# Counting fixed points of Boolean networks

## Algorithmic complexity

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# Boolean networks: Interaction graph and Fixed points

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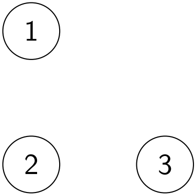

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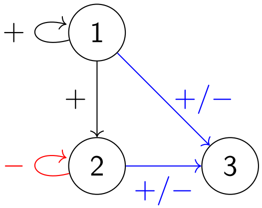

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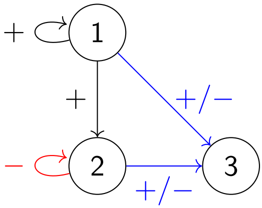


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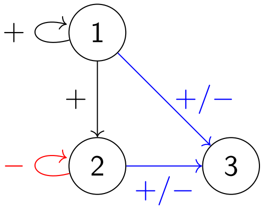


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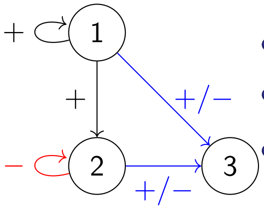


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- $f(000) = 000$. Thus, 000 is a fixed point.
- $f(010) = 001$. Thus, 010 is not.

# Presentation of the problems and results

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Table: Results

| *Problem* | $\Delta(G) < d$ | 1 | $k \geq 2$ | $k$ in argument |
|---|---|---|---|---|
| $\max(\Phi(G)) \geq k$ | yes | P | NP | $NP^{PP}$ |
| | no | | | NEXPTIME |
| $\min(\Phi(G)) < k$ | yes | $NP^{NP}$ | | $NP^{PP}$ |
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For any interaction graph $G$. $\max(\Phi(G)) \geq 1$ if and only if there is a positive cycle in every initial strongly connected component of $G$.

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## Theorem

The problem $\max(\Phi(G)) \geq 1$ is in $\mathrm{P}$.

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$\psi : (\bar\lambda_1 \vee \lambda_2 \vee \lambda_3) \wedge (\lambda_1 \vee \lambda_2 \vee \lambda_3) \wedge (\bar\lambda_2 \vee \bar\lambda_3)$.
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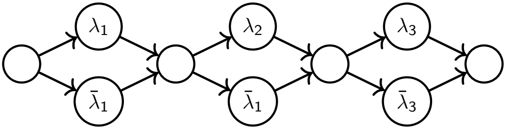

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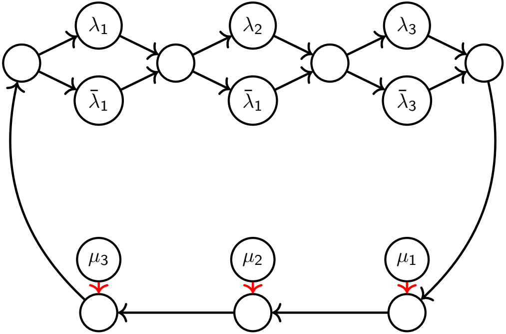

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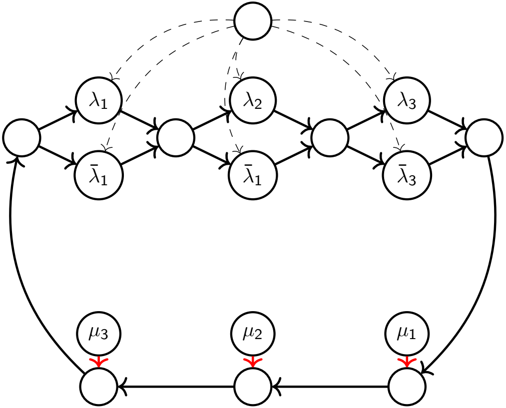

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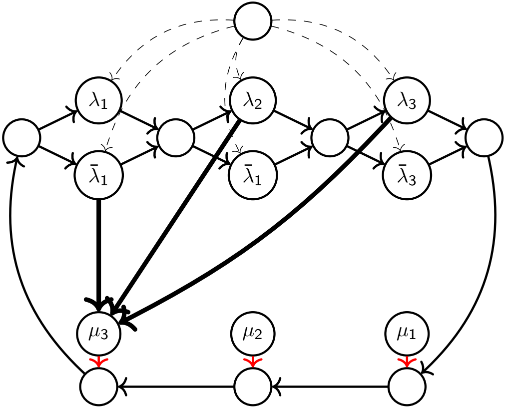

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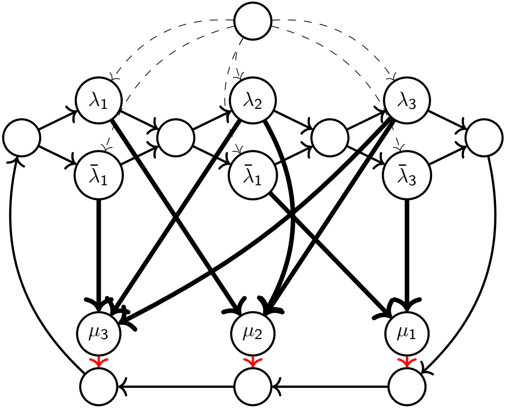

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## Theorem

The problem $\max(\Phi(G_\psi)) \geq 2$ is NP-complet.

# Conclusion

Table: Results

<table>
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