On self-assembly of aperiodic tilings

Galanov Ilya
Laboratoire d'Informatique de Paris Nord
Université Paris 13

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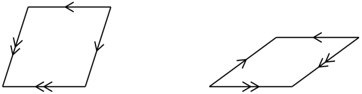
Tiling

- Tiling: covering of the plane by interior disjoint tiles;
- Aperiodic tiling: no invariance by translation;
- Vertex-atlas $\mathcal{A}(r)$: all the patterns of radius r;
- Local rules: a finite set of patterns that characterize the tiling.



Tiling

- Tiling: covering of the plane by interior disjoint tiles;
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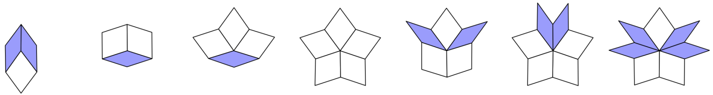
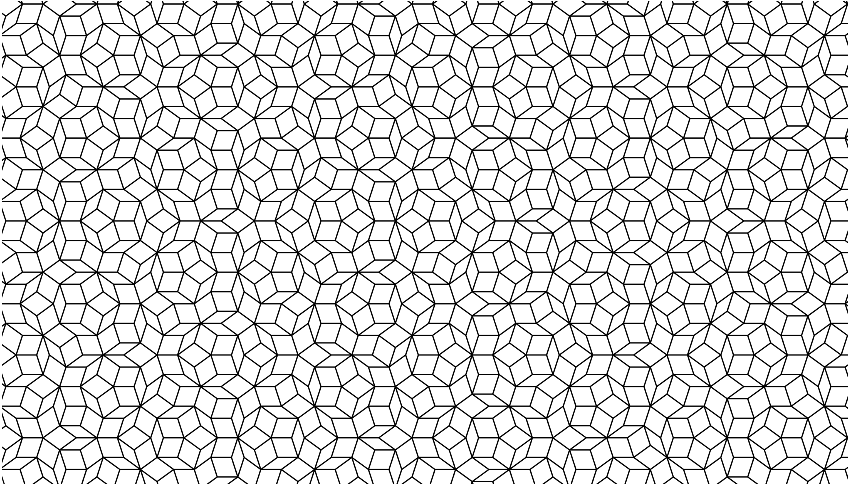


Penrose Tiling



Question

Is it possible to grow an aperiodic tiling locally?

The meaning of the locality constraint:

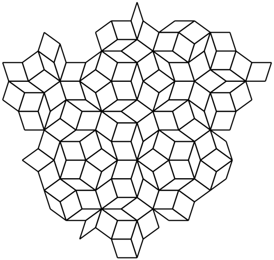
- units of the growing cluster must be added one by one;
- decisions are local, i.e. according to tiles within a fixed distance;
- no information must be stored between the steps.

Motivation

- Rapid development of aperiodic tilings started after discovery of quasicrystals in 1982 by Dan Shechtman (Nobel prize in 2011);
- The atomic arrangement of a quasicrystal breaks the periodicity (no translational symmetry);
- Due to specific local structure of these materials the growth process of such crystals is still poorly understood.

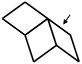
Main Obstacle: Deceptions

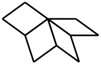
- Deceptions: patterns allowed by local rules which cannot be extended to a tiling of entire plane;



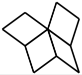
- Deceptions exist for all aperiodic tilings.

Some Tiles Are *Forced* by Vertex-Atlas:



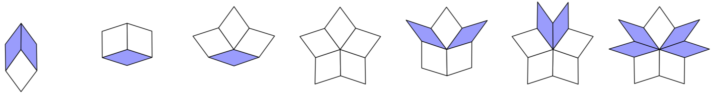


(a)



(b)

- (a) is allowed;
- (b) is forbidden.



Self-Assembly Algorithm (Socolar, 1991)

- Start with a finite pattern of Penrose tiling;
- Keep adding the forced tiles one by one until it is possible;
- When there are none left, add a thick tile to a *special* site;
- Repeat.

Theorem (Socolar, 1991)

The algorithm can build any Penrose tiling.

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- Start with a finite pattern of Penrose tiling;
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Theorem (Socolar, 1991)

The algorithm can build any Penrose tiling.

- However, this algorithm is *not* local.

Demonstration

Demonstration: Penrose.

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Cut-and-project

Definition (Planar tiling)

Let E be a d-dim. affine space in $\mathbb{R}^n$ called the slope.
Select the d-dim. faces with vertices in $\mathbb{Z}^n$ lying in $E + [0,1]^n$.
Project them onto E to get a so-called *planar* $n \to d$ tiling.

Example: Planar 2 $\to$ 1 Tiling

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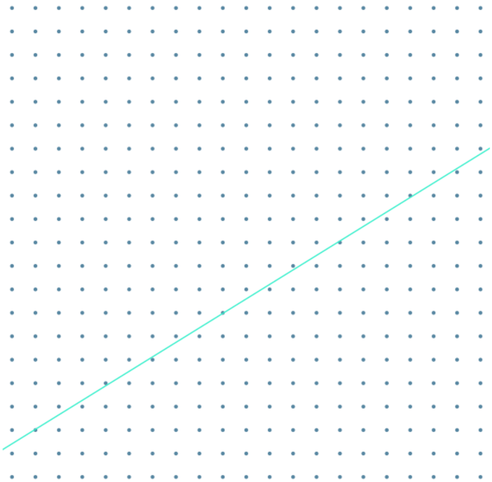
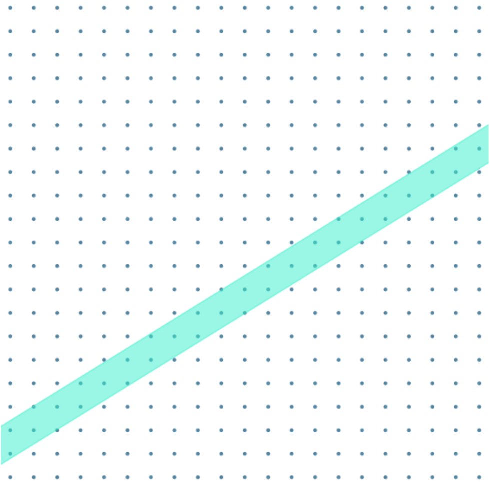
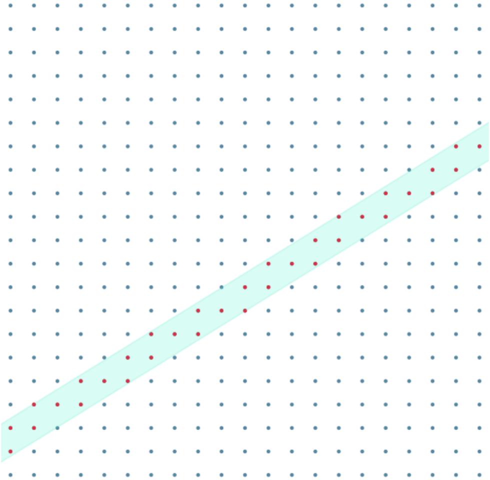


Example: Planar $2 \to 1$ Tiling



Example: Planar 2 $\to$ 1 Tiling



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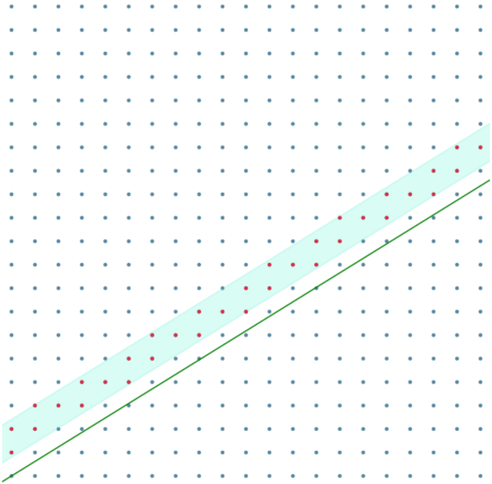
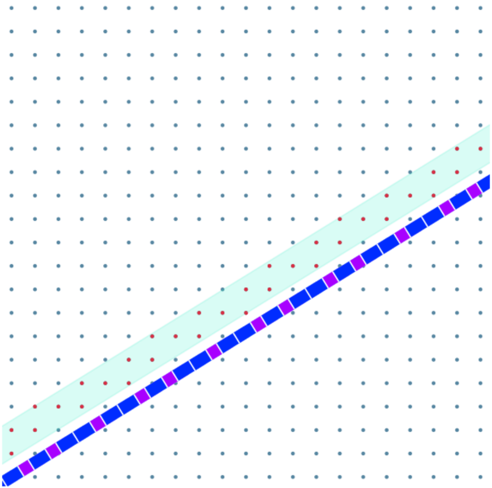


Example: Planar $2 \to 1$ Tiling



Example: Planar 2 → 1 Tiling

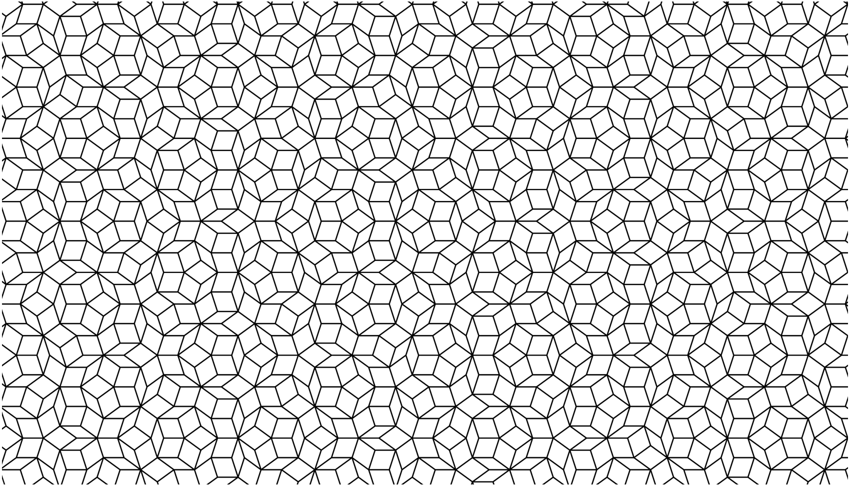
Cut-and-project

Theorem (De Bruijn, 1981)

Penrose tiling is planar $5 \to 2$ *with the slope generated by*

$$u = \begin{pmatrix} 1 \\ \cos(2\pi/5) \\ \cos(4\pi/5) \\ \cos(6\pi/5) \\ \cos(8\pi/5) \end{pmatrix} \quad v = \begin{pmatrix} 0 \\ \sin(2\pi/5) \\ \sin(4\pi/5) \\ \sin(6\pi/5) \\ \sin(8\pi/5) \end{pmatrix}$$

Example: Penrose Tiling

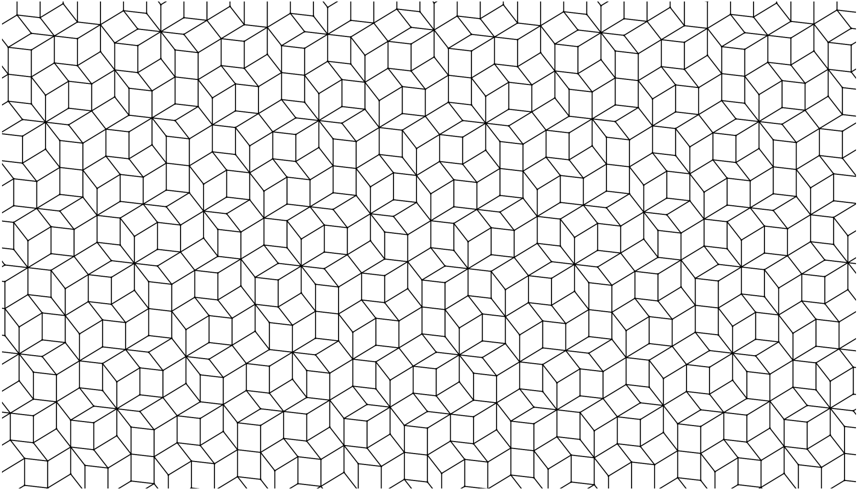


Example: Golden-Octagonal

Golden-Octagonal tiling is planar $4 \to 2$ with the slope generated by

$$u = \begin{pmatrix} -1 \\ 0 \\ \phi \\ \phi \end{pmatrix} \quad v = \begin{pmatrix} 0 \\ 1 \\ \phi \\ 1 \end{pmatrix}$$

Example: Golden-Octagonal

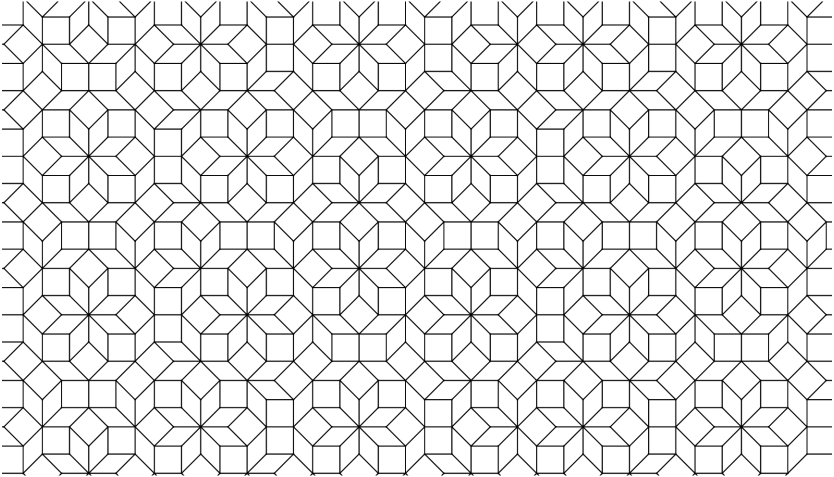


Example: Ammann-Beenker

Ammann-Beenker tiling is planar $4 \to 2$ with the slope generated by

$$u = \begin{pmatrix} 1 \\ \cos(\pi/4) \\ \cos(2\pi/4) \\ \cos(3\pi/4) \end{pmatrix} \quad v = \begin{pmatrix} 0 \\ \sin(\pi/4) \\ \sin(2\pi/4) \\ \sin(3\pi/4) \end{pmatrix}$$

Example: Ammann-Beenker



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Local Rules

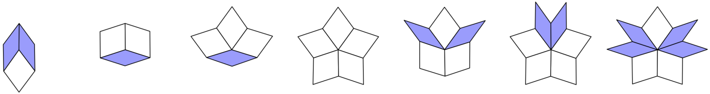
Definition (Local rules)

A d-plane $E \subset \mathbb{R}^n$ is said to admit *local rules* if there exists a vertex-atlas $\mathcal{A}(r)$ so that any $n \to d$ tiling with the same atlas is planar with the slope parallel to E.

Theorem (Bedaride, Fernique, 2017)

A planar $4 \to 2$ tiling admits local rules if and only if it is determined by its subperiods (easily checked on the generating vectors).

Examples



- Penrose tilling has local rules.
- Golden-Octagonal tiling has local rules.
- Ammann-Beenker tiling does not have local rules!

Proposition

In order to have a local self-assembly algorithm for a planar tiling it is necessary for the slope of the tiling to admit local rules.

Is it sufficient?

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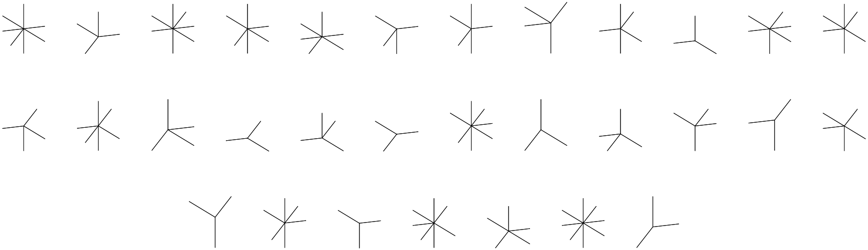
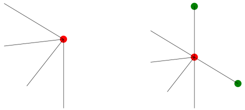
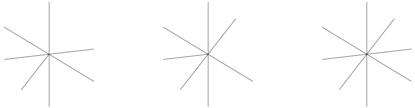
1-Atlas of Golden-Octagonal Tilings



Forced Vertex Example:



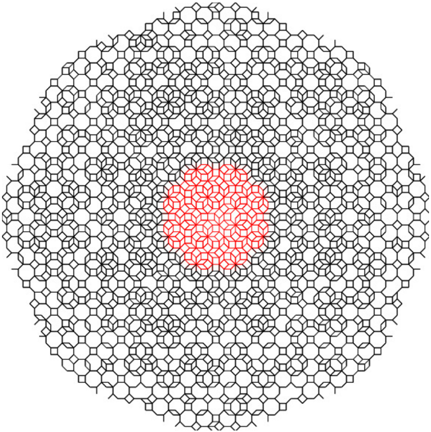


Local Algorithm

Given $r > 0$, a vertex-atlas $\mathcal{A}(r)$ and a finite pattern S:

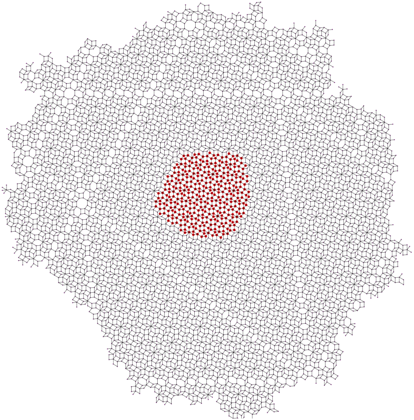
- pick at random a vertex v in S and let $P(v, r)$ be the subpattern of radius r and center v;
- consider the set F of all the elements in the vertex-atlas $\mathcal{A}(r)$ that *matches* with the subpattern $P(v, r)$;
- add to S all the vertices that appear in every pattern of F;
- Repeat.

Ammann-Beenker



Ammann-Beenker tiling does not have local rules and will not grow.

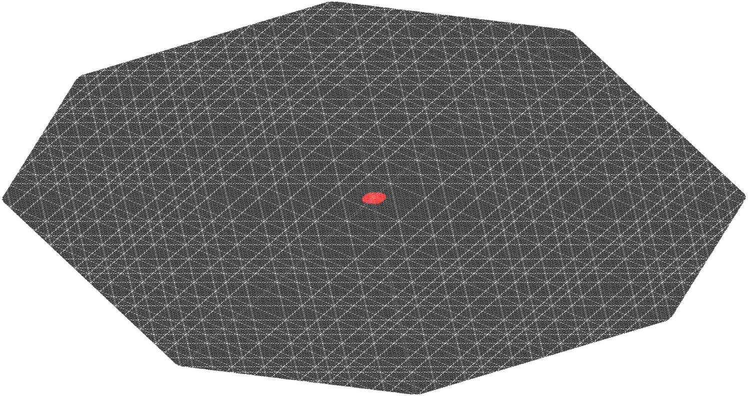
But Golden-Octagonal will



Demonstration

Demonstration: Golden-Octagonal.

Golden-Octagonal



Main Conjecture

Conjecture

Consider a planar tiling $\mathcal{T}$ with local rules. For any $\varepsilon > 0$ there exists an input data, such that the above algorithm generates proportion $(1-\varepsilon)$ of the tiles of a planar tiling with slope parallel to the slope of $\mathcal{T}$.

- The algorithm is local but it misses some tiles (*conway worms*).

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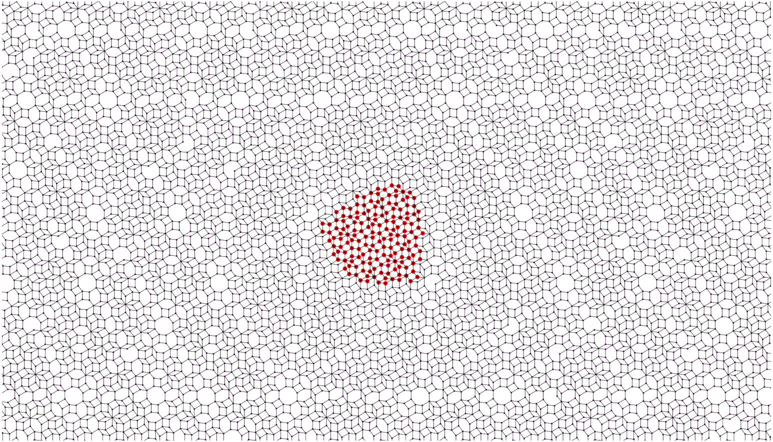
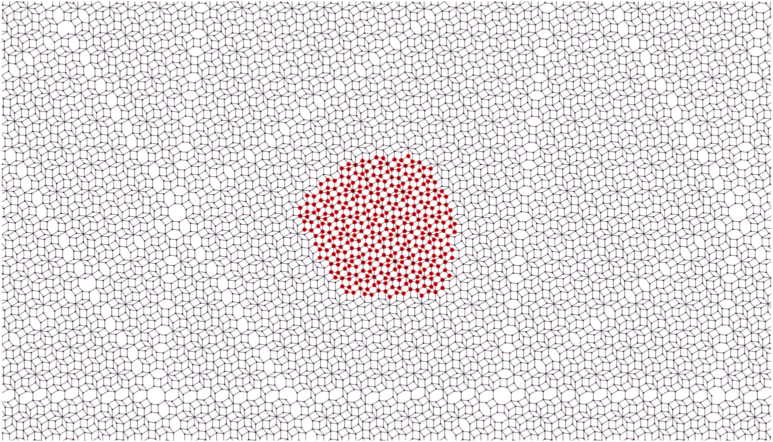
Smaller Seed

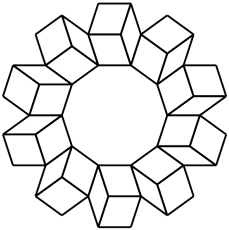


Bigger Seed



Defective Seeds

With a *correct* seed it is impossible to get all the tiles, but with a *defective* seed one can grow a tiling of the entire plane except for a finite region!



The *decapod*, an example of such a seed for Penrose tiling.

Demonstration

Demonstration.

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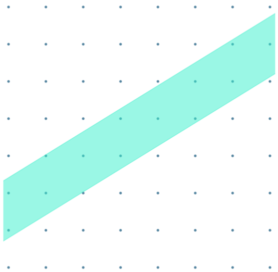
Window

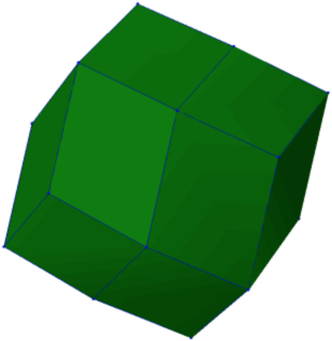
Definition (Window)

The *window* W of a planar tiling with a slope $E \subset \mathbb{R}^n$ is the orthogonal projection of $[0,1]^n$ onto $E^\perp$, where $E^\perp$ is a complementary space to E

$$W = \pi^\perp([0,1]^n).$$



Window



The window for Penrose tiling.

Region in the Window

Proposition

To every pattern of a tiling we can assign a region in the window:

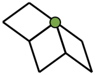
$$R(P) = \bigcap_{x:\pi(x)\in P} (W - \pi^{\perp}(x)).$$

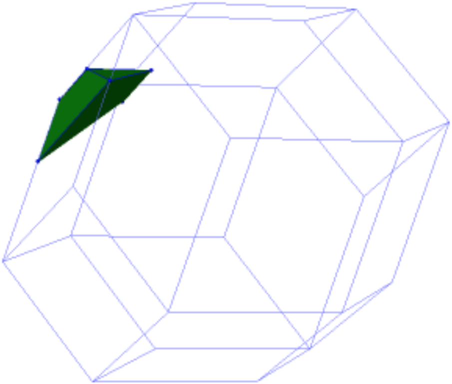
Corollary

In order for a pattern P to appear in a tiling it is necessary that

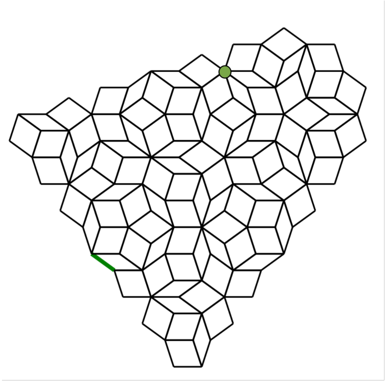
$$R(P) \neq \varnothing.$$

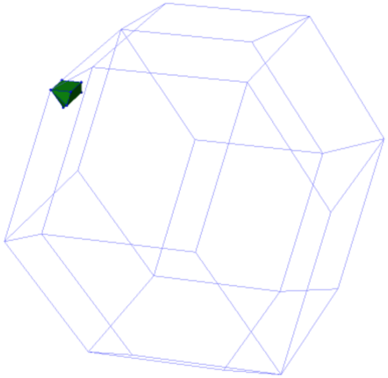
Examples





Examples



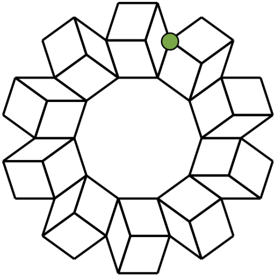


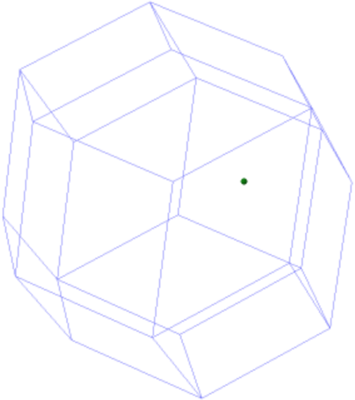
Examples

$$R(tiling) = \{point\}.$$

Examples

$$R(decapod) = \{point\}$$





Defective Seeds

Conjecture

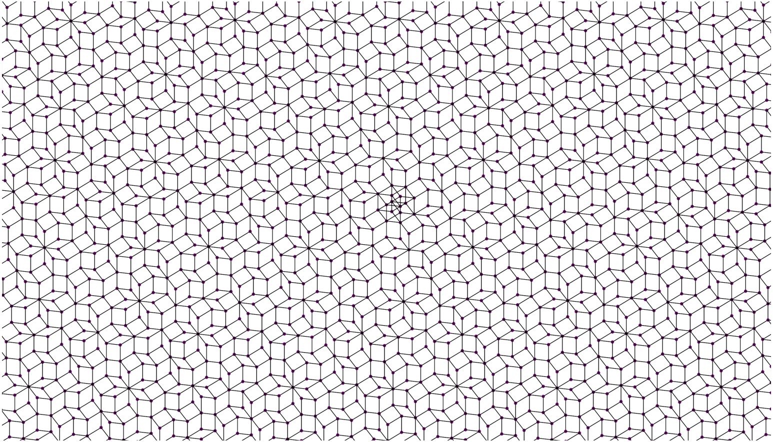
For all the planar tilings with local rules there is a set of defective seeds such that the growth with such seeds will produce a tiling of the entire plane except for a finite region.

Lemma

For any tiling with local rules $\mathcal{T}$ *and for any* $R > \lceil \max(||p_i||_1) \rceil$*, where* $\{p_i\}$ *is the set of subperiods of* $\mathcal{T}$*, there exist a seed* D *with following properties:*

- *every subpattern of* D *of radius* R *is correct (i.e. it is a subset of a tiling with the same slope)*
- $R(D) = \{point\}$

Defective Seed for Golden-Octagonal



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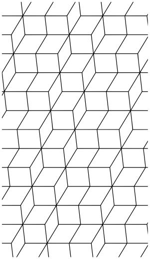
Shadows

Definition

The *ijk*-shadow of a $4 \to 2$ planar tiling is the orthogonal projection of its *lift* to the space generated by e_i, e_j and e_k.



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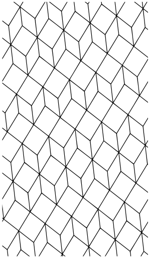

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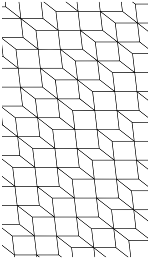

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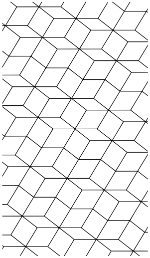

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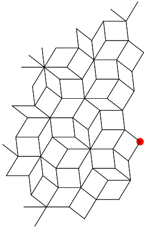
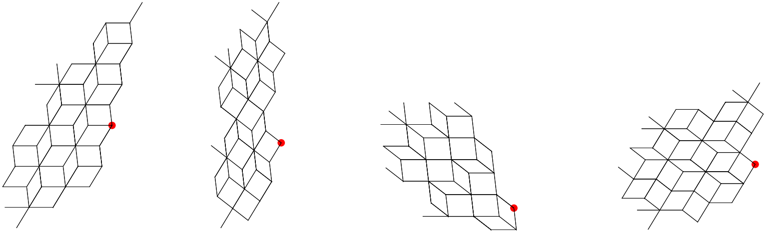
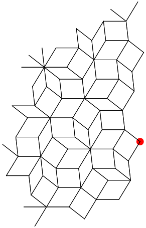
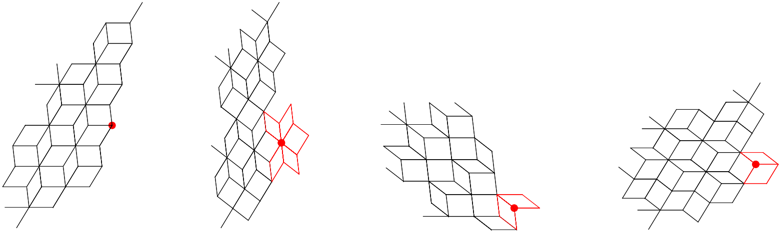
Shadows





Shadows Can Vote!





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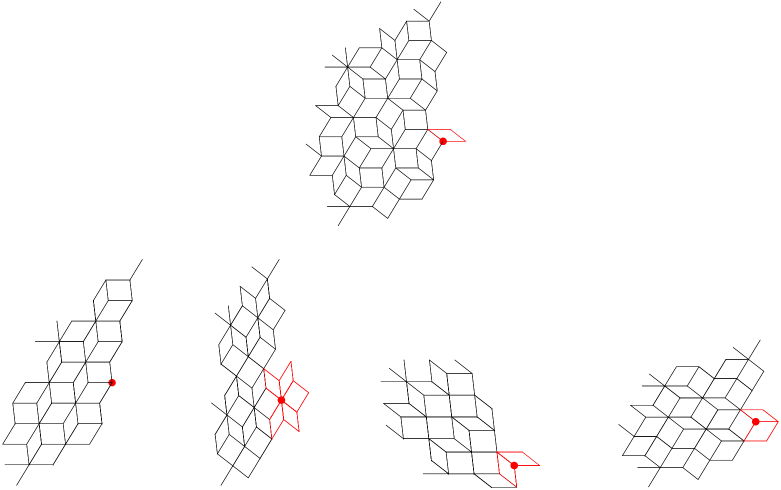
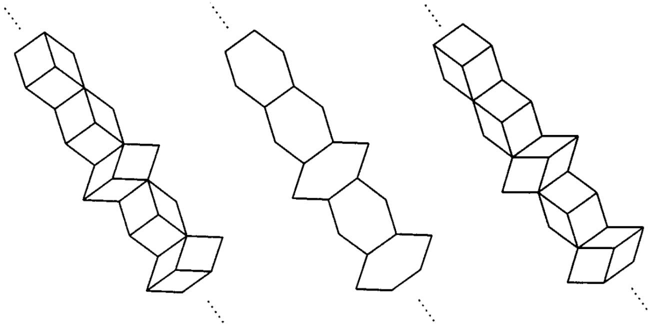


Thank you for your attention!

Conway worms



now to Construct The Defective Seeds?