

Time-bounded Kolmogorov complexity provides an obstacle to soficness of multidimensional shifts

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- **Def** (A **shift** X in dimension 1) :
 - ▶ An alphabet Σ
 - ▶ A set F of forbidden patterns
 - ▶ X : the set of all configurations in $\Sigma^{\mathbb{Z}}$ without any pattern from F

- **Def** : An **admissible pattern** of a shift X is a pattern that appears somewhere in a configuration of X

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 - ▶ An alphabet Σ
 - ▶ A set F of forbidden patterns
 - ▶ X : the set of all configurations in $\Sigma^{\mathbb{Z}}$ without any pattern from F
- **Type 1** : **SFT**, can be defined by a finite set F

Ex. SFT : alternated black and white squares

$$\Sigma = \{\square, \blacksquare\}, F = \{\square\square, \blacksquare\blacksquare\}$$


$$X = \{\dots \square\blacksquare\square\blacksquare\square \dots, \dots \blacksquare\square\blacksquare\square \dots\}$$

- **Def** (A shift X in dimension 1) :
 - ▶ An alphabet Σ
 - ▶ A set F of forbidden patterns
 - ▶ X : the set of all configurations in $\Sigma^{\mathbb{Z}}$ without any pattern from F
- **Type 2** : **sofic** shift, can be defined by a regular language F

Ex. sofic shift : even number of white squares

$$\Sigma = \{\square, \blacksquare\}, F = \{ \blacksquare\square (\square\square)^* \blacksquare \}$$

$$X = \{ \dots, \dots \square \blacksquare \square \square \square \dots \square \blacksquare \square \dots, \dots \}$$


 even number
of white squares

- **Def** (A shift X in dimension 1) :
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 - ▶ X : the set of all configurations in $\Sigma^{\mathbb{Z}}$ without any pattern from F

- **Type 3** : **effective** shift, can be defined by a computable set F


Ex. effective shift : the mirror shift

$$\Sigma = \{\square, \blacksquare, \color{red}\blacksquare\}, F = \{\square \color{red}\blacksquare \blacksquare, \blacksquare \color{red}\blacksquare \square, \square \square \color{red}\blacksquare \blacksquare, \dots\}$$

$$X = \{\dots, \dots \square \blacksquare \color{orange}\blacksquare \blacksquare \square \blacksquare \blacksquare \color{red}\blacksquare \square \blacksquare \color{orange}\square \square \blacksquare \blacksquare \square \blacksquare \dots, \dots\}$$

- **Def** (A shift X in dimension 1) :
 - ▶ An alphabet Σ
 - ▶ A set F of forbidden patterns
 - ▶ X : the set of all configurations in $\Sigma^{\mathbb{Z}}$ without any pattern from F
- **Type 2 (equivalent definition)** : **sofic** shift. There exists :
 - ▶ An SFT $Y \subseteq \Delta^{\mathbb{Z}}$
 - ▶ A mapping $\phi : \Delta \rightarrow \Sigma$
 - ▶ $X = \phi(Y)$

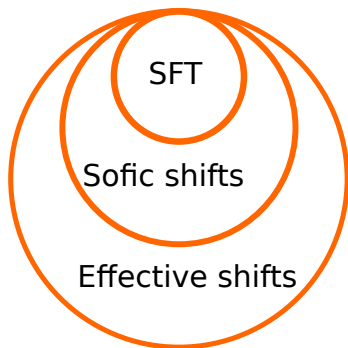
Ex. sofic shift: $Y = \{ \dots, \dots \boxed{b} \blacksquare \boxed{a} \boxed{b} \boxed{a} \dots \boxed{b} \blacksquare \boxed{a} \dots, \dots \}$


 even number of white squares

$\Sigma = \{ \square, \blacksquare \}$ $X = \{ \dots, \dots \square \blacksquare \square \square \square \dots \square \blacksquare \square \dots, \dots \}$

- How to distinguish between sofic and effective shifts ?

- ▶ Pumping lemma
- ▶ Particular properties of sofic shift (entropy)



- Properties of SFT and sofic shifts are quite well understood in dimension 1

- Can we generalize those definitions in higher dimensions ?
 - ▶ Straightforward for SFT and effective shifts
 - ▶ More difficult for sofic shifts
- **Def** (A **shift** X in dimension $d > 1$, $d = 2$) :
 - ▶ An alphabet Σ
 - ▶ A set F of forbidden patterns
 - ▶ X : the set of all configurations in $\Sigma^{\mathbb{Z}^d}$ without any pattern from F

- **Def** (A shift X in dimension $d > 1$, $d = 2$) :
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- **Type 1** : **SFT**, can be defined by a finite set F

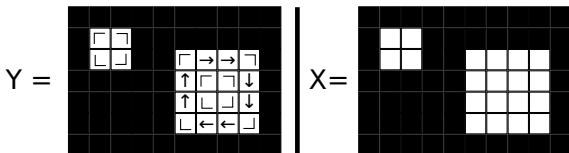
Ex. SFT : horizontally alternated black and white squares

$$\Sigma = \{\square, \blacksquare\}, F = \{ \square\square, \blacksquare\blacksquare \}$$

$$X = \{ \dots, \dots \begin{array}{cccc} \dots & & & \\ \square & \blacksquare & \square & \blacksquare & \square & \blacksquare & \square \\ \blacksquare & \square & \blacksquare & \square & \blacksquare & \square & \blacksquare \\ \dots & & & & & & \end{array} \dots, \dots \}$$

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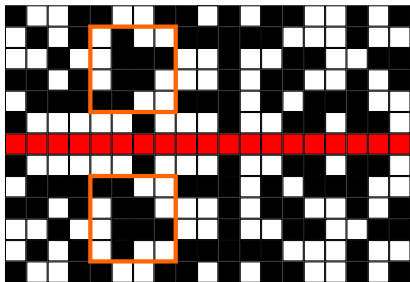
Ex. sofic shift :
 white squares
 $\Sigma = \{\square, \blacksquare\}$



- **Def** (A shift X in dimension $d > 1$, $d = 2$) :
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- **Type 3** : **effective** shift, can be defined by a computable set F

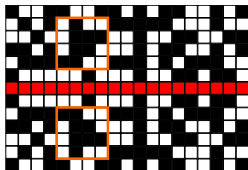
Ex. effective shift :
the mirror shift

$$\Sigma = \{\square, \blacksquare, \blacksquare\}$$



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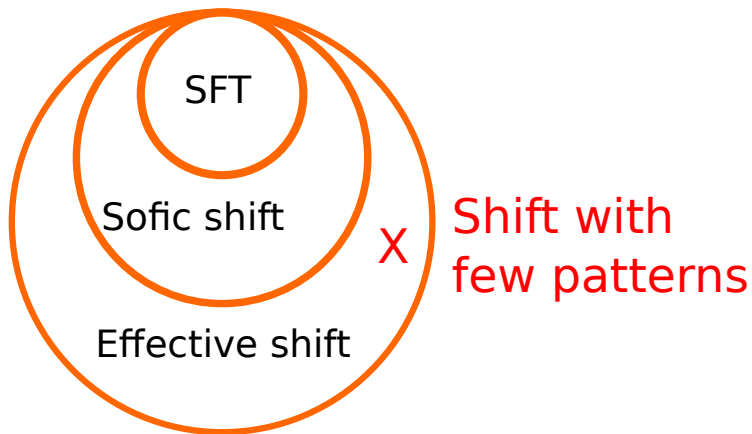
Claim

This shift is not sofic

An intuition behind the usual proof : Quadratic information (description of $n \times n$ -patterns) must traverse the linear boundary (orange line around patterns)

We need a lot of $n \times n$ -patterns ($\gg 2^n$) to prove non-soficness

- Can we construct an effective non-sofic shift with few patterns ?



Theorem 1 [implicitly in Durand, Levin & Shen 2008]

For every sofic shift X , for every n , there exists an admissible $n \times n$ - pattern P such that a program of size $O(n)$ prints P

Intuitively :

- The size of the program measures the information in P
- If every pattern contains too much information, the shift is not sofic

Formal statement involves Kolmogorov complexity

Theorem 1 [implicitly in Durand, Levin & Shen 2008]

For every sofic shift X , for every n , there exists an admissible $n \times n$ - pattern P such that a program of size $O(n)$ prints P

Theorem 2 [Rumyantsev & Ushakov, 2010]

There exists an effective shift such that for all admissible $n \times n$ -patterns P , every program which print P have size $\Omega(n^2)$.

Corollary

Shift from Theorem 2 is effective and non-sofic

Theorem 1'

For every sofic shift X , for every n , there exists an admissible $n \times n$ - pattern P such that a program of size $O(n)$ prints P in time $2^{O(n^2)}$.

Intuitively :

- P can be computed fast from a short description
- If no pattern can be computed fast from a short description, the shift is not sofic

Formal statement involves Time-bounded Kolmogorov complexity

Theorem 2' (main result)

There exists an effective shift such that :

(i) For all admissible $n \times n$ -patterns P , every program which print P in time $2^{O(n^2)}$ have size $\Omega(n^{1.5})$.

(ii) For all admissible $n \times n$ -patterns P , there exists a program of size $O(\log n)$ which prints P (very slowly)

Theorem 1'

For every sofic shift X , for every n , there exists an admissible $n \times n$ - pattern P such that a program of size $O(n)$ prints P in time $2^{O(n^2)}$.

Theorem 2' (main result)

There exists an effective shift such that :

- (i) For all admissible $n \times n$ -patterns P , the programs which print P in time $2^{O(n^2)}$ have size $\Omega(n^{1.5})$.**
- (ii) ...**

Fact 1

Shift from Theorem 2' is effective and non-sofic.

Theorem 2' (main result)

There exists an effective shift such that :

(i) ...

(ii) For all admissible $n \times n$ -patterns P , there exists a program of size $O(\log n)$ which prints P (very slowly)

Fact 1

Shift from Theorem 2' is effective and non-sofic.

Fact 2

Shift from Theorem 2' has $\text{poly}(n)$ $n \times n$ -patterns

Corollary

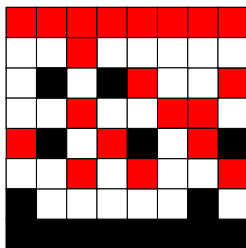
There exists an effective non-sofic shift with only $\text{poly}(n)$ admissible $n \times n$ -patterns

This is the non-technical version of our main result.

- A more general technique to prove non-soficness : see proceedings.
- We translate in the language of Kolmogorov complexity the proofs for most previously known examples and suggest several new ones.

Ex. The shift from
Kaas & Madden :

$$\Sigma = \{\square, \blacksquare, \color{red}\blacksquare\}$$



A forbidden pattern

Key points of the talk

- Unexpected use of time-bounded Kolmogorov complexity
- A new tool to prove the non-soficness of a shift
- Examples of effective non-sofic shifts, with few patterns

Thanks for your attention

