Towards the "right" generalization of descriptive set theory to uncountable cardinals

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Classical descriptive set theory

According to the introduction of Kechris' book *Classical descriptive set theory* (1995)

> *Descriptive set theory is the study of definable sets in Polish (i.e. separable completely metrizable) spaces".*

Part of the success experienced by this theory is arguably due to its wide applicability: Polish spaces are ubiquitous in mathematics!

Examples. $\mathbb{R}^n$, $\mathbb{C}^n$, ${}^\omega 2$, ${}^\omega\omega$, $\mathcal{K}(X)$ (= hyperspace of compact subsets of X with the Vietoris topology), any separable Banach space, ...

There has been various attempts to generalize classical DST to different setups, usually first varying the space(s) under consideration, and then naturally adapting (some of) the relevant definitions to the new context.

Baire spaces (A. H. Stone, *Non-separable Borel sets*, 1962)

Work on the so-called **Baire spaces**, i.e. spaces of the form $\prod_{n \in \omega} T_n$ endowed with the product of the discrete topology on each T_n.

Up to homeomorphism, this reduces to the study of definable sets in

$$B(\lambda) = {}^{\omega}\lambda$$

with λ and arbitrary cardinal.

Remark. If $\mathrm{cof}(\lambda) = \omega$ one could also consider the natural generalization of the Cantor space

$$C(\lambda) = \prod_{i \in \omega} \lambda_i,$$

where the λ_i's are increasing and cofinal in λ (in symbols, $\lambda_i \nearrow \lambda$). However, Stone proved that if $\lambda > \omega$ then $C(\lambda) \approx B(\lambda)$.

Generalized DST: version 1 (Vaught 1974, Mekler-Väänänen 1993, ...)

Study of definable sets in

$$^{\kappa}2$$

endowed with the *bounded topology*, which is generated by

$$\boldsymbol{N}_s = \{x \in {}^{\kappa}2 \mid s \sqsubseteq x\}, \qquad s \in {}^{<\kappa}2,$$

usually under the assumption $\kappa^{<\kappa} = \kappa$ (equivalently, κ regular + $2^{<\kappa} = \kappa$).

Remark. Regularity of κ causes the loss of metrizability when $\kappa > \omega$: indeed, ${}^{\kappa}2$ is (completely) metrizable iff ${}^{\kappa}2$ is first-countable iff $\mathrm{cof}(\kappa) = \omega$.

The resulting theory is extremely rich and interesting, but quite different from the classical one.

Generalized DST: version 2 (Woodin, *Suitable extender models II*, 2012)

Study of definible sets in the space

$$\mathrm{V}_{\lambda+1}$$

with the topology generated by

$$O_{a,\alpha} = \{X \in V_{\lambda+1} \mid X \cap \mathrm{V}_\alpha = a\}, \qquad \alpha < \lambda,\, a \subseteq \mathrm{V}_\alpha$$

under $\mathsf{I0}(\lambda)$ ($= \exists j \colon \mathrm{L}(\mathrm{V}_{\lambda+1}) \prec \mathrm{L}(\mathrm{V}_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$).

In this context, $\mathrm{V}_{\lambda+1}$ is a large cardinal version of ${}^{\omega}2$: V_λ is considered an analogue of $\mathrm{V}_\omega \approx \omega$, so that $V_{\lambda+1} = \mathcal{P}(\mathrm{V}_\lambda)$ is the analogue of $\mathcal{P}(\omega) \approx {}^{\omega}2$.

A general trend has emerged (with exceptions!):

the theory of $\mathcal{P}(\mathrm{V}_{\lambda+1})$ *in* $\mathrm{L}(\mathrm{V}_{\lambda+1})$ *under* $\mathsf{I0}(\lambda)$

is reminiscent of

the theory of $\mathcal{P}(\mathbb{R})$ *in* $\mathrm{L}(\mathbb{R}) = \mathrm{L}(\mathrm{V}_{\omega+1})$ *under* AD.

Different generalizations?

- If $\mu = \mathrm{cof}(\lambda) < \lambda$ and $2^{<\lambda} = \lambda$ (equivalently, λ is strong limit) then

$$^{\lambda}2 \approx {}^{\mu}\lambda \approx \prod_{i<\mu} \lambda_i,$$

where $\lambda_i \nearrow \lambda$. (Products are endowed with the $< \mu$-supported product topology; when $\mu = \omega$ this is just the product topology.)

- Therefore, if we further have $\mu = \omega$ then

$$B(\lambda) \approx C(\lambda) \approx {}^{\lambda}2.$$

- I0(λ) implies that $\mathrm{cof}(\lambda) = \omega$ and λ is a limit of inaccessible cardinals. It easily follows that

$$\mathrm{V}_{\lambda+1} \approx {}^{\lambda}2 \approx B(\lambda).$$

Different generalizations?

So all these generalizations virtually deal with the **generalized Cantor space** ${}^{\lambda}2$ for different uncountable cardinals λ (Usually concentrating on the two extreme cases $\mathrm{cof}(\lambda) = \omega$ and $\mathrm{cof}(\lambda) = \lambda$).

Remark 1

A "generalized DST at λ" should arguably concern λ-**spaces**, i.e. spaces of weight λ. The assumption

$$(\dagger) \qquad 2^{<\lambda} = \lambda$$

is needed to guarantee that ${}^{\lambda}2$ has this property. Thus $(\dagger)$ *will be assumed throught the rest of this talk*.

Remark 2

One might wonder what should be the **generalized Baire space**. The choice ${}^{\lambda}\lambda$ is natural, but such space is a λ-space if and only if λ is regular. The correct option in the general case seems to be ${}^{\mu}\lambda$, where $\mu = \mathrm{cof}(\lambda)$.

A criticism...

All of this is quite interesting (and fun!) for set theorists, but to play the devil's advocate one could point out that

> *unlike the classical case, generalized DST concentrates on just one very specific space.*

This could become an issue when moving from the theoretical side to that of finding applications elsewhere...

Main question

Is there a more general notion of "Polish-like space" for which we can develop a decent (generalized) DST?

(...of course it is questionable what "decent" means.)

Generalized Polish spaces: the $\mathrm{cof}(\lambda) = \omega$ case

(joint work with Dimonte and Shi, *unpublished*)

The countable cofinality case

The setup

λ uncountable with $\mathrm{cof}(\lambda) = \omega$ and $2^{<\lambda} = \lambda$ (i.e. λ is strong limit).

Definition

A topological space is λ-**Polish** if it is a completely metrizable λ-space.

Examples. ${}^{\lambda}2$, $B(\lambda) = {}^{\omega}\lambda$, $\mathrm{V}_{\lambda+1}$ with λ limit of inaccessibles, $\mathcal{K}(X)$ for a λ-Polish X, any Banach space of density λ, ...

Definition

- λ^+-**Borel sets**: smallest λ^+-algebra generated by open sets.
- λ-**analytic sets**: continuous images of λ-Polish spaces or, equivalently, continuous images of (closed subsets of) ${}^{\omega}\lambda$.

Basic properties

Proposition (closure properties)

The class of λ-Polish spaces is closed under

- disjoint sums of size $\leq \lambda$;
- countable products;
- G_δ subspaces (and this is optimal: $Y \subseteq X$ is λ-Polish iff Y is G_δ).

Remark. In the latter we really mean *countable* intersections of open sets (and not $\leq \lambda$-intersections!).

Theorem (surjective universality of ${}^\omega\lambda$)

For every λ-Polish X there is a continuous bijection $f \colon C \to X$ with $C \subseteq {}^\omega\lambda$ closed (and f^{-1} is λ^+-Borel). If moreover $X \neq \emptyset$, then there is a continuous surjection ${}^\omega\lambda \twoheadrightarrow X$. The same for λ^+-Borel subsets of X.

λ-Perfect spaces

Definition

A point $x \in X$ is **λ-isolated** if it admits an open neighborhood of size $< \lambda$. The space X is **λ-perfect** if it has no λ-isolated point. A subset of X is **λ-perfect** if it is a closed λ-perfect subspace of X.

Theorem (embedding ${}^{\lambda}2$ into λ-perfect spaces)

Every nonempty λ-perfect λ-Polish space contains a *closed* set homeomorphic to ${}^{\lambda}2$ ($\approx \prod_{i<\omega} \lambda_i$).

Here we crucially use that for any metric space Y TFAE:

1. $|Y| < \lambda$
2. Y has weight $< \lambda$
3. there is $\kappa < \lambda$ such that all *spaced* subsets of Y are of size $\leq \kappa$.

   [$A \subseteq Y$ is spaced if there is $r > \omega$ such that $d(x, y) \geq r$ for all $x, y \in A$.]

This holds only under the hypothesis that $\kappa^{\omega} < \lambda$ for every $\kappa < \lambda$.

λ-Perfect spaces

Theorem (generalized Cantor-Bendixson)

Every λ-Polish space X *uniquely* decomposes as a disjoint union $X = P \sqcup C$, where P is λ-perfect and C is open of size $\leq \lambda$. The subspace P is called the λ-**perfect kernel** of X.

Remark. The λ-perfect kernel can equivalently be recovered as the set of λ-*accumulation points* (i.e. points all of whose open neighborhoods have size $> \lambda$) or through *Cantor-Bendixon derivatives*.

Corollary (topological CH_λ for λ-Polish spaces)

Let X be λ-Polish. Either $|X| \leq \lambda$ or ${}^{\lambda}2 \hookrightarrow X$ (as a closed set).

Corollary (generalized Borel isomorphism theorem)

Two λ-Polish spaces X, Y are λ^+-Borel isomorphic iff $|X| = |Y|$.

Zero-dimensionality

Definition (Lebesgue covering dimension 0)

Let X be a λ-Polish space. Then $\dim(X) = 0$ iff every open cover of X has a refinement consisting of disjoint (cl)open sets.

Examples. ${}^{\lambda}2$, ${}^{\omega}\lambda$, $\mathrm{V}_{\lambda+1}$, ...

Remark. For metrizable spaces X we have in general $\mathrm{ind}(X) \leq \dim(X) = \mathrm{Ind}(X)$ (Katětov); if X is also separable, then the three dimensions coincide, otherwise this is not the case.

Proposition (universality of ${}^{\omega}\lambda$ for zero-dimensional)

Every λ-Polish space X with $\dim(X) = 0$ is homeomorphic to a closed subset of ${}^{\omega}\lambda$.

Moreover

- every closed $F \subseteq X$ is a retract of X;
- every G_δ subset of ${}^{\omega}\lambda$ is homeomorphic to a closed subset of it.

Characterization of ${}^{\lambda}2$

Theorem (generalized Brouwer/Alexandrov-Urysohn theorem)

Up to homeomorphism, the generalized Cantor space ${}^{\lambda}2$ is the unique topological space X such that

(1) X is λ-Polish

(2) $\dim(X) = 0$

(3) X is λ-perfect.

Moreover, the last condition can be replaced with any of the following:

(3′) U has weght λ for every (cl)open $U \subseteq X$.

(3″) every open $U \subseteq X$ has arbitrarily large (below λ) discrete subsets.

Corollary (canonical form for zero-dimensional spaces)

All zero-dimensional λ-Polish spaces of size $> \lambda$ are of the form ${}^{\lambda}2 \sqcup C$ with C open of size $\leq \lambda$.

Lusin-Souslin theorem

Generalized Lusin's separation theorem

If A, B are disjoint analytic subsets of a λ-Polish space, then A can be separated from B by a λ^+-Borel set.

Generalized Souslin's theorem

A subsets of a λ-Polish space is λ-bianalytic iff it is λ^+-Borel.

This has many consequences, such as:

- a function is λ^+-Borel iff its graph is λ-analytic, iff its graph is λ^+-Borel;
- the injective λ^+-Borel image of a λ^+-Borel set is still λ^+-Borel;
- a set is λ^+-Borel iff it is the *injective* continuous image of a closed subset of ${}^\omega\lambda$;
- ...

λ-Perfect Set Property

Definition (λ-PSP)

A subset A of a λ-Polish space has the λ-**Perfect Set Property** (in symbols λ-PSP(A)) if either $|A| \leq \lambda$ or ${}^{\lambda}2 \hookrightarrow A$ (as a closed set).

Theorem

Let X be λ-Polish. If $A \subseteq X$ is λ-analytic, then λ-PSP(A).

Cramer (essentially) proved that assuming I0(λ), all subsets of $\mathrm{V}_{\lambda+1}$ belonging to $\mathrm{L}(\mathrm{V}_{\lambda+1})$ have the λ-PSP. Thus we get

Theorem

Assume I0(λ) *and let* X *be an arbitrary* λ*-Polish space. If* $A \subseteq X$ *is in* $\mathrm{L}(\mathrm{V}_{\lambda+1})$ *then* λ-PSP(A). *In particular, all projective sets have the* λ-PSP.

Silver's dichotomy

Theorem (generalized Silver's dichotomy)

Suppose that λ is limit of measurable cardinals. Let E be a λ-coanalytic equivalence relation on a λ-Polish space. Then either E has $\leq \lambda$-many equivalence classes, or there is a λ-perfect set of E-inequivalent elements.

Strategy for the proof

- Assume that there are $> \lambda$-many E-equivalence classes, and let $f \colon C \to X$ be a continuous bijection with $C \subseteq {}^{\omega}\lambda$ closed.
- For $x, y \in {}^{\omega}\lambda$ set $x \mathrel{E'} y \iff x, y \notin C \vee (x, y \in C \wedge f(x) \mathrel{E} f(y))$. The equivalence relation E' is still λ-coanalytic.
- *(Difficult part)* Show that there is $\varphi \colon {}^{\omega}\lambda \to {}^{\omega}\lambda$ such that $\neg(\varphi(x) \mathrel{E'} \varphi(y))$ for $x \neq y$.
- Conclude that $f(\mathrm{rng}(\varphi) \cap C)$ is a λ-analytic subset of X consisting of E-inequivalent elements and of size $> \lambda$, thus it contains a λ-perfect set.

Generalized Polish spaces: the regular case

(Coskey-Shlicht, *Generalized Choquet spaces*, Fund. Math. 2016
and
M.-Schlicht, *unpublished*)

The regular case

The setup

κ uncountable regular and $2^{<\kappa} = \kappa$ (i.e. $\kappa^{<\kappa} = \kappa$).

If X is second-countable, then X is Polish iff X is Hausdorff, regular and strong Choquet (i.e. II wins $G^{\omega}_{\mathrm{Ch}}(X)$). This motivates the following definitions.

The strong κ-Choquet game on X

$G^{\kappa}_{\mathrm{Ch}}(X)$ is a game of length κ of the form

$$\begin{array}{c||ccccccccc} \mathrm{I} & x_0, U_0 & & x_1, U_1 & & x_2, U_2 & & \dots & \\ \hline \mathrm{II} & & V_0 & & V_1 & & V_2 & & \dots \end{array}$$

where $x_\alpha \in V_\alpha \subseteq U_\alpha \subseteq \bigcap_{\beta<\alpha} V_\beta$ and U_α, V_α are open relatively to $\bigcap_{\beta<\alpha} U_\beta$.

II *wins if* $\bigcap_{\alpha<\rho} U_\alpha \neq \emptyset$ *for all limit* $\rho \leq \kappa$.

The regular case

Definition

X is **strong κ-Choquet** if it is a Hausdorff, regular (κ-)space and II wins $G^{\kappa}_{\mathrm{Ch}}(X)$.

Examples. ${}^{\kappa}2$, ${}^{\kappa}\kappa$, $({}^{\kappa}2, \leq_{\mathrm{lex}})$, ...

Definition

- κ^+-**Borel sets**: smallest κ^+-algebra generated by open sets.
- κ-**analytic sets**: continuous images of strong κ-Choquet spaces or, equivalently, continuous images of *closed subsets* of ${}^{\kappa}\kappa$.

Basic properties

Proposition (closure properties)

- Continuous open images of strong κ-Choquet spaces are strong κ-Choquet.
- Strong κ-Choquet spaces are closed under disjoint unions of size κ and products of size κ (equipped with the $< \kappa$-supported product topology).

No closure under subspaces (except for open sets): there are arbitrarily complex subsets of ${}^{\kappa}2$ which are strong κ-Choquet, and there are closed subsets of ${}^{\kappa}2$ which are not strong κ-Choquet spaces.

Theorem (surjective universality of ${}^{\kappa}\kappa$)

For every strong κ-Choquet space X there is a continuous bijection $f \colon C \to X$ with $C \subseteq {}^{\kappa}\kappa$ closed (and f^{-1} is κ^+-Borel). If moreover $X \neq \emptyset$, then there is a continuous surjection ${}^{\kappa}\kappa \twoheadrightarrow X$. The same for κ^+-Borel subsets of X.

κ-Perfect spaces

Definition

A point $x \in X$ is κ-**isolated** if there is a family $\mathcal{U}$ of open sets such that $|\mathcal{U}| < \kappa$ and $\bigcap \mathcal{U} = \{x\}$. The space X is κ-**perfect** if it has no κ-isolated point. A subset of X is κ-**perfect** if it is a closed κ-perfect subspace of X.

Theorem (embedding ${}^{\kappa}2$ into κ-perfect spaces)

Every nonempty κ-perfect strong κ-Choquet space contains a closed set homeomorphic to ${}^{\kappa}2$.

Corollary (generalized Borel isom. theorem for κ-perfect spaces)

Any two κ-perfect strong κ-Choquet spaces are κ^+-Borel isomorphic.

However, one cannot remove the κ-perfectness condition from the above results. There are two main problems. First, Cantor-Bendixson derivatives and κ-accumulation points can give different "κ-perfect kernels", and both may fail to be strong κ-Choquet. Even more seriously...

κ-Perfect spaces

No CH$_\kappa$-like theorem for arbitrary strong κ-Choquet spaces

(Agostini-M.) For any $A \subseteq {}^\kappa 2$ such that κ-PSP(A) fails there is a strong κ-Choquet space $X \subseteq {}^\kappa 2$ of size $|A|$ such that ${}^\kappa 2 \not\hookrightarrow X$.

Sketch of the proof

Given $\tilde{x} \in {}^\kappa 3$, let $x \in {}^{\leq\kappa} 2$ be obtained from $\tilde{x}$ by removing all occurrences of 2. A sequence $\tilde{x} \in {}^{\leq\kappa} 3$ is *almost binary* if there are only finitely many entries in $\tilde{x}$ taking value 2. Given $A \subseteq {}^\kappa 2$ set

$$\tilde{A} = \{\tilde{x} \in {}^\kappa 3 \mid \tilde{x} \text{ is almost binary and } x \in A\} \cup \{\underbrace{s_0{}^\frown 2{}^\frown s_1{}^\frown 2{}^\frown s_2{}^\frown 2{}^\frown \dots}_{\omega \text{ many}}{}^\frown\langle 22222222\dots\rangle \in {}^\kappa 3 \mid s_i \in {}^{<\kappa} 2\}.$$

$\tilde{A}$ is clearly strong κ-Choquet and ${}^\kappa 2 \hookrightarrow \tilde{A}$ if and only if ${}^\kappa 2 \hookrightarrow A$. $\square$

Zero-dimensionality

We now consider strong κ-Choquet spaces X with $\mathrm{ind}(X) = 0$. When X is also κ-additive, this is equivalent to $\dim(X) = 0$ (and also to $\mathrm{Ind}(X) = 0$).

Theorem (universality of ${}^\kappa\kappa$ for zero-dimensional)

Every zero-dimensional κ-*additive* strong κ-Choquet space is homeomorphic to a closed subset of ${}^\kappa\kappa$.

No characterization of ${}^\kappa 2$ and/or ${}^\kappa\kappa$ yet...

Zero-dimensionality, κ-additivity and κ-perfectness are not enough.

In fact

Proposition

There are 2^κ-many pairwise non-homeomorphic zero-dimensional κ-additive strong κ-Choquet spaces.

Lusin-Souslin theorem, κ-PSP, SIlver's dichotomy, ...

Here starts the nightmare...

...a paradise for set theorists!

- No separation for κ-analytic sets.
- There are κ-bianalytic sets which are not κ^+-Borel.
- The rest is mostly either false or independent (even for closed sets and closed equivalence relations).

Generalized Polish spaces: the general case

(joint work with Agostini, *work in progress*)

The general case

The setup

λ uncountable with $\mu = \mathrm{cof}(\lambda)$ and $2^{<\lambda} = \lambda$.

Which are the "nice" spaces to be considered? (1) We obviously concentrate on Hausdorff regular λ-spaces. (2) We have to give up with metrizability when $\mu \neq \omega$. (3) μ-strong Choquet-ness ensures a form of completeness (...to be discussed later).

Is this enough?

The missing condition...

X is μ-**Nagata-Smirnov** (shortly, μ-**NS**) if it admits a base $\mathcal{B} = \bigcup_{i<\mu} \mathcal{B}_i$ such that for every $x \in X$ and $i < \mu$ there exists an open neighborhood U of x such that $\{V \in \mathcal{B}_i \mid V \cap U \neq \emptyset\}$ has cardinality $< \mu$.

- X is metrizable iff it is Hausdorff regular and ω-NS.
- If λ is regular (i.e. $\mu = \lambda$), any λ-space is λ-NS.

About the generalized strong Choquet game...

Recall that the strong μ-Choquet property is not enough to ensure that the "perfect kernel" stays strong μ-Choquet.

The strong (λ, μ)-Choquet game on X

$G_{\mathrm{Ch}}^{\lambda,\mu}(X)$ is a game of length μ of the form

$$\begin{array}{c||ccccccccc} \mathrm{I} & x_0, U_0 & & x_1, U_1 & & x_2, U_2 & & \dots & \\ \hline \mathrm{II} & & V_0 & & V_1 & & V_2 & & \dots \end{array}$$

with the same rules as $G_{\mathrm{Ch}}^{\mu}(X)$ plus $|U_\alpha|, |V_\alpha| \geq \lambda$ for all α's.

II wins if

- the run does not stop before μ (i.e. $|\bigcap_{\beta<\alpha} U_\beta| \geq \lambda$ for all $\beta < \mu$), and
- $\bigcap_{\alpha<\mu} U_\alpha \neq \emptyset$.

A space X is strong (λ, μ)-Choquet if II wins $G_{\mathrm{Ch}}^{\lambda,\mu}(X)$.

Polish-like spaces for an arbitrary uncountable λ

Definition (tentative)

A Hausdorff regular topological space is called λ-**Descriptive Set Theoretic space** (briefly, λ-**DST space**) if it is a λ-space which is both μ-NS and strong (λ, μ)-Choquet.

Examples. ${}^{\lambda}2$, ${}^{\mu}\lambda$, $\prod_{i<\mu} \lambda_i$ for $\lambda_i \nearrow \lambda$, $\mathrm{V}_{\lambda+1}$ (if λ is limit of inaccessibles), $\mathcal{K}(X)$ for a λ-DST space X, ...

Definition

- λ^+-**Borel sets**: smallest λ^+-algebra generated by open sets.
- λ-**analytic sets**: continuous images of λ-DST spaces or, equivalently, continuous images of closed subsets of ${}^{\mu}\lambda$.

Basic properties

Proposition (closure properties)

The class of λ-DST spaces is closed under

- disjoint sums of size $\leq \lambda$;
- products of size $\leq \mu$ (endowed with the $< \mu$-supported product topology).

No closure under subspaces (except for open sets): there are arbitrarily complex subsets of ${}^{\lambda}2$ which are λ-DST, and there are closed subsets of ${}^{\lambda}2$ which are not λ-DST.

Theorem (surjective universality of ${}^{\mu}\lambda$)

For every λ-DST space X there is a continuous bijection $f \colon C \to X$ with $C \subseteq {}^{\mu}\lambda$ closed (and f^{-1} is λ^{+}-Borel). If moreover $X \neq \emptyset$, then there is a continuous surjection ${}^{\mu}\lambda \twoheadrightarrow X$. The same for λ^{+}-Borel subsets of X.

(λ, μ)-Perfect spaces

Definition

A point $x \in X$ is (λ, μ)-**isolated** if there is a family $\mathcal{U}$ of open sets such that $|\mathcal{U}| < \mu$, $x \in \bigcap \mathcal{U}$, and $|\bigcap \mathcal{U}| < \lambda$. The space X is (λ, μ)-**perfect** if it has no (λ, μ)-isolated point. A subset of X is (λ, μ)-**perfect** if it is a closed (λ, μ)-perfect subspace of X.

Theorem (embedding ${}^{\lambda}2$ into λ-perfect spaces)

Every nonempty (λ, μ)-perfect λ-DST space contains a closed set homeomorphic to ${}^{\lambda}2$ ($\approx \prod_{i<\mu} \lambda_i$ if $\mu < \lambda$).

- We carefully combine the techniques used in the countable cofinality case and in the regular case.
- When $\omega < \mu < \lambda$, a crucial role is played by the "(previously) missing condition" of being μ-NS.

(λ, μ)-Perfect spaces

Theorem (generalized Cantor-Bendixson)

Every λ-Polish space X *uniquely* decomposes as a disjoint union $X = P \sqcup C$, where P is a λ-perfect λ-DST space and C is open of size $\leq \lambda$. The subspace P is called the λ-**perfect kernel** of X.

Corollary (topological CH_λ for λ-DST spaces)

Let X be a λ-DST space. Either $|X| \leq \lambda$ or ${}^{\lambda}2 \hookrightarrow X$ (as a closed set).

Corollary (generalized Borel isomorphism theorem)

Two λ-DST spaces X, Y are λ^+-Borel isomorphic iff $|X| = |Y|$.

Case $\mu = \lambda$: comparing with strong λ-Choquet spaces

- The μ-NS is for free (follows from regularity of λ).
- A point $x \in X$ is (λ, μ)-isolated iff it is λ-isolated, hence (λ, μ)-perfectness coincides with λ-perfectness.
- λ-DST spaces $\subsetneq$ strong λ-Choquet spaces. In fact, there may be strong λ-Choquet spaces which are not even λ^+-Borel isomorphic to a λ-DST space.
- However, if X is λ-perfect (equivalently, (λ, μ)-perfect), then X is λ-DST iff it is strong λ-Choquet.
- Arguably, λ-DST spaces are the right ones: if $X = P \sqcup C$ with C open of size λ and P λ-perfect and λ-DST (equivalently, strong λ-Choquet), then X was already λ-DST.


COMING
SOON

Coming soon...

- Zero-dimensionality (Lebesgue covering dimension!), characterization of ${}^{\lambda}2$, ...
- λ-Perfect Set Property, Silver's dichotomy, ...
- Applications to classification problems for uncountable structures and nonseparable spaces (both in the regular and in the singular case).
- Interaction with other fields (e.g. combinatorics of singular cardinals).

The end

Thank you for your attention!