.

M_1 = the minimal proper class mouse with a Woodin cardinal $\delta = \delta^{M_1}$.

Then:

- $M_1 = L[\mathbb{E}^{M_1}] = L[\mathbb{E}]$ with $\mathbb{E} \subseteq M_1|\delta$.
- Write Σ_{M_1} for the (unique) normal iteration strategy for M_1.
- Is M_1 closed under Σ_{M_1}?
- If $\mathcal{T}$ is a normal tree on M_1, we say $\mathcal{T}$ is maximal iff

$$L[M(\mathcal{T})] \models \text{“}\delta(\mathcal{T}) \text{ is Woodin”}.$$

- Let $\mathcal{T}$ be maximal and $b = \Sigma_{M_1}(\mathcal{T})$. Then

$$M_b^{\mathcal{T}} = L[M(\mathcal{T})],$$

$$i_b^{\mathcal{T}}(\delta^{M_1}) = \delta(\mathcal{T}).$$

- M_1 computes correct branches through non-maximal trees.

But there are maximal trees $\mathcal{U} \in M_1$ such that:

– $L[M(\mathcal{U})]$ is a ground of M_1,
– $\delta(\mathcal{U})$ is a successor cardinal of M_1,
– so $i_b^{\mathcal{U}} \notin M_1$,
– so $b \notin M_1$

(uses Woodin's genericity iterations).

(Corollary: M_1 has proper grounds.)

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A strategy mouse* is an iterable structure $M = L_\alpha[\mathbb{E}, \Sigma]$, where:

- $\mathbb{E}$ is a good sequence of extenders,
- Σ encodes a partial iteration strategy for M.

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Question: How much iteration strategy can be added to a mouse M, without adding reals?

Definition 1.5.

For $\eta \in \mathrm{OR}$, let

$$\Sigma_\eta = \Sigma_{M_1} \restriction (M_1 | \eta).$$

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Theorem 1.6 (Woodin).

Let $\kappa = \kappa_0^{M_1}$. Then $M_1[\Sigma_\kappa]$ is a nice extension.
But $M_1^\# \in L[M_1, \Sigma_{M_1}]$ (construct by closing under Σ_{M_1}).

So δ^{M_1} is countable in $L[M_1, \Sigma_{M_1}]$.

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- The mantle $\mathbb{M}$ is the intersection of all grounds.
- Let η be a cardinal. The η-mantle $\mathbb{M}_\eta$ is the intersection of all W such that W is a ground via some forcing $\mathbb{P}$ of cardinality $< \eta$.

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Higher mice...Fuchs and Schindler extended this to a wider class of mice M lacking strong cardinals.

A strong cardinal changes the picture:

Definition 1.7.

$M_{\mathrm{sw}} =$ the minimal proper class mouse N with ordinals $\delta < \kappa$ such that

$$N \models \text{“}\delta \text{ is Woodin and } \kappa \text{ is strong”}.$$

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  - $= \mathrm{HOD}^{M[G]}$*, for M-generic $G \subseteq \mathrm{Col}(\omega, \lambda)$, for all large λ,*
  - *= the mantle of M,*
  - *= the least ground of M.*

Definition 1.9.

$M_{\rm swsw}$ = minimal proper class mouse $M = L[\mathbb{E}]$ with

$$\delta_0 < \kappa_0 < \delta_1 < \kappa_1$$

with δ_n Woodin and κ_n strong in M, for $n = 0, 1$.

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Theorem 1.10 (Sargsyan, Schindler, S.).

Assume $M^{\#}_{\mathrm{swsw}}$ exists. Then there is a class $\mathscr{V}_2$ of $M = M_{\mathrm{swsw}}$ such that:

- *$\mathscr{V}_2$ is a strategy mouse, closed under its strategy,*
- *$\mathscr{V}_2 \models$ "There are 2 Woodin cardinals"*
- *The universe of $\mathscr{V}_2$:*
  - $= \mathrm{HOD}^{M[G]}$, *for M-generic $G \subseteq \mathrm{Col}(\omega, \lambda)$, for large λ,*
  - $=$ *the mantle of M,*
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(To appear in Varsovian models II)

Theorem 1.11 (S.).

The κ_0-mantle of M_{swsw} is a strategy mouse $\mathscr{V}_1$.

Definition 1.12.

$M_{\mathrm{sw}\omega}$ = minimal proper class mouse $M = L[\mathbb{E}]$ with

$$\delta_0 < \kappa_0 < \delta_1 < \kappa_1 < \ldots < \delta_n < \kappa_n < \ldots$$

for $n < \omega$, with δ_n Woodin and κ_n strong in M.

Definition 1.12.

$M_{\mathrm{sw}\omega}$ = minimal proper class mouse $M = L[\mathbb{E}]$ with

$$\delta_0 < \kappa_0 < \delta_1 < \kappa_1 < \ldots < \delta_n < \kappa_n < \ldots$$

for $n < \omega$, with δ_n Woodin and κ_n strong in M.

Theorem 1.13 (S.).

Assume $M^{\#}_{\mathrm{sw}\omega}$ exists. Then there is a class $\mathscr{V}_\omega$ of $M = M_{\mathrm{sw}\omega}$ such that:

- *$\mathscr{V}_\omega$ is a strategy mouse, closed under its strategy,*
- *$\mathscr{V}_\omega \models$ "There are ω Woodin cardinals",*
- *The universe of $\mathscr{V}_\omega$:*
  - $= \mathrm{HOD}^{M[G]}$, *for M-generic $G \subseteq \mathrm{Col}(\omega, \lambda)$, for large λ,*
  - $=$ *the mantle of M, and*
  - $=$ *the least ground of M.*

But back to M_1...

What is the largest nice extension of M_1? Recall $M_1[\Sigma_{\kappa_0^{M_1}}]$ is nice (Woodin).

Theorem 1.14 (S.).

Let $\kappa = \kappa_0^{M_1}$ and $\kappa_+ = (\kappa^+)^{M_1}$. Then

$$M_1[\Sigma_\eta] \text{ is nice iff } \eta \leq \kappa_+.$$

In fact,

1. $M_1[\Sigma_{\kappa_+}] = M_1[\Sigma_\kappa]$.
2. *Let $\mathcal{U} \in M_1$ be the M_1-genericity iteration at κ_+. Let $b = \Sigma_{M_1}(\mathcal{U})$. Then*

$$M_1[b] = M_1[\Sigma_{\mathrm{OR}}] = L[M_1^\#].$$

What is the η-mantle of M_1? Does it model ZFC?

Write (for good enough η):

- $M_{\infty\eta}$ = the direct limit of all iterates of M_1 via maximal trees in $M_1|\eta$, and
- $\Gamma_{\infty\eta} = \Sigma_{j(\eta)}^{M_{\infty\eta}}$ = the corresponding strategy fragment for $M_{\infty\eta}$,*
- $j : M_1 \to M_{\infty\eta}$ is the iteration map.

Woodin showed that $M_{\infty\eta}[\Gamma_{\infty\eta}]$ is nice when $\eta \leq \kappa_0^{M_1}$.

Theorem 1.15 (S.).

Let $\kappa = \kappa_0^{M_1}$. Then the κ-mantle of M_1 is $M_{\infty\kappa}[\Gamma_{\infty\kappa}] \models \mathrm{ZFC}$.

Remark: Let $\eta = (\delta^{+\omega})^{M_1}$.
Then the strategy mouse "at η" is a proper subset of the η-mantle of M_1:

$$M_{\infty\eta}[\Gamma_{\infty\eta}] \subsetneq \mathbb{M}_\eta^{M_1}.$$

Proof of Theorem 1.14 part 2 (failure of niceness):

Let $\mathcal{T} = \mathcal{U}^\frown b$ be M_1-genericity iteration, first iterating least measurable out to $\kappa = \kappa_0^{M_1}$.
Let $P = L[M(\mathcal{U})] =$ last model of $\mathcal{T}$, and $j : M_1 \to P$ the iteration map.
Properties:

- $\mathcal{U} \in M_1$, with $\mathcal{U}$ of length κ_+.
- $\mathcal{U}$ and $M(\mathcal{U})$ are definable without parameters over $M_1|\kappa_+$.
- P is a ground of M_1 via extender algebra.
- $j(\delta^{M_1}) = \delta^P = \kappa_+$.
- $j(\kappa_n) = \kappa_{n+1}$ for $n < \omega$.
- $\langle \kappa_n \rangle_{n<\omega} \in M_1[j] = M_1[b]$.
- $M_1^\# \in M_1[b]$.
- $M_1[b] = L[M_1^\#] = M_1[\Sigma_{\mathrm{OR}}]$.

QED.

Proof of Theorem 1.14 part 1 ($M_1[\Sigma_{\kappa_+}] = M_1[\Sigma_\kappa]$):

Let $\mathcal{T} \in M_1|\kappa_+$ be a maximal tree, $b = \Sigma_{M_1}(\mathcal{T})$.
Let $j : M_1 \to M_1$ be elementary with $\mathrm{cr}(j) = \kappa$.

Claim 1.

$\delta \leq \mathrm{cof}^{M_1}(\mathrm{lh}(\mathcal{T}))$.

Proof.

There is no maximal tree $\mathcal{U} \in M_1|\kappa$ with $\mathrm{cof}^{M_1}(\mathrm{lh}(\mathcal{U})) < \delta$ (as $M_1[\Sigma_\kappa]$ is nice and iteration maps continuous at δ^{M_1}).

But j lifts this to $M_1|j(\kappa)$, hence to $M_1|\kappa_+$. □

– $\mathcal{T} \in M_1|\kappa_+$ is maximal, $b = \Sigma_{M_1}(\mathcal{T})$.
– $j : M_1 \to M_1$ is elementary with $\mathrm{cr}(j) = \kappa$.

CLAIM 2.

$\delta \leq \mathrm{cof}^{M_1}(\mathrm{lh}(\mathcal{T})) < \kappa$.

Proof.

Supose $\mathrm{cof}^{M_1}(\mathrm{lh}(\mathcal{T})) = \kappa$. Let $c =$ wellfounded branch through $j(\mathcal{T})$. Then

$$M_b^{\mathcal{T}} = L[M(\mathcal{T})] \text{ and } M_c^{j(\mathcal{T})} = L[M(j(\mathcal{T}))] \text{ are classes of } M_1.$$

Let $k = j \restriction M_b^{\mathcal{T}}$. Then

$$k : M_b^{\mathcal{T}} \to M_c^{j(\mathcal{T})}$$

is elementary, k discontinuous at $\delta(\mathcal{T})$. Embeddings

$$i_b^{\mathcal{T}} : M_1 \to M_b^{\mathcal{T}},$$

$$i_c^{j(\mathcal{T})} : M_1 \to M_c^{j(\mathcal{T})},$$

and j, k all fix infinitely many indiscernibles I', are continuous at δ^{M_1}. But

$$\mathrm{Hull}^{M_1}(I') \text{ is cofinal in } \delta^{M_1},$$

a contradiction. □

We show $M_1[\Sigma_{\kappa_+}] \subseteq M_1[\Sigma_\kappa]$.

Work in M_1. Let $\mathcal{T} \in M_1|\kappa_+$ be maximal tree.

Let $\eta = \text{cof}(\text{lh}(\mathcal{T}))$. So $\delta \leq \eta < \kappa$.

Let θ be large, and take an elementary

$$\pi : H \to M_1|\theta$$

with $H \in M_1$ transitive, $\text{card}(H) = \eta$, $\text{cr}(\pi) > \eta$, π cofinal in η, $\pi(\bar{\mathcal{T}}) = \mathcal{T}$.

So $\text{cof}^{M_1}(\text{lh}(\bar{\mathcal{T}})) = \eta$.

Now in V, let $\bar{b} = \Sigma_{M_1}(\bar{\mathcal{T}})$. Then $\pi``\bar{b}$ yields a $\mathcal{T}$-cofinal branch b through $\mathcal{T}$.

Case 1: $\bar{\mathcal{T}}$ is non-maximal. Then $\bar{b} \in M_1$, and $\pi \in M_1$, so $b \in M_1$. Since $\text{cof}^{M_1}(\eta) > \omega$, it follows that $M_b^{\mathcal{T}}$ is wellfounded, so $b = \Sigma_{M_1}(\mathcal{T})$.

Case 2: $\bar{\mathcal{T}}$ is maximal. Then $\bar{b} \in M_1[\Sigma_\kappa]$, a nice extension. By maximality, $\text{cof}^{M_1[\Sigma_\kappa]}(\eta) = \delta > \omega$, so again, $b = \Sigma_{M_1}(\mathcal{T})$.

QED.

Proof of Theorem 1.15 (κ-mantle of M_1):

Let $\mathscr{F}_\kappa$ be the collection of all κ-grounds of M_1 which are iterates of M_1.

Then $\mathscr{F}_\kappa$ is dense in the κ-grounds.

Every $W \in \mathscr{F}_\kappa$ computes

$$M_{\infty\kappa}[\Gamma_{\infty\kappa}],$$

in the same manner. So $M_{\infty\kappa}[\Gamma_{\infty\kappa}] \subseteq \mathbb{M}_\kappa^{M_1}$.

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in the same manner. So $M_{\infty\kappa}[\Gamma_{\infty\kappa}] \subseteq \mathbb{M}_\kappa^{M_1}$.

Notation: For $P, Q \in \mathscr{F}_\kappa$, write $P \leq Q$ iff Q is an iterate of P. For $P \leq Q$,

$$i_{PQ} : P \to Q$$

is the iteration map.

Say that P is α-stable iff for all $Q \in \mathscr{F}_\kappa$ with $P \leq Q$, we have

$$i_{PQ}(\alpha) = \alpha.$$

It remains to show

$$\mathbb{M}_\kappa^{M_1} \subseteq M_{\infty\kappa}[\Gamma_{\infty\kappa}].$$

Let $X \in \mathbb{M}_\kappa^{M_1}$.
Let $j : M_1 \to M_1$ be elementary with $\mathrm{cr}(j) = \kappa$. Then

$$j(X) \in \mathbb{M}_{j(\kappa)}^{M_1}.$$

But

$$M_{\infty\kappa}[\Gamma_{\infty\kappa}] = \mathrm{HOD}^{M_1[G]}$$

is a ground of M_1, via Vopenka, a forcing of size $< j(\kappa)$. So

$$j(X) \in M_{\infty\kappa}[\Gamma_{\infty\kappa}].$$

So there is an ordinal γ and formula φ such that

$$\alpha \in j(X) \iff M_1[G] \models \varphi(\gamma, \alpha).$$

Then for all $P \in \mathscr{F}_\kappa$ we have

$$j(X) = \{\alpha \in \mathrm{OR} \mid P \models \text{“}\mathrm{Col}(\omega, <\kappa) \text{ forces } \varphi(\gamma, \alpha)\text{”}\}.$$

Let $P \in \mathscr{F}_\kappa$ be γ-stable. Note $i_{QR}(j(X)) = j(X)$ for all $Q, R \in \mathscr{F}_\kappa$ with $P \leq Q \leq R$.

Claim 3.

$i_{QR}(X) = X$ for all such $Q, R \in \mathscr{F}_\kappa$ with $P \leq Q \leq R$.

Proof.

Let $Y = i_{QR}(X)$. We have

$$j \circ i_{QR} = i_{QR} \circ j \restriction Q.$$

Therefore

$$j(Y) = j(i_{QR}(X)) = i_{QR}(j(X)) = j(X),$$

so $Y = X$. □

Claim 4.

$X \in M_{\infty\kappa}[\Gamma_{\infty\kappa}]$.

Proof.

For $Q \in \mathscr{F}_\kappa$, write

$$i_{Q\infty} : Q \to M_{\infty\kappa}$$

for the iteration map. Let

$$X^* = i_{P\infty}(X)$$

where P is as before.
The usual $*$-map map

$$\alpha \mapsto \alpha^* = \min\{i_{Q\infty}(\alpha) \mid Q \in \mathscr{F}_\kappa\}$$

is in $M_{\infty\kappa}[\Gamma_{\infty\kappa}]$.
But

$$\alpha \in X \iff i_{Q\infty}(\alpha) = i_{Q\infty}(X) \iff \alpha^* \in X^*,$$

where Q is α-stable. So $X \in M_{\infty\kappa}[\Gamma_{\infty\kappa}]$. □

QED.

Consider $\eta = (\delta^{+\omega})^{M_1}$.
One shows that

$$M_{\infty\eta}[\Gamma_{\infty\eta}] \subseteq \mathbb{M}_{\eta}^{M_1}$$

and $(\delta^{+\omega})^{M_1}$ is the least measurable of $M_{\infty\eta}[\Gamma_{\infty\eta}]$, much as before.
But

$$\left\langle (\delta^{+n})^{M_1} \right\rangle_{n<\omega} \in \mathbb{M}_{\infty\eta},$$

(and this sequence is Prikry generic over $M_{\infty\eta}[\Gamma_{\infty\eta}]$).

Question: Does $\mathbb{M}_{\eta}^{M_1} \models$ ZFC?