Results on set mappings

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A *set mapping* is $F : \kappa \to \mathcal{P}(\kappa)$ for some infinite cardinal κ.

A set $A \subseteq \kappa$ is *free* if $y \notin F(u)$ for $u \in A$, $y \in A - \{u\}$.

Paul Turán asked in 1934, if $f : \mathbb{R} \to [\mathbb{R}]^{<\omega}$ does there exist an infinite free set.

Fundamental Theorem on Set Mappings.
(Hajnal) If $\kappa > \mu$, $F : \kappa \to [\kappa]^{<\mu}$ then there is a free set of size κ.

Theorem. (Bagemihl) If f is a set mapping on $\mathbb{R}$ with $f(x)$ nowhere dense for $x \in \mathbb{R}$ then there is an everywhere dense free set.

If $\kappa^{<\kappa} = \kappa$, let $\mathbb{R}_\kappa$ be the set of all nonconstant $f : \kappa \to \{0, 1\}$ with no last 0.
Order $\mathbb{R}_\kappa$ lexicographically, then we have the notions of noweher dense, everywhere dense, etc.

Theorem. (Bagemihl) (GCH) If f is a set mapping on $\mathbb{R}_\kappa$ with $f(x)$ nwd for $x \in \mathbb{R}_\kappa$, then there is a free set of cardinality κ.

Theorem. (K, with a little help from S.) ($\kappa^{<\kappa} = \kappa$) If f is a set mapping on $\mathbb{R}_\kappa$ with $f(x)$ nwd for $x \in \mathbb{R}_\kappa$, then there is an everywhere dense free set.

A *set mapping* is $F : [\kappa]^r \to [\kappa]^{<\mu}$ for some finite r, infinite cardinals κ and μ.

A set $A \subseteq \kappa$ is *free* if $y \notin F(u)$ for $u \in [A]^r$, $y \in A - u$.

Theorem. (Erdős–Hajnal) If $F : [\exp_{r-1}(\kappa)^+]^r \to [\exp_{r-1}(\kappa)^+]^{<\kappa}$ is a set mapping, then there is a free set of cardinality κ^+.

Theorem. (Kuratowski) If $F : [\omega_n]^n \to [\omega_n]^{<\omega}$ is a set mapping, then there is a free set of size $n+1$.

Theorem. (Sierpiński) There is a set mapping $F : [\omega_{n-1}]^n \to [\omega_{n-1}]^{<\omega}$ with no free set of size $n+1$.

Theorem. (Hajnal–Máté) If $f : [\omega_2]^2 \to [\omega_2]^{<\omega}$, then there are arbitrarily large finite free sets.

Theorem. (Hajnal) If $f : [\omega_3]^3 \to [\omega_3]^{<\omega}$, then there are arbitrarily large finite free sets.

$t_0 = 5,\ t_1 = 7,\ t_{n+1}$ is the least number that $t_{n+1} \to (t_n, 7)^5$.

Theorem. (Komjáth–Shelah) It is consistent that there is a set mapping $f : [\omega_n]^4 \to [\omega_n]^{<\omega}$ with no free set of cardinality t_n.

s_n is the minimum number such that $s_n \to (5)^3_{3^n}$.
Roughly a triple exponential.

Theorem. (S. Mohsenipour, S. Shelah) It is consistent that there is a set mapping $F : [\omega_n]^4 \to [\omega_n]^\omega$ with no free set of size s_n.

Theorem. (Gillibert) If $F : [\omega_n]^n \to [\omega_n]^{<\omega}$ is a set mapping, then there is a free set of size $n+2$.

Theorem. (Gillibert–Wehrung) If $F:[\omega_n]^r\to[\omega_n]^{<\omega}$ is a set mapping, then there is a free set of size

$$2^{\left\lfloor \frac{1}{2}\left(1-\frac{1}{2^r}\right)^{-\frac{n+1}{2^r}}\right\rfloor}.$$

For $r=4$, this is about $2^{1.016^n}$.

Theorem. (Hajnal–Máté) Let $F : [\omega_2]^2 \to [\omega_2]^{<\omega}$ be a set mapping
(a) if $\beta < f(\alpha, \beta)$ $(\alpha < \beta < \omega_2)$, then there is a free set of size $\aleph_2$;
(b) if $f(\alpha, \beta) \subseteq (\alpha, \beta)$ $(\alpha < \beta < \omega_2)$, then there is an infinite free set.

Definition. If λ is an infinite cardinal, $1 \leq r < \omega$, we call a set mapping $f : [\lambda]^r \to \mathcal{P}(\lambda)$ *of order* $(\mu_0, \mu_1, \ldots, \mu_r)$, if the following holds. For every $s \in [\lambda]^r$ with increasing enumeration $s = \{\alpha_0, \ldots, \alpha_{r-1}\}$ we have
$|f(s) \cap \alpha_0| < \mu_0$,
$|f(s) \cap (\alpha_i, \alpha_{i+1})| < \mu_{i+1}$ $(i < r - 1)$, and
$|f(s) \cap (\alpha_{r-1}, \lambda)| < \mu_r$.

Theorem. (GCH) Assume that $0 < r < \omega$, $\lambda = \kappa^{+r}$. Let $f : [\lambda]^r \to \mathcal{P}(\lambda)$ be a set mapping of order $(\kappa, \kappa^+, \kappa^{++}, \dots, \kappa^{+r})$. Then there is a free set of order type $\kappa^+ + r - 1$.

Theorem. If $1 \leq r < \omega$ and κ is infinite, then there is a set mapping $f_r : [\kappa^{+r}]^r \to \mathcal{P}(\kappa^{+r})$ of order $(0, \kappa^+, \kappa^{++}, \dots, \kappa^{+r-1}, 0)$, with no free set of order type $\kappa^+ + r$.

Theorem. If $1 \leq r < \omega$, κ is infinite, then there is a set mapping $f : [\kappa^{+r}]^r \to \mathcal{P}(\kappa^{+r})$ of order $(\kappa^+, 0, 0, \ldots, 0)$ such that f has no free set of order type

$$\begin{cases} 2, & (r = 1) \\ \omega, & (r = 2) \\ \omega_{r-3} + 1, & (3 \leq r < \omega). \end{cases}$$

Two methods of decomposing vector spaces into the union of countably many parts each omitting some configuration.

Theorem. (Rado) Each vector space over $\mathbb{Q}$ is the union of ctbly many pieces, each omitting a 3-AP.

Proof. Let V be a vector space over $\mathbb{Q}$, and $B = \{b_i : i \in I\}$ a basis with I ordered. If $x \in V$ write as

$$x = \lambda_1 b_{i_1} + \cdots + \lambda_n b_{i_n}$$

where $i_1 < \cdots < i_n$. Let $\langle \lambda_1, \ldots \lambda_n \rangle$ be the color of x.

Assume that x, y, z get the same color $\langle \lambda_1, \dots, \lambda_n \rangle$ and $x + z = 2y$.
Then
$x = \lambda_1 b_{i_1^x} + \cdots + \lambda_n b_{i_n^x}$, where $i_1^x < \cdots < i_n^x$,
$y = \lambda_1 b_{i_1^y} + \cdots + \lambda_n b_{i_n^y}$, where $i_1^y < \cdots < i_n^y$,
$z = \lambda_1 b_{i_1^z} + \cdots + \lambda_n b_{i_n^z}$, where $i_1^z < \cdots < i_n^z$.

Let $i = \min\{i_1^x, i_1^y, y_1^z\}$. Then the coefficients of x, y, z in b_i are 0 or λ_1, one of them is λ_1 and they form a 3-AP. This is only possible, if all are equal to λ_1 and so $i_1^x = i_1^y = i_1^z$.
Proceed to i_2^x, i_2^y, i_2^z, etc. Eventually, $x = y = z$.

Definition. If S is a set, $\mathcal{H}$ is a set system on S, then the coloring number of $\mathcal{H}$ is countable, $\mathrm{Col}(\mathcal{H}) \leq \omega$, if there is a well ordering $<$ of S such that for each $x \in S$, x is the largest element of finitely many sets in $\mathcal{H}$.

If a Rado-type proof gives that for some vector space V and configuration system $\mathcal{H}$ on V, V is the union of countably many parts omitting configurations in $\mathcal{H}$, do we have $\mathrm{Col}(\mathcal{H}) \leq \omega$?

Theorem. If V is a vector space over $\mathbb{Q}$, $|V| = \aleph_n$, then there is a well ordering such that each element is the last member of only finitely many arithmetic progressions of length $n+1$. Consequently, there is a set mapping $f : V \to [V]^{<\omega}$ with no free arithmetic progression of length $n+1$.

Theorem. If V is a vector space over $\mathbb{Q}$ with $|V| = \aleph_{n-1}$, $f : V \to [V]^{<\omega}$ is a set mapping, then there is a free arithmetic progression of length n.

Some old Erdős–Hajnal problems

Problem 1. Is the following consistent? GCH plus if $f : [\omega_2]^3 \to \omega_2$ then there is an uncountable free set.

Problem 2. Is the following consistent? GCH plus if $f : [\omega_3]^3 \to [\omega_3]^{<\omega}$ then there is an uncountable free set.

Thank you for your patience!