



# NIVAT'S CONJECTURE

## CONFIGURATIONS

$$\mathcal{A} = \{ \text{cyan square}, \text{green square}, \text{purple square}, \text{orange square} \}$$

$$w \in \mathcal{A}^{\mathbb{Z}}$$



# CONFIGURATIONS, COMPLEXITY

$P_w(n)$  = number of patterns of size  $n$



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$$P_w(1) = 4$$

$$P_w(2) \geq 11$$

$$P_w(4) \geq 18$$

# CONFIGURATIONS, COMPLEXITY, PERIODICITY



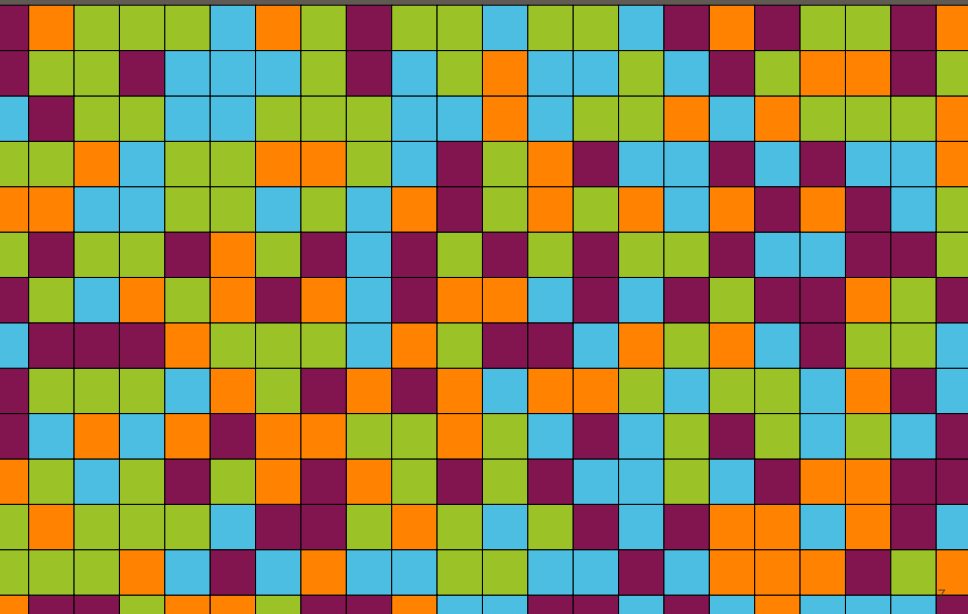
$$P_w(4) = 4$$

## CONFIGURATIONS, COMPLEXITY, PERIODICITY

## Theorem (Morse, Hedlund, 1938)

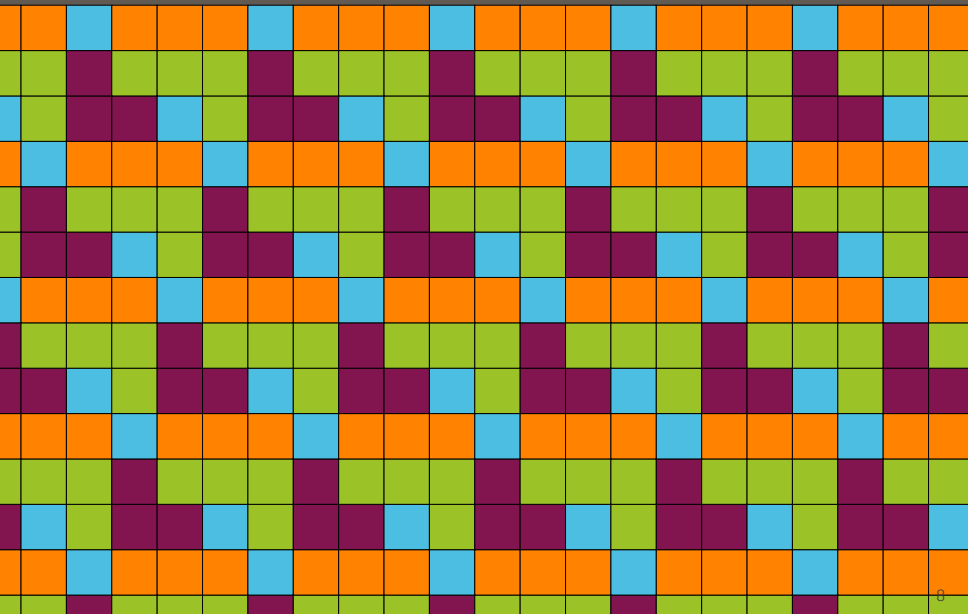
$$\forall w \in \mathcal{A}^{\mathbb{Z}}, \exists n > 0,$$
$$P_w(n) \leq n \Rightarrow w \text{ periodic}$$

# 2D CONFIGURATIONS





## 2D CONFIGURATIONS



## 2D: NIVAT'S CONJECTURE

Conjecture (Nivat, 1997)

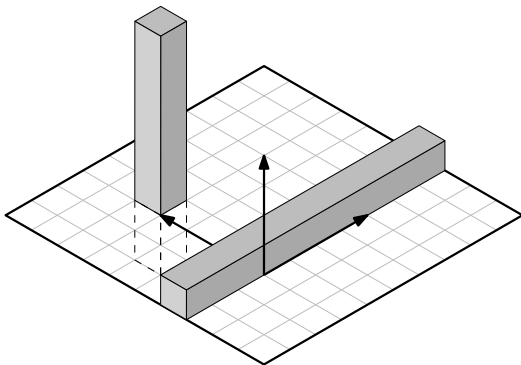
$$\forall c \in \mathcal{A}^{\mathbb{Z}^2}, \exists m, n > 0,$$
$$P_c(m, n) \leq mn \Rightarrow c \text{ periodic}$$

## HIGHER DIMENSION ?

## Theorem

$$\exists c \in \mathcal{A}^{\mathbb{Z}^3}, \forall n > 0,$$
$$P_c(n, n, n) \leq n^3 \text{ and } c \text{ not periodic}$$

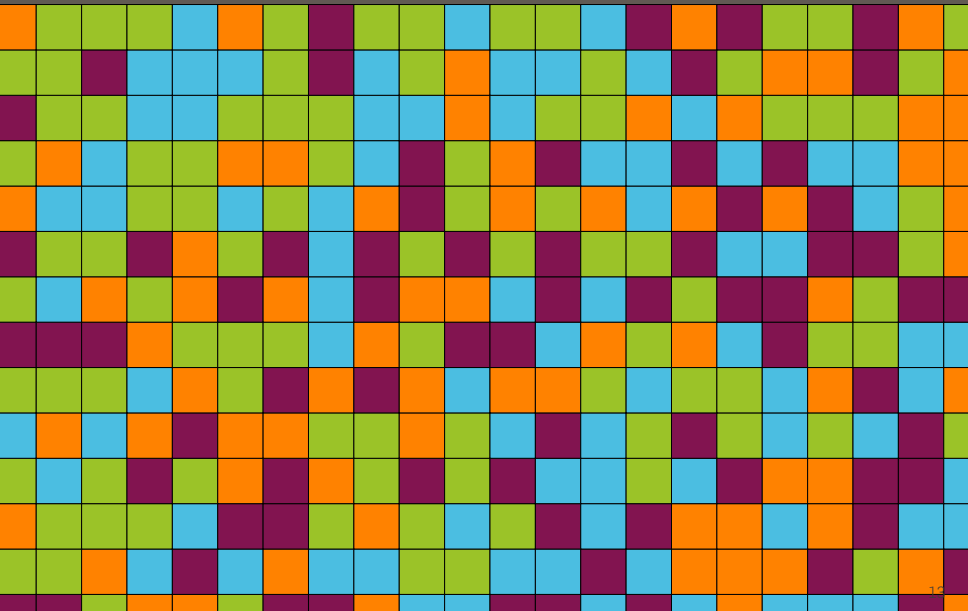
## HIGHER DIMENSION ?



$$P_c(n, n, n) = 2n^2 + 1 < n^3$$

# ALGEBRAIC TOOLS

# CONFIGURATIONS ARE (LAURENT) SERIES



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3	1	1	1	0	3	1	2	1	1	0	1	1	0	2	3	2	1	1	2	3	1
1	1	2	0	0	0	1	2	0	1	3	0	0	1	0	2	1	3	3	2	1	3
2	1	1	0	0	1	1	1	0	0	3	0	1	1	3	0	3	1	1	1	3	3
1	3	0	1	1	3	3	1	0	2	1	3	2	0	0	2	0	2	0	0	3	3
3	0	0	1	1	0	1	0	3	2	1	3	1	3	0	3	2	3	2	0	1	3
2	1	1	2	3	1	2	0	2	1	2	1	2	1	1	2	0	0	2	2	1	3
1	0	3	1	3	2	3	0	2	3	3	0	2	0	2	1	2	2	3	1	2	2
2	2	2	3	1	1	1	0	3	1	2	2	0	3	1	3	0	2	1	1	0	0
1	1	1	0	3	1	2	3	2	3	0	3	3	1	0	1	1	0	3	2	0	3
0	3	0	3	2	3	3	1	1	3	1	0	2	0	1	2	1	0	1	0	2	1
1	0	1	2	1	3	2	3	1	2	1	2	0	0	1	0	2	3	3	2	2	0
3	1	1	1	0	2	2	1	3	1	0	1	2	0	2	3	3	0	3	2	0	0
1	1	3	0	2	0	3	0	0	1	1	0	0	2	0	3	3	3	2	1	3	2

# CONFIGURATIONS ARE (LAURENT) SERIES

-10,5	c-9,5	c-8,5	c-7,5	c-6,5	c-5,5	c-4,5	c-3,5	c-2,5	c-1,5	c0,5	c1,5	c2,5	c3,5	c4,5	c5,5	c6,5	c7,5	c8,5	c9,5	c10,5	c11,5
-10,4	c-9,4	c-8,4	c-7,4	c-6,4	c-5,4	c-4,4	c-3,4	c-2,4	c-1,4	c0,4	c1,4	c2,4	c3,4	c4,4	c5,4	c6,4	c7,4	c8,4	c9,4	c10,4	c11,4
-10,3	c-9,3	c-8,3	c-7,3	c-6,3	c-5,3	c-4,3	c-3,3	c-2,3	c-1,3	c0,3	c1,3	c2,3	c3,3	c4,3	c5,3	c6,3	c7,3	c8,3	c9,3	c10,3	c11,3
-10,2	c-9,2	c-8,2	c-7,2	c-6,2	c-5,2	c-4,2	c-3,2	c-2,2	c-1,2	c0,2	c1,2	c2,2	c3,2	c4,2	c5,2	c6,2	c7,2	c8,2	c9,2	c10,2	c11,2
-10,1	c-9,1	c-8,1	c-7,1	c-6,1	c-5,1	c-4,1	c-3,1	c-2,1	c-1,1	c0,1	c1,1	c2,1	c3,1	c4,1	c5,1	c6,1	c7,1	c8,1	c9,1	c10,1	c11,1
-10,0	c-9,0	c-8,0	c-7,0	c-6,0	c-5,0	c-4,0	c-3,0	c-2,0	c-1,0	c0,0	c1,0	c2,0	c3,0	c4,0	c5,0	c6,0	c7,0	c8,0	c9,0	c10,0	c11,0
-10,-1	c-9,-1	c-8,-1	c-7,-1	c-6,-1	c-5,-1	c-4,-1	c-3,-1	c-2,-1	c-1,-1	c0,-1	c1,-1	c2,-1	c3,-1	c4,-1	c5,-1	c6,-1	c7,-1	c8,-1	c9,-1	c10,-1	c11,-1
-10,-2	c-9,-2	c-8,-2	c-7,-2	c-6,-2	c-5,-2	c-4,-2	c-3,-2	c-2,-2	c-1,-2	c0,-2	c1,-2	c2,-2	c3,-2	c4,-2	c5,-2	c6,-2	c7,-2	c8,-2	c9,-2	c10,-2	c11,-2

$$c = \sum_{i=-\infty}^{\infty} c_{i,j} X^i Y^j$$

-10,-7	c-9,-7	c-8,-7	c-7,-7	c-6,-7	c-5,-7	c-4,-7	c-3,-7	c-2,-7	c-1,-7	c0,-7	c1,-7	c2,-7	c3,-7	c4,-7	c5,-7	c6,-7	c7,-7	c8,-7	c9,-7	c10,-7	c11,-7
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## OPERATIONS: SUM

$$c + d = \sum_{i=-\infty}^{\infty} (c_{i,j} + d_{i,j}) X^i Y^j$$

Formal sum  $\leftrightarrow$  Sum of configurations

## OPERATIONS: MULTIPLICATION

$$X^a Y^b c = \sum_{i=-\infty}^{\infty} c_{i,j} X^{i+a} Y^{j+b}$$

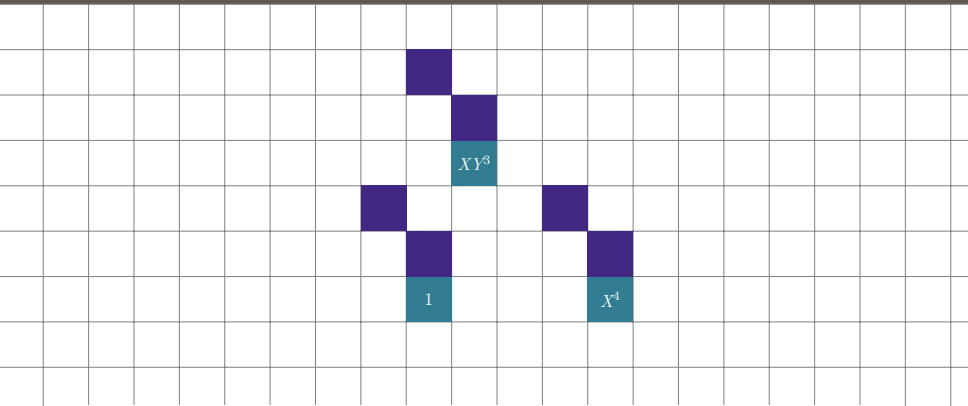
Multiplication by  $X^a Y^b \leftrightarrow$  Translation of vector  $(a, b)$

(PARENTHESIS: SEE YOUR POLYNOMIALS LIKE YOU NEVER DID)

		$Y^2$
		$XY$
$(0,0)$		$X$

$$X + XY + Y^2$$

(PARENTHESIS: SEE YOUR POLYNOMIALS LIKE YOU NEVER DID)



$$(X + XY + Y^2)(1 + X^4 + XY^3)$$

## EXPRESSING PERIODICITY

$$(X^a Y^b - 1)c = 0$$

$$\Leftrightarrow$$

$$X^a Y^b c = c$$

$$\Leftrightarrow$$

$c$  periodic of period  $(a, b)$

## ALGEBRA IS COMING

$$\text{Ann}(c) = \{p \mid pc = 0\}$$

$$c \text{ periodic} \Leftrightarrow \exists a, b \in \mathbb{Z}, (X^a Y^b - 1) \in \text{Ann}(c)$$

$\text{Ann}(c)$  is a **polynomial ideal**:

$$\rightarrow 0 \in I$$

$$\rightarrow f, g \in I \Rightarrow f + g \in I$$

$$\rightarrow f \in I \text{ and } h \text{ any polynomial} \Rightarrow fh \in I$$

SUPER NICE RESULTS

## WARMING UP

$c$  of low complexity

Theorem (Kari, Szabados, 2015)

$$\exists p \neq 0 \in \text{Ann}(c)$$



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Theorem (Kari, Szabados, 2015)

$$\exists a_1, b_1, a_2, b_2 \dots, a_r, b_r \in \mathbb{Z}$$

$$\left( X^{a_1} Y^{b_1} - 1 \right) \left( X^{a_2} Y^{b_2} - 1 \right) \dots \left( X^{a_r} Y^{b_r} - 1 \right) \in \text{Ann}(c)$$

# ASYMPTOTIC NIVAT

Let  $c$  be non-periodic.

## Conjecture (Nivat)

$P_c(m, n) > mn$  for all  $m, n \in \mathbb{N}$ .

## Theorem (Kari, Szabados, 2015)

$P_c(m, n) > mn$  holds for all but finitely many choices of  $m, n \in \mathbb{N}$ .

## NIVAT FOR ALGEBRAIC SUBSHIFTS

Let  $c$  such that  $\exists$  polynomial  $f$  with 0 or 1 line polynomial factor such that

$$fc = 0 \pmod{2}$$

**Theorem (Kari, M., 2019)**

If  $\exists D, P_c(D) \leq |D|$ .

Then  $c$  is **periodic**.

# LOW COMPLEXITY SUBSHIFTS

$F$  finite set of patterns.

$$X_F = \{c \mid \text{All patterns of } c \text{ are in } F\}$$

If  $F$  is defined by  $mn$  rectangular patterns of size  $m \times n$ .

(Therefore for any  $c \in X_F$ ,  $P_c(m, n) \leq mn$ )

## Conjecture (Nivat)

All  $c \in X_F$  are periodic.

## Theorem (Kari, M.)

There exists  $c \in X_F$  periodic.

## LOW COMPLEXITY SUBSHIFTS

**Domino Problem (DP):**

« Given  $F$ , is  $X_F$  empty ? »

In general, DP is **undecidable**.

In many cases, DP is undecidable.

If Nivat is true,

DP is **decidable** for  $X$  of low complexity.

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## WHAT NEXT ?

- Nivat's conjecture
- Use algebraic tools for similar things (Cellular Automata ?)

Thank you !



# HSRM THEME



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