Lower Bounds for the Perfect Subtree Property at Weakly Compact Cardinals

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κ-Trees and Upper Bounds

A First Lower Bound and Sealed Trees

Below Woodin cardinals

Non-domestic Mouse from the κ-PSP

κ-Trees

Definition

Let κ be a regular cardinal. A tree T of height κ is called a *normal κ-tree* if

- each level of T has size $<\kappa$,
- each level has at least one split,
- for every limit ordinal $\alpha < \kappa$ and every branch up to α there is at most one least upper bound in T, and
- for every $t \in T$ and $\alpha < \kappa$ above the height of t, there is some t' of level α in T such that $t <_T t'$.

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- For $\kappa > \omega$, if there are no κ-Kurepa trees, then $\kappa^+ \notin \mathfrak{S}_\kappa$.
- For $\kappa > \aleph_1$, if the tree property holds at κ, then $\min(\mathfrak{S}_\kappa) = \kappa$.

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The following gives an upper bound.

Proposition

Let κ be $<\mu$-supercompact, where μ is strongly inaccessible. Then, there is a forcing extension in which κ is weakly compact, $\mathfrak{S}_\kappa = \{\kappa, \kappa^{++}\}$.

Proof idea: Consider $\mathrm{Col}(\kappa, <\mu) \times \mathrm{Add}(\kappa, \mu^+)$.

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Question

Is this optimal?

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How about strongly sealed κ-trees with at least κ^+ many branches?

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A Sealed Tree in K

Theorem (Hayut, M.)

Let us assume that there is no inner model with a Woodin cardinal. Then for every regular cardinal κ, there is a strongly sealed κ-tree with exactly $(\kappa^+)^K$ many branches. In particular, if κ is weakly compact, then there is a strongly sealed tree on κ with κ^+ many branches.

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- **Nodes:** $\langle \bar{M}, \bar{x} \rangle$, where $\bar{M} = \mathrm{trcl}(\mathrm{Hull}^{K_{\kappa^+}}(\rho \cup \{x\}))$ for some $\rho < \kappa$, $x \in K_{\kappa^+} \cap {}^{\kappa}2$ and x collapses to $\bar{x}$.

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- **Tree order:** $\langle M_0, x_0 \rangle \leq \langle M_1, x_1 \rangle$ if there is some ordinal ρ such that $M_0 = \mathrm{trcl}(\mathrm{Hull}^{M_1}(\rho \cup \{x_1\}))$ and x_1 collapses to x_0.

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Claim

$\mathbb{T}(K_{\kappa^+})$ is a tree of height κ with at least $(\kappa^+)^K$ many branches.

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- Argue that each branch in V is in fact already in K, so $|[T]| = (\kappa^+)^K$ (use maximality and universality of K).
- Use forcing absoluteness to see that T is strongly sealed.
- Use covering to obtain $(\kappa^+)^K = (\kappa^+)^V$.

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Observation

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Why? Woodin's stationary tower forcing with critical point κ^+ will introduce new branches to any κ-tree T, while preserving the regularity of κ, as well as many large cardinal properties of κ.

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Lemma (Folklore)

Let κ be a cardinal. The following are equivalent for a tree T of height κ:

1. *T has a perfect subtree.*
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What is the consistency strength of this statement?

Proposition

Let κ be $< \mu$-supercompact, where μ is strongly inaccessible. Then, there is a forcing extension in which κ is weakly compact, $\mathfrak{S}_\kappa = \{\kappa, \kappa^{++}\}$ and the Perfect Subtree Property holds at κ.

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Let κ be a weakly compact cardinal and let us assume that

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Proof idea:

- Consider the tree $\mathbb{T}(\mathcal{S})$ for $\mathcal{S} = \mathcal{S}(\kappa)$ the stack of mice on $K^c||\kappa$ (cf. Andretta-Neeman-Steel and Jensen-Schimmerling-Schindler-Steel).

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- $\mathbb{T}(\mathcal{S})$ has exactly $(\kappa^+)^V$ many branches (using covering as in JSSS).
- $\mathbb{T}(\mathcal{S})$ does not have a perfect subtree (argue that set of branches does not change in an $\mathrm{Add}(\kappa, 1)$-generic extension of V).

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"Never say there is nothing beautiful in the world anymore. There is always something to make you wonder in the shape of a tree, the trembling of a leaf."

Albert Schweitzer