ALGEBRA, TILINGS AND NIVAT

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NIVAT'S CONJECTURE

CONFIGURATIONS

$\mathcal{A} = \{$  $\}$

$w \in \mathcal{A}^{\mathbb{Z}}$

CONFIGURATIONS, COMPLEXITY

$P_w(n) =$ number of patterns of size n

CONFIGURATIONS, COMPLEXITY

$$P_w(n) = \text{ number of patterns of size } n$$

$$P_w(1) = 4$$

$$P_w(2) \geq 11$$

$$P_w(4) \geq 18$$

CONFIGURATIONS, COMPLEXITY, PERIODICITY

$$P_w(4) = 4$$

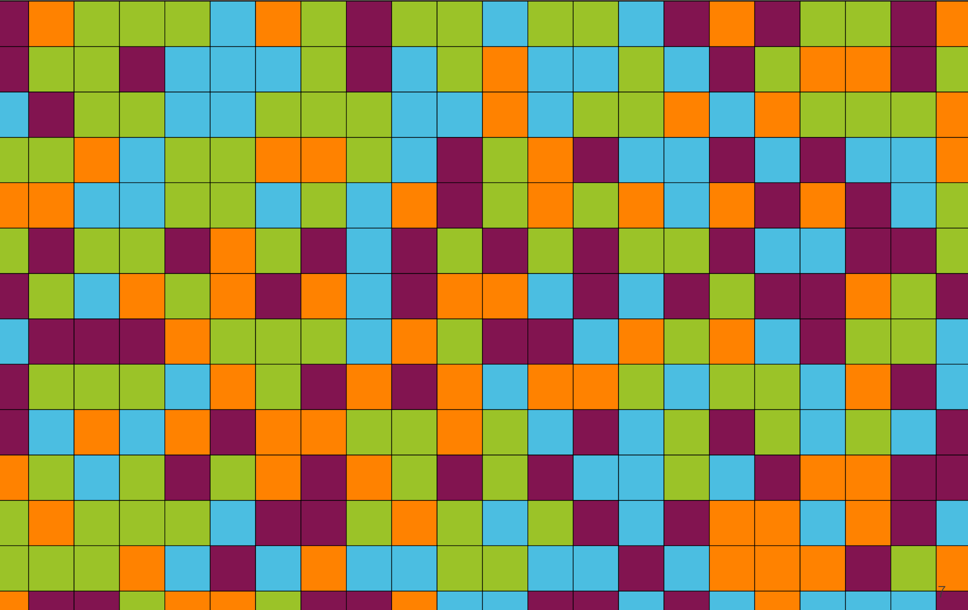
CONFIGURATIONS, COMPLEXITY, PERIODICITY

Theorem (Morse, Hedlund, 1938)

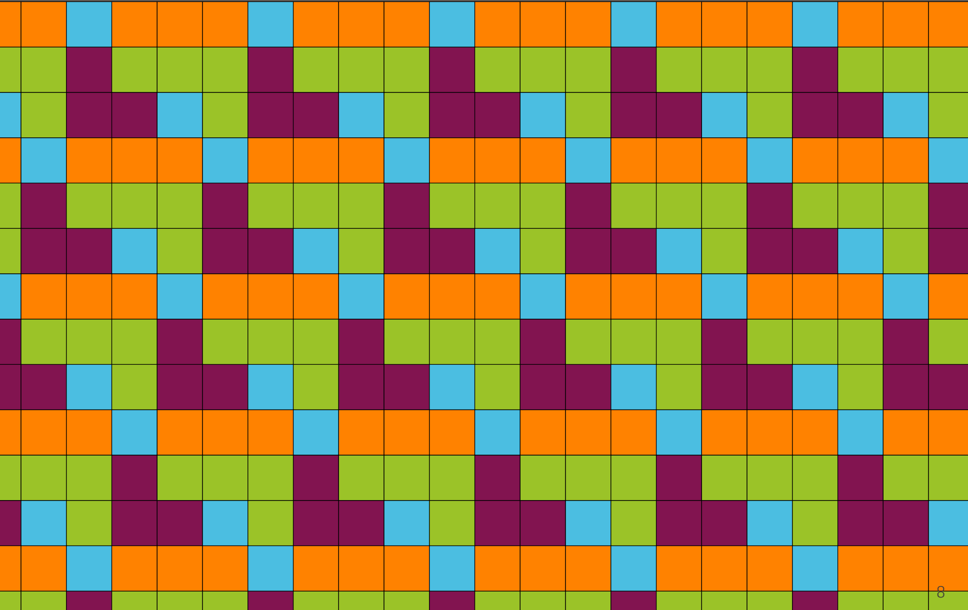
$$\forall w \in \mathcal{A}^{\mathbb{Z}}, \exists n > 0,$$

$$P_w(n) \leq n \quad \Rightarrow \quad w \textit{ periodic}$$

2D CONFIGURATIONS



2D CONFIGURATIONS



2D: NIVAT'S CONJECTURE

Conjecture (Nivat, 1997)

$$\forall c \in \mathcal{A}^{\mathbb{Z}^2}, \exists m, n > 0,$$

$$P_c(m, n) \leq mn \quad \Rightarrow \quad c \text{ periodic}$$

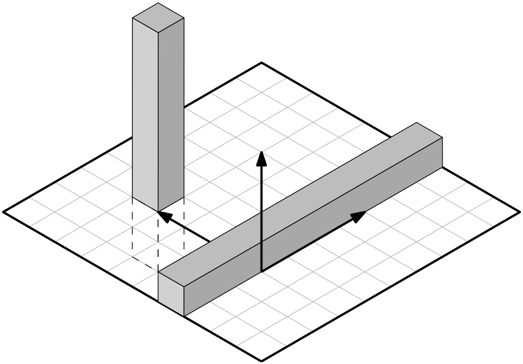
HIGHER DIMENSION ?

Theorem

$$\exists c \in \mathcal{A}^{\mathbb{Z}^3}, \forall n > 0,$$

$$P_c(n,n,n) \leq n^3 \quad \textit{and} \quad c \textit{ not periodic}$$

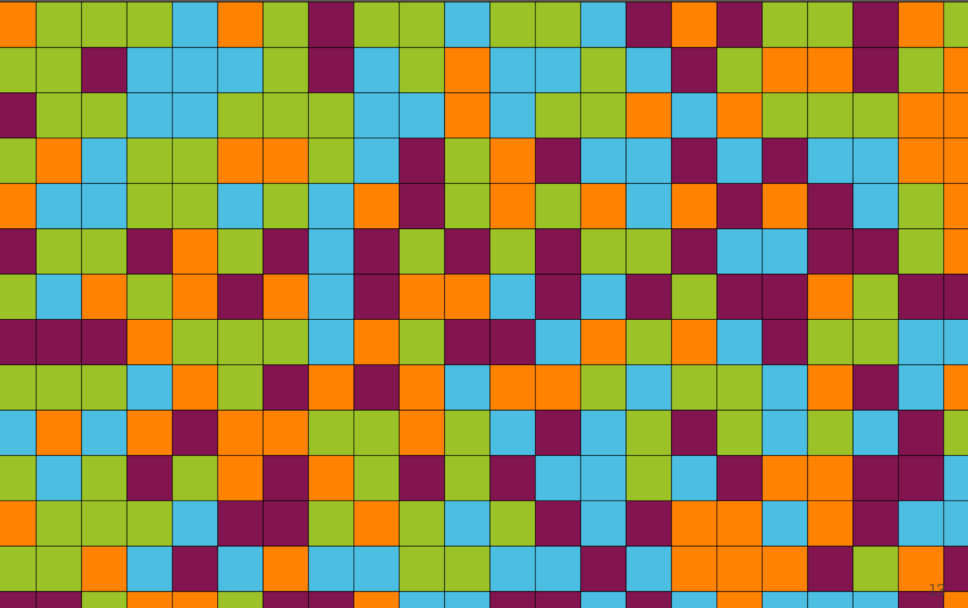
HIGHER DIMENSION ?



$$P_c(n, n, n) = 2n^2 + 1 < n^3$$

ALGEBRAIC TOOLS

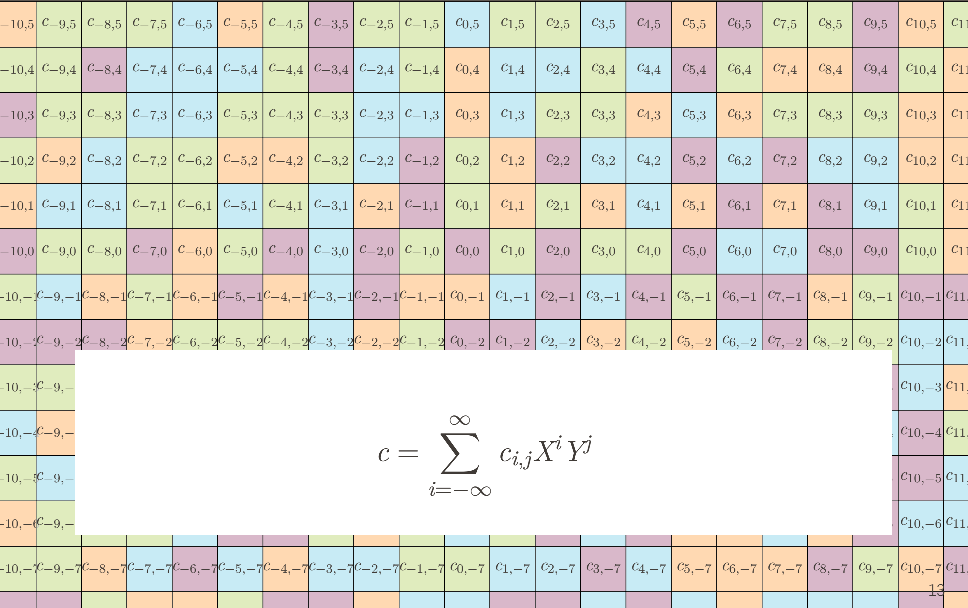
CONFIGURATIONS ARE (LAURENT) SERIES



CONFIGURATIONS ARE (LAURENT) SERIES

3	1	1	1	0	3	1	2	1	1	0	1	1	0	2	3	2	1	1	2	3	1
1	1	2	0	0	0	1	2	0	1	3	0	0	1	0	2	1	3	3	2	1	3
2	1	1	0	0	1	1	1	0	0	3	0	1	1	3	0	3	1	1	1	3	3
1	3	0	1	1	3	3	1	0	2	1	3	2	0	0	2	0	2	0	0	3	3
3	0	0	1	1	0	1	0	3	2	1	3	1	3	0	3	2	3	2	0	1	3
2	1	1	2	3	1	2	0	2	1	2	1	2	1	1	2	0	0	2	2	1	3
1	0	3	1	3	2	3	0	2	3	3	0	2	0	2	1	2	2	3	1	2	2
2	2	2	3	1	1	1	0	3	1	2	2	0	3	1	3	0	2	1	1	0	0
1	1	1	0	3	1	2	3	2	3	0	3	3	1	0	1	1	0	3	2	0	3
0	3	0	3	2	3	3	1	1	3	1	0	2	0	1	2	1	0	1	0	2	1
1	0	1	2	1	3	2	3	1	2	1	2	0	0	1	0	2	3	3	2	2	0
3	1	1	1	0	2	2	1	3	1	0	1	2	0	2	3	3	0	3	2	0	0
1	1	3	0	2	0	3	0	0	1	1	0	0	2	0	3	3	3	2	1	3	2

CONFIGURATIONS ARE (LAURENT) SERIES



$$c = \sum_{i=-\infty}^{\infty} c_{i,j} X^i Y^j$$

OPERATIONS: SUM

$$c + d = \sum_{i=-\infty}^{\infty} (c_{i,j} + d_{i,j})\, X^i Y^j$$

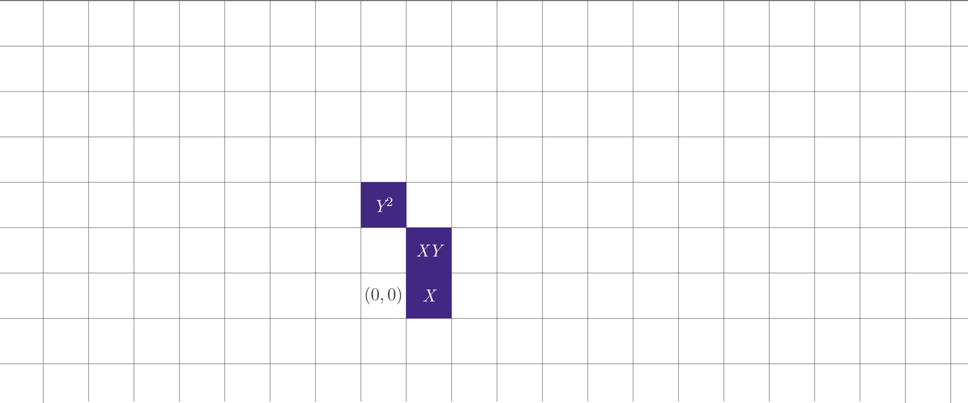
Formal sum $\leftrightarrow$ Sum of configurations

OPERATIONS: MULTIPLICATION

$$X^a Y^b c = \sum_{i=-\infty}^{\infty} c_{i,j} X^{i+a} Y^{j+b}$$

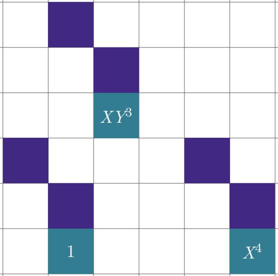
Multiplication by $X^a Y^b \leftrightarrow$ Translation of vector (a, b)

(PARENTHESIS: SEE YOUR POLYNOMIALS LIKE YOU NEVER DID)



$$X + XY + Y^2$$

(PARENTHESIS: SEE YOUR POLYNOMIALS LIKE YOU NEVER DID)



$$(X + XY + Y^2)(1 + X^4 + XY^3)$$

EXPRESSING PERIODICITY

$$(X^a Y^b - 1)c = 0$$

$$\Leftrightarrow$$

$$X^a Y^b c = c$$

$$\Leftrightarrow$$

c periodic of period (a, b)

ALGEBRA IS COMING

$$\text{Ann}(c) = \{p \mid pc = 0\}$$

$$c \text{ periodic } \Leftrightarrow \exists a, b \in \mathbb{Z}, (X^a Y^b - 1) \in \text{Ann}(c)$$

Ann(c) is a **polynomial ideal**:

- $0 \in I$
- $f, g \in I \Rightarrow f + g \in I$
- $f \in I$ and h any polynomial $\Rightarrow fh \in I$

SUPER NICE RESULTS

WARMING UP

c of low complexity

Theorem (Kari, Szabados, 2015)

$\exists p \neq 0 \in \mathrm{Ann}(c)$

WARMING UP

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Theorem (Kari, Szabados, 2015)

$\exists a_1, b_1, a_2, b_2 \ldots, a_r, b_r \in \mathbb{Z}$

$$\left(X^{a_1} Y^{b_1} - 1\right)\left(X^{a_2} Y^{b_2} - 1\right) \cdots \left(X^{a_r} Y^{b_r} - 1\right) \in \text{Ann}(c)$$

ASYMPTOTIC NIVAT

Let c be non-periodic.

Conjecture (Nivat)

$P_c(m, n) > mn$ for all $m, n \in \mathbb{N}$.

Theorem (Kari, Szabados, 2015)

$P_c(m, n) > mn$ holds for all but finitely many choices of $m, n \in \mathbb{N}$.

NIVAT FOR ALGEBRAIC SUBSHIFTS

Let c such that $\exists$ polynomial f with 0 or 1 line polynomial factor such that

$$fc = 0 \mod 2$$

Theorem (Kari, M., 2019)

If $\exists D, P_c(D) \leq |D|$.
Then c is **periodic**.

LOW COMPLEXITY SUBSHIFTS

F finite set of patterns.

$$X_F = \{c \mid \text{All patterns of } c \text{ are in } F\}$$

If F is defined by mn rectangular patterns of size $m \times n$.

(Therefore for any $c \in X_F$, $P_c(m, n) \leq mn$)

Conjecture (Nivat)

All $c \in X_F$ are periodic.

Theorem (Kari, M.)

There exists $c \in X_F$ periodic.

LOW COMPLEXITY SUBSHIFTS

Domino Problem (DP):

« Given F, is X_F empty ? »

In general, DP is **undecidable**.

In many cases, DP is undecidable.

If Nivat is true,
DP is **decidable** for X of low complexity.

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Theorem (Kari, M.)

~~If Nivat is true,~~
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WHAT NEXT ?

- → Nivat's conjecture
- → Use algebraic tools for similar things (Cellular Automata ?)

Thank you !

HSRM THEME