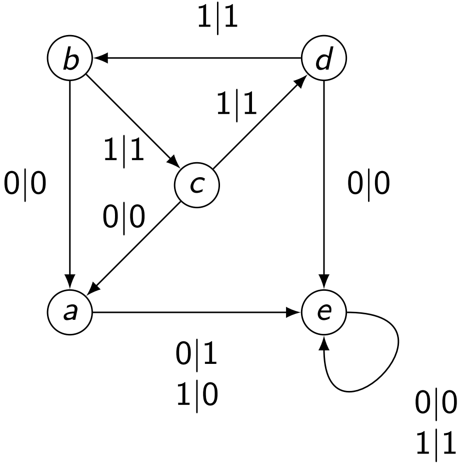
Mealy automata & random groups

Thibault Godin

EJCIM, Marseille March 8, 2019

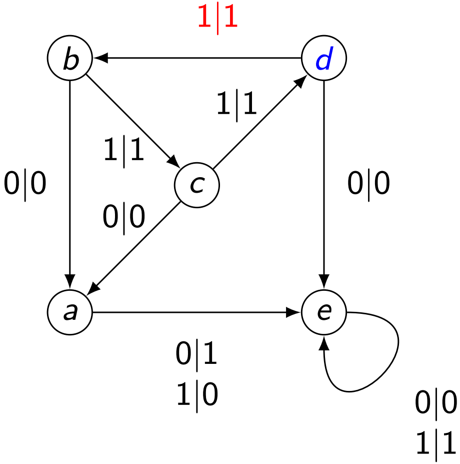
IMAG
INSTITUT MONTPELLIERAIN
ALEXANDER GROTHENDIECK

Institut ÉLIE CARTAN



Mealy automaton $\mathcal{G}$

$$\mathcal{A} = (Q, \Sigma, \delta, \rho)$$



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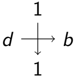

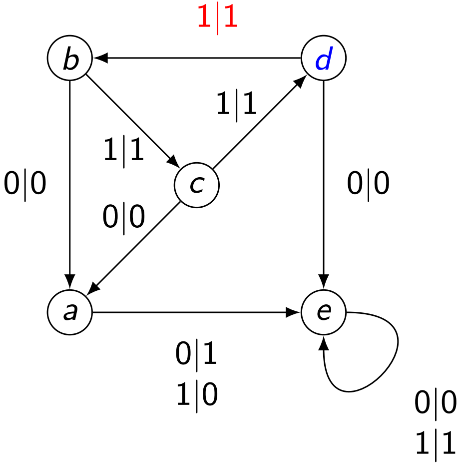

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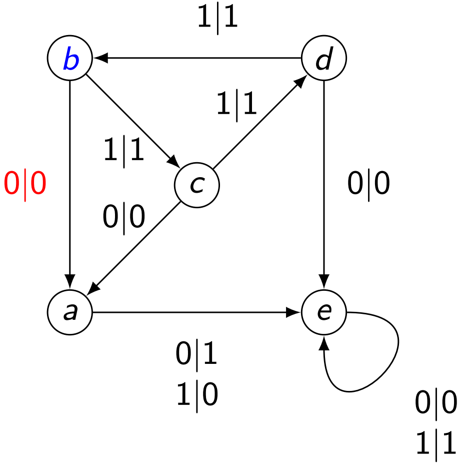
1 0 0 0 1

d



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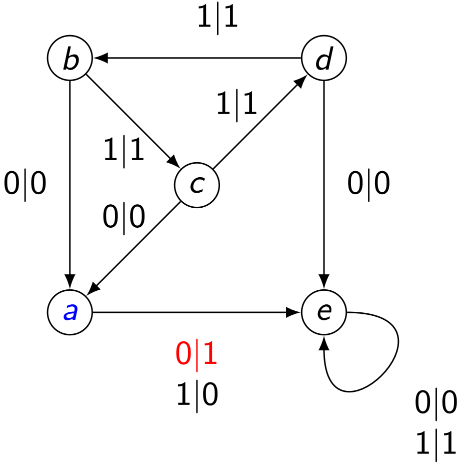

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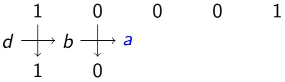

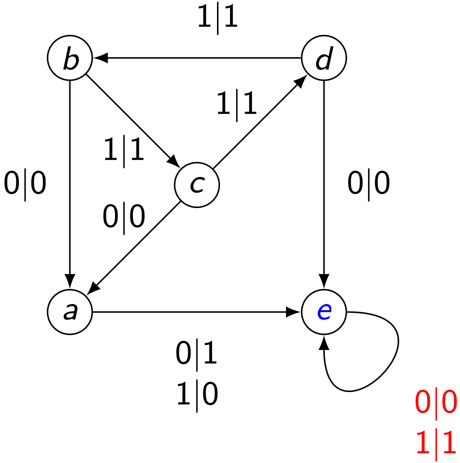

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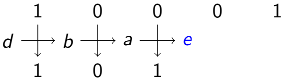

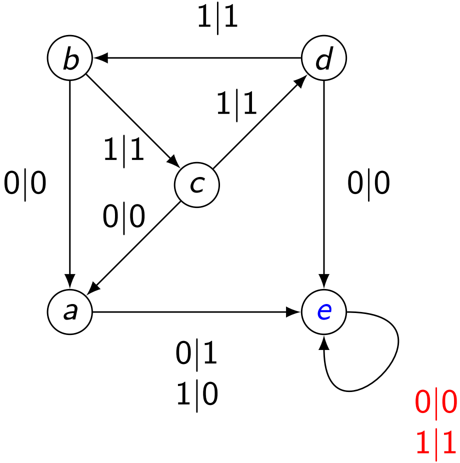

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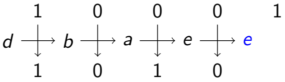

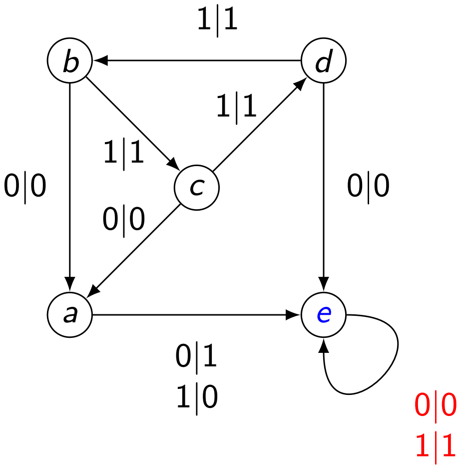

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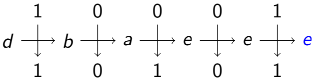

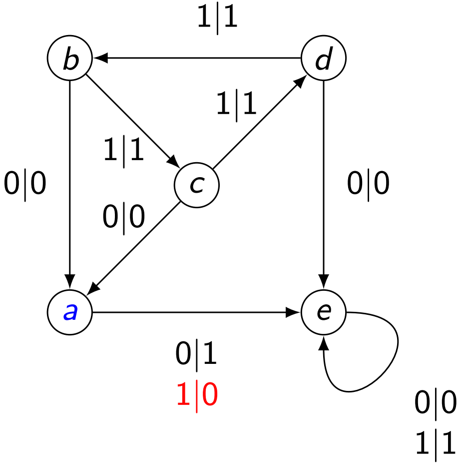

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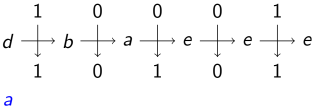

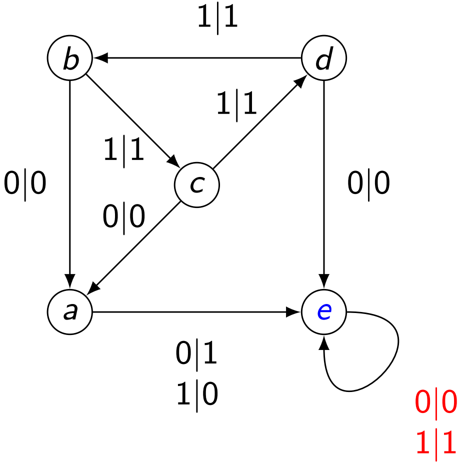

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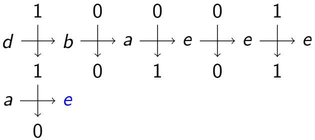

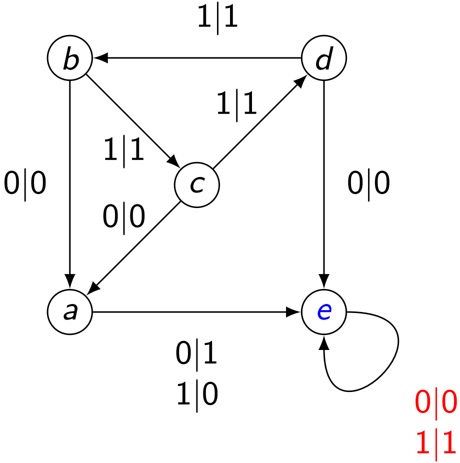

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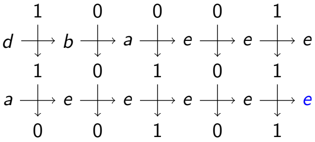

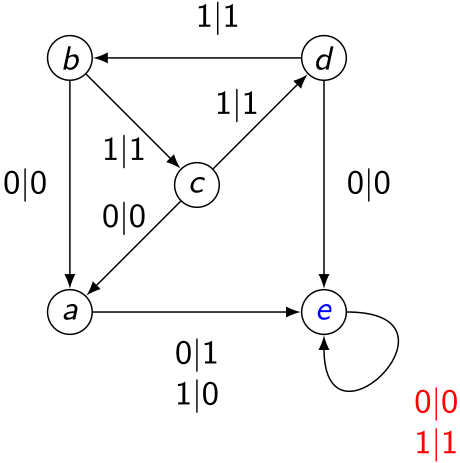

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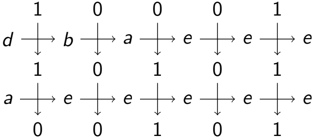
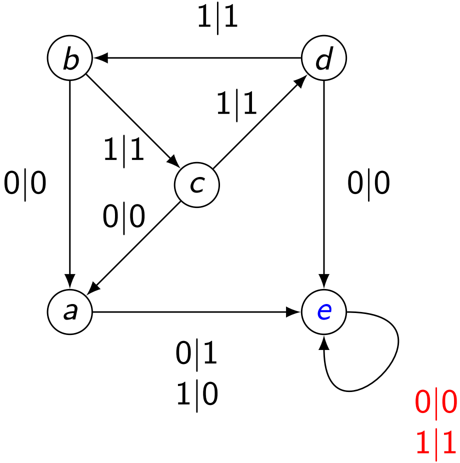


$$\rho_{da}(10001) = \rho_a(\rho_d(10001))$$



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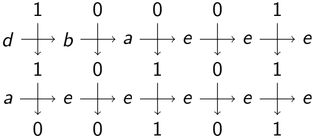

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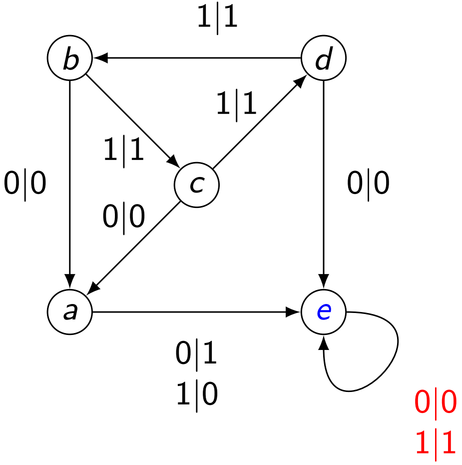

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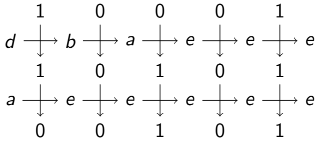

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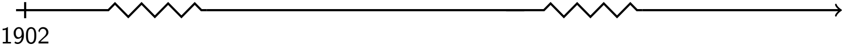
Burnside

Can an infinite group have all element of finite order?



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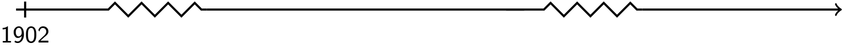


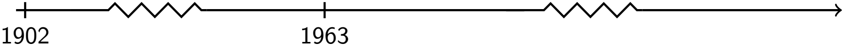
Burnside Problem

Burnside

Can an infinite group have all element of finite order?

Golod Shafarevich

Yes! (complicated)



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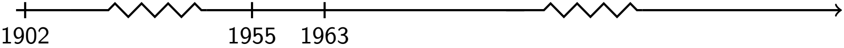
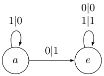


Burnside Problem

Mealy



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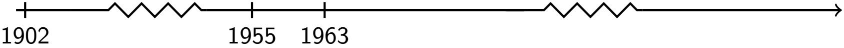
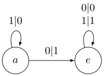


Burnside Problem

Glushkov

Let's use Mealy automata to get (semi)groups!

Mealy



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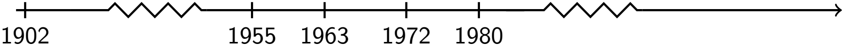

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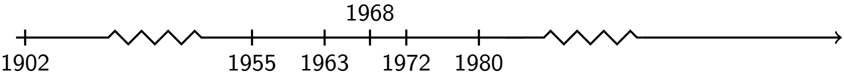
Grigorchuk-Aleshin

Burnside automaton group



Burnside Problem

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Growth

Cayley Graph: $\Gamma(G, S)$

$$g \xrightarrow{s} g.s$$

Growth

Cayley Graph: $\Gamma(G, S)$ $ex : G = \mathbb{Z}^2,\ S = \{a = (0,1), b = (1,0)\}$

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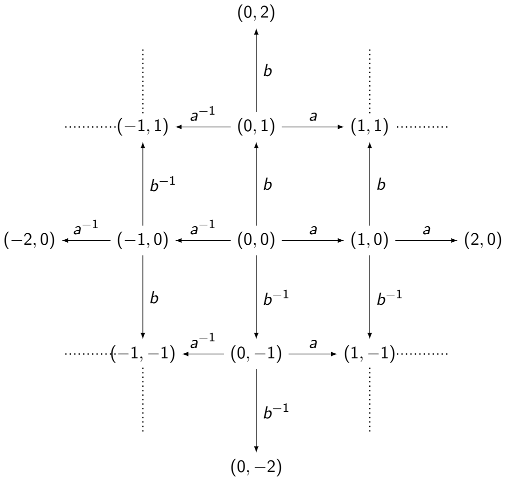

Growth

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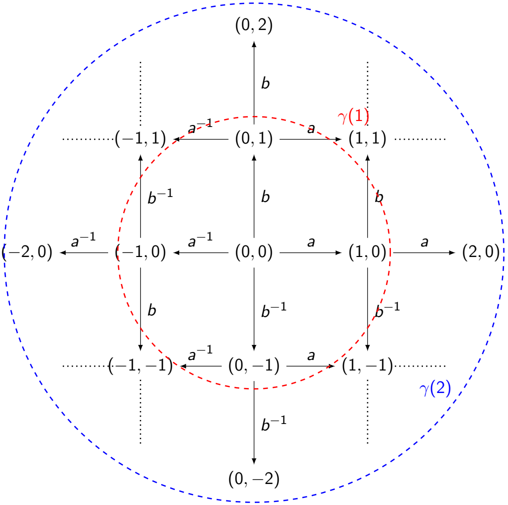
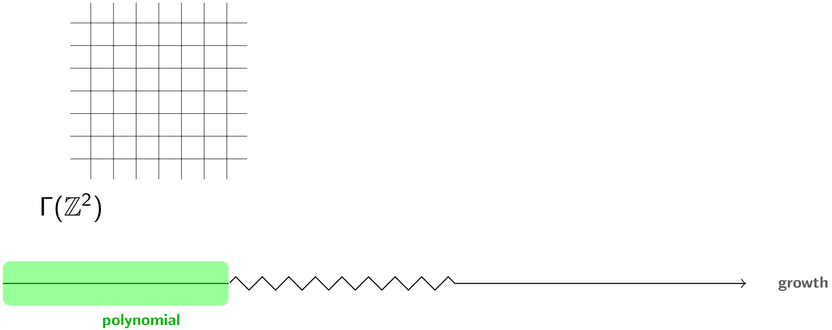


$\gamma(0) = 1$
$\gamma(1) = 5$
$\gamma(2) = 13$
$\vdots$
$\gamma(n) = 2n^2 + 2n + 1$

Milnor's Problem



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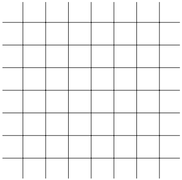
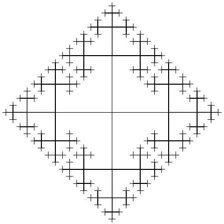
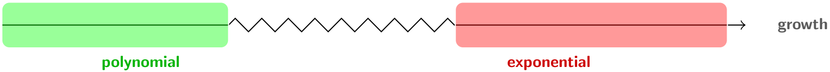


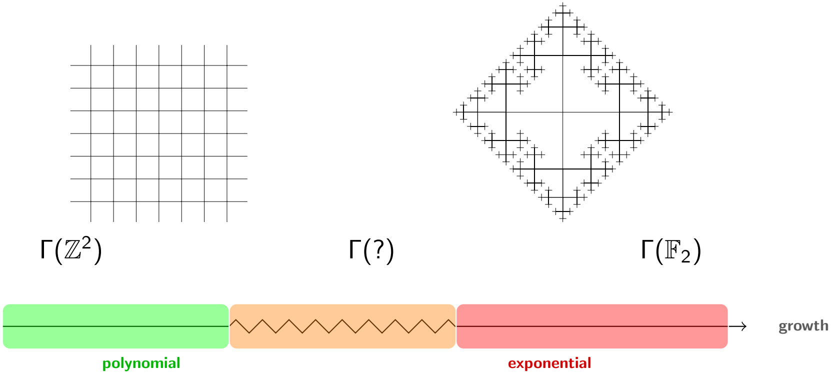
$\Gamma(\mathbb{Z}^2)$



$\Gamma(\mathbb{F}_2)$



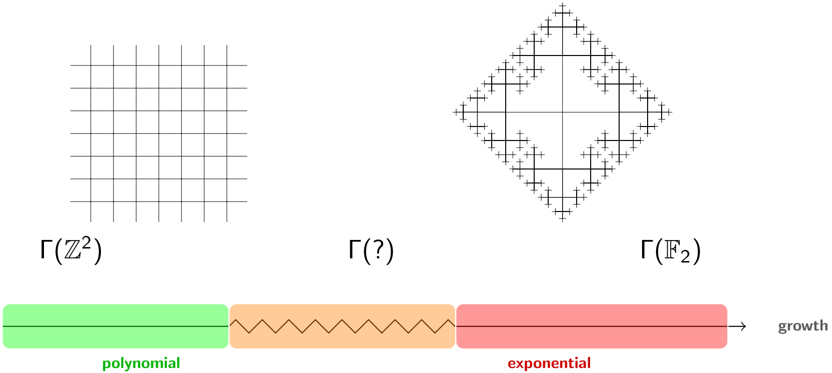
Milnor's Problem



Milnor's Problem

Milnor's Problem (1968):

Do groups of growth between polynomial and exponential exist?



1968

1902 1955 1963 1972 1980

Burnside Problem

Milnor Problem

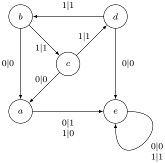
Grigorchuk-Aleshin

Burnside automaton group

1968 1983

1902 1955 1963 1972 1980

Burnside Problem



Milnor Problem

Grigorchuk

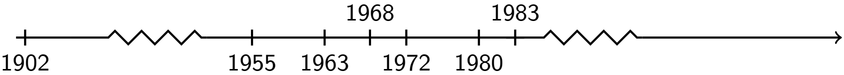
Automaton group of intermediate growth

Grigorchuk-Aleshin

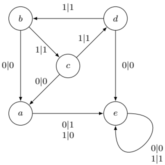
Burnside automaton group

(semi)Group theorists

Gromov problem
Day problem
Attiyah problem
...



Burnside Problem



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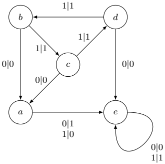
1902 — 1955 — 1963 — 1968 — 1972 — 1980 — 1983 — 2000

Burnside Problem



Milnor Problem

Grigorchuk

Automaton group of intermediate growth

Transfert of struct. prop.

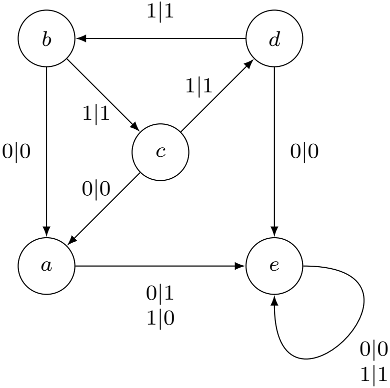
Grigorchuk-Aleshin

Burnside automaton group

Small automata, interesting groups

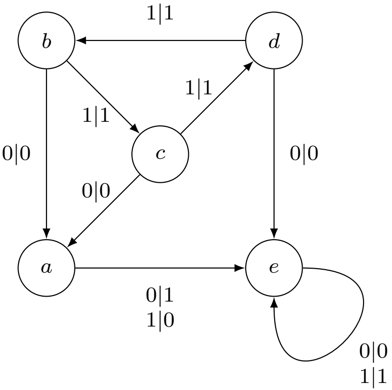
Automaton

Group



Small automata, interesting groups

Automaton



Group

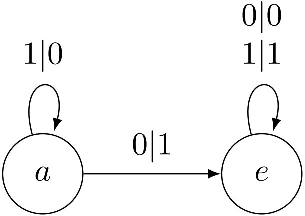
infinite Burnside

amenable not elementary amenable

intermediate growth

Small automata, interesting groups

Automaton



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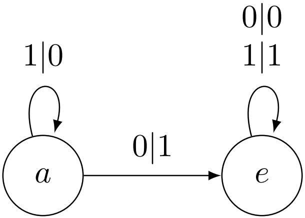
Automaton



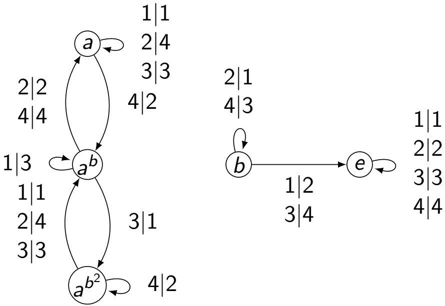
Group

integer $\mathbb{Z}$

Small automata, interesting groups

Automaton

Group



Small automata, interesting groups

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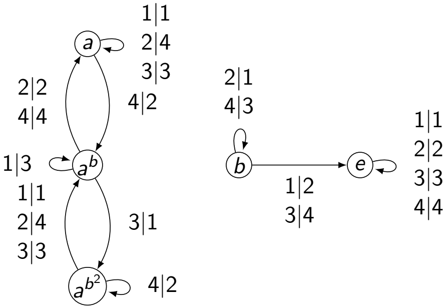
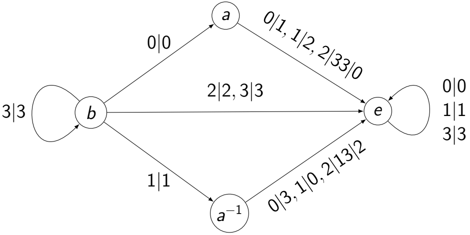


Heisenberg $H_3(\mathbb{Z})$

distortion

Small automata, interesting groups

Automaton



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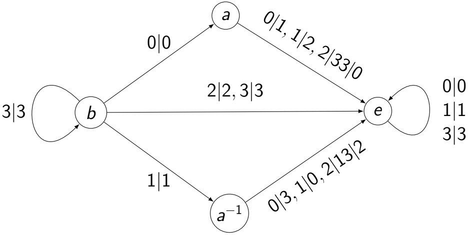

Group

infinite Burnside

Classify?

- 2-letters 2-states $\rightarrow$ 64 automata

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  $\to$ 6 groups:

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  $\rightarrow$ 6 groups: $\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z} \bigoplus \mathbb{Z}/2\mathbb{Z}, \quad \mathbb{Z}, D_\infty, \quad \mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$

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- 2-letters 3-states $\rightarrow$ $5,832$ automata

Classify?

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- 2-letters 3-states $\rightarrow 5,832$ automata (≤ 122 non isomorphic)

Classify?

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lots of computations, lots of troubles

Typical behaviour?

How does a random group looks like?

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How does a random group looks like?

random permutations

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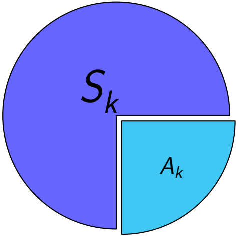
Theorem (Dixon'69)

$$\text{generically } \langle \sigma, \tau \rangle = \begin{cases} S_k \\ A_k \end{cases}$$



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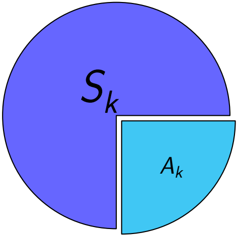

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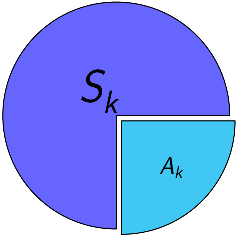

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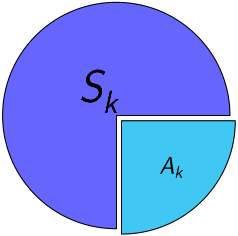

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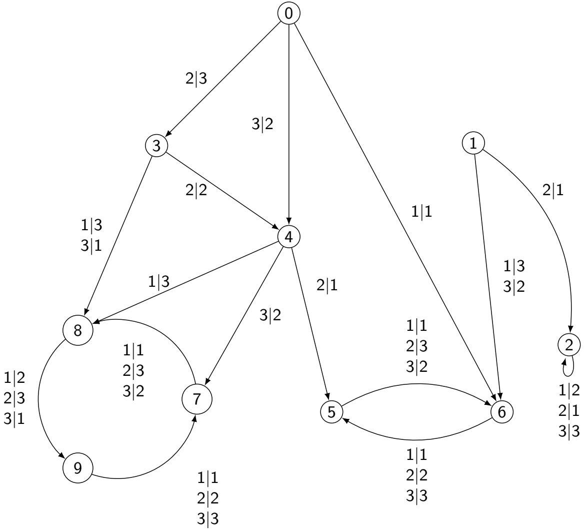
$\rightarrow$Random Mealy automata

Structural constraints



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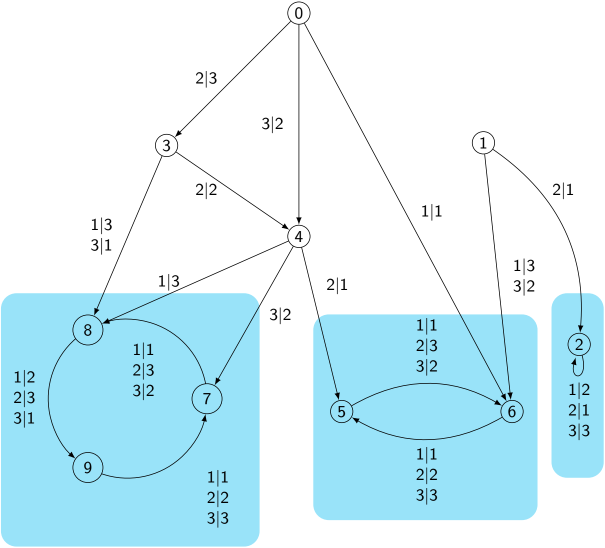
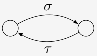


Random 2-state cyclic automata

Theorem

$\langle$ 

$\rangle =$

Random 2-state cyclic automata

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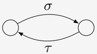
generically $\langle$  $\rangle$

$$= \begin{cases} S_k \times S_k \\ (A_k \times A_k) \rtimes \langle(\pi, \pi)\rangle \\ A_k \times A_k \end{cases}$$

Random 2-state cyclic automata

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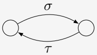
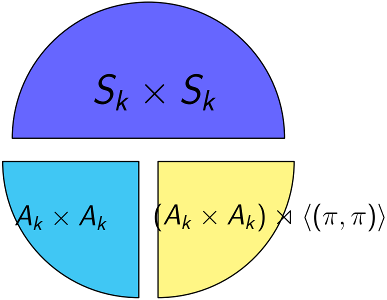
generically $\langle$  $\rangle = \begin{cases} S_k \times S_k \\ (A_k \times A_k) \rtimes \langle(\pi,\pi)\rangle \\ A_k \times A_k \end{cases}$



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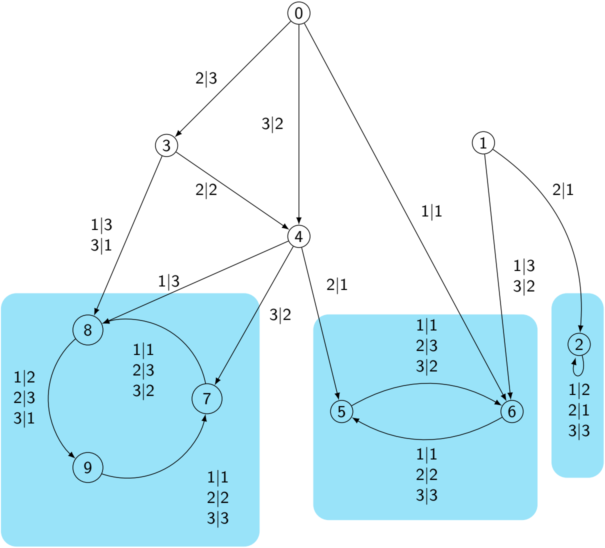

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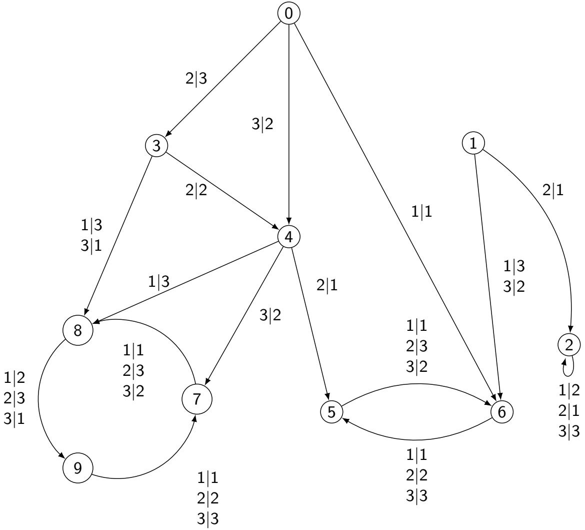

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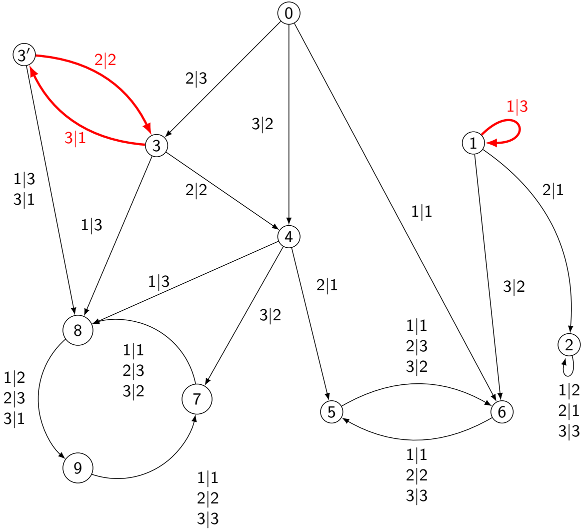
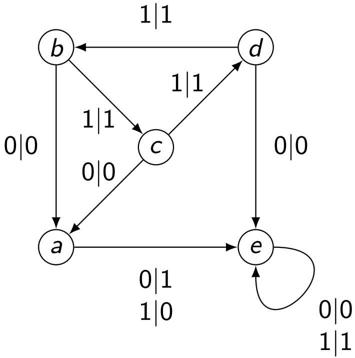


Polynomial automata



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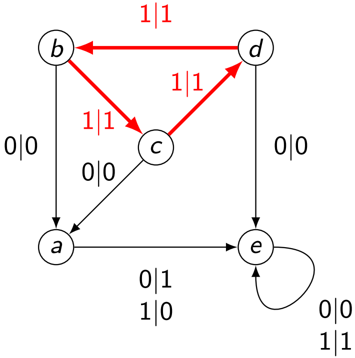

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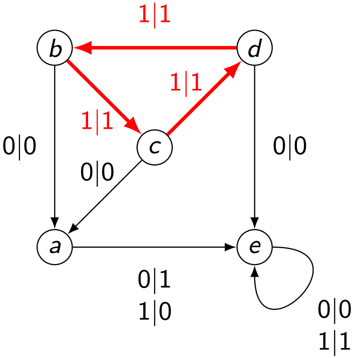

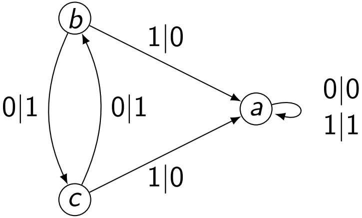

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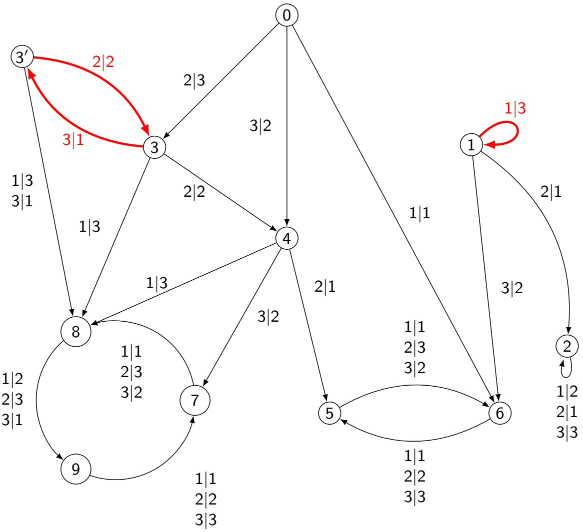

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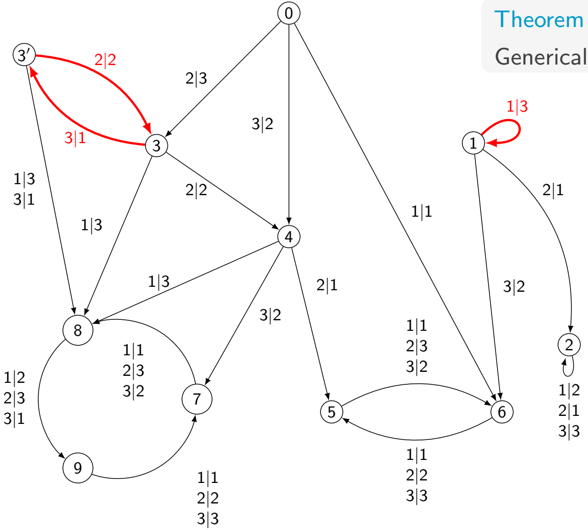
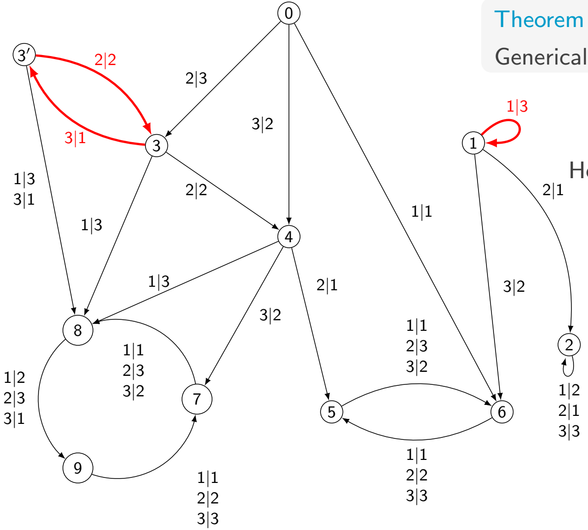


Theorem

Generically element of infinite order

Structural constraints



Theorem

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Hence generically

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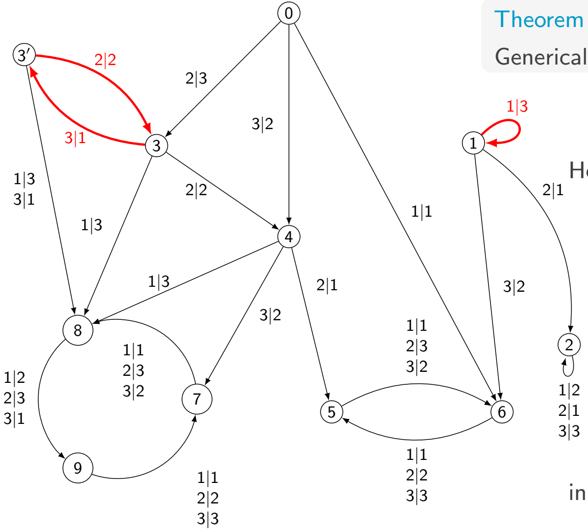

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in other classes:

- elements of infinite order
- exponential growth

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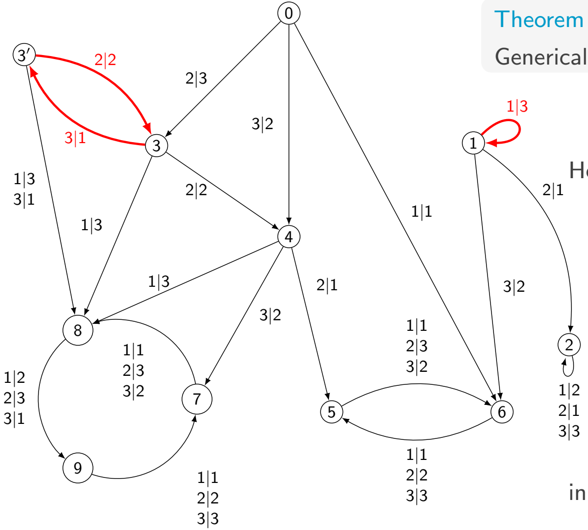

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Thanks!

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