Avoiding additive powers - Algorithmic proofs

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Powers

An infinite word $w = 31410421030342432334121432132 14 \cdots$

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w contains an abelian square : same blocks up to a permutation

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These notions can naturally be extended to higher powers, such as cubes ...

Avoidability

Objective

Construct infinite words over finite alphabets avoiding such patterns

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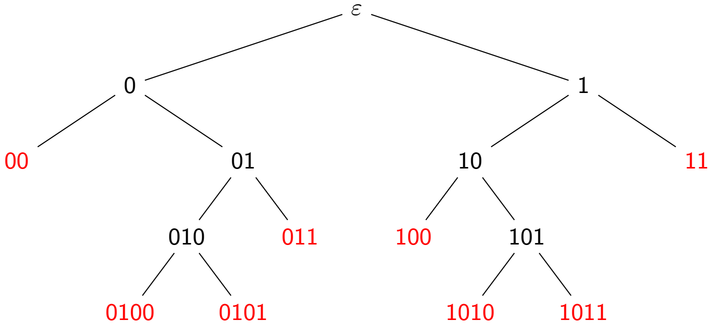

But it is possible over $\{0, 1, 2\}$ (A. Thue, 1912)

State of the art

Problem : Find an infinite word avoiding pure/abelian/additive powers

	Pure	**Abelian**	**Additive**	
cubes	2 letters 1906	3 letters 1979	4 letters 2014	3 letters 2015
squares	3 letters 1912	4 letters 1992	?	

1906 - A.Thue
Über unendliche Zeichenreihen,
Skrifter udgivne af Videnskabsselskabet i Christiania : Mathematisk-naturvidenskabelig Klasse, 1-22, 1906

1912 - A.Thue
Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihen,
Skrifter udgivne af Videnskabsselskabet i Christiania : Mathematisk-naturvidenskabelig Klasse, 1-67, 1912

1979 - F.M. Dekking
Strongly non-repetitive sequences and progression-free sets,
In *Journal of Combinatorial Theory, Series A*, Volume 27, 181-185, 1979

1992 - V. Keränen
Abelian squares are avoidable on 4 letters,
In *Automata, Languages and Programming*, July 13 – 17, 41-52, 1992

2014 - J. Cassaigne, J. D. Currie, L. Schaeffer, J. Shallit
Avoiding Three Consecutive Blocks of the Same Size and Same Sum,
In *Journal of the ACM* , Volume 61, issue no.2, April 2014

2015 - M. Rao
On some generalizations of abelian power avoidability,
In *Theoretical Computer Science*, (601) 39-46, 2015

State of the art

A 4-letter morphism avoiding additive cubes [J. Cassaigne *et* al. 2014]

$$\varphi_0 : 0 \mapsto 03, \quad 1 \mapsto 43, \quad 3 \mapsto 1, \quad 4 \mapsto 01$$

$$\varphi_0^\infty(0) = 03143011034343031011011031430343430343430314301\cdots$$

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It is possible to avoid additive **cubes** over a **4-letter alphabet** with a **morphism of size 2**

State of the art

A 3-letter word without additive cubes [M. Rao. 2015]

$$s_{\{0,1,5\}} : \quad \{0,1,3,4\}^\star \quad \longrightarrow \quad \{0,1,5\}^\star$$

$$\begin{array}{ccl}
0 & \longmapsto & \{005015100100115010115, 005015100100115100115\} \\
1 & \longmapsto & \{005015100100105055115, 050015100100105055115\} \\
3 & \longmapsto & \{005015101155155055115, 050015101155155055115\} \\
4 & \longmapsto & \{005015155055155055115, 050015155055155055115\}
\end{array}$$

$s_{\{0,1,5\}}(\varphi_0^\infty(0))$ avoids additive cubes

M. Rao (2015)
On some generalizations of abelian power avoidability,
In *Theoretical Computer Science*, (601) 39-46, 2015

Questions

	Pure	Abelian	Additive	
cubes	2 letters 1906	3 letters 1979	4 letters 2014	3 letters 2015
squares	3 letters 1912	4 letters 1992	?	

Do there exist :

- many 4-letter morphisms avoiding additive cubes ?
- morphic words without additive cubes but with non-abelian additive squares ?

$$\mathbf{w} = 6021062260101 \cdot \overbrace{06026}^{\Sigma=14} \cdot \overbrace{22622}^{\Sigma=14} \cdot 6021060101060101 \cdots$$

$$\mathbf{w_0} = 0314301103434 \cdot \underbrace{30310}_{\Sigma=7} \cdot \underbrace{11011}_{\Sigma=4} \cdot 0314303434303434 \cdots$$

Apply it to other morphisms

$$\varphi_0(0) = 03 \qquad \varphi_0(1) = 43$$
$$\varphi_0(3) = 1 \qquad \varphi_0(4) = 01$$

The corresponding incidence matrix :

$$\mathrm{Mat}(\varphi_0) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

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$$\text{Mat}(\varphi_0) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\varphi(6) = 60 \qquad \varphi(2) = 10$$
$$\varphi(0) = 2 \qquad \varphi(1) = 62$$

$$\mathbf{w} = \lim_{n\to\infty} \varphi^n(6) = 6021062260101060262262260210601010601 01\cdots$$

Apply previous proof to other morphisms

Lemma

If a morphism is similar to φ_0, then it fits the informatic proof developped by Cassaigne *et al.* in 2014.

Theorem (Jamet, L., Stoll)

Let w be a fixed point of a morphism similar to φ_0. The following propositions are decidable :

- w avoids additive cubes
- in w, all additive squares are abelian squares

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M. Rao, M. Rosenfeld (2018)
Avoiding Two Consecutive Blocks of Same Size and Same Sum over $\mathbb{Z}^2$,
In *Siam Journal on Discrete Mathematics*, Volume 32, Number 4, 2381-2397, 2018

Sketch of the proof

Why do we choose morphic words ?

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- $\varphi^1(6) = 60$

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- $\varphi^2(6) = 602$

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- $\varphi^3(6) = 60210$

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- $\varphi^3(6) = 60210$
- $\varphi^4(6) = 60210622$

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- $\varphi^2(6) = 602$
- $\varphi^3(6) = 60210$
- $\varphi^4(6) = 60210622$
- $\varphi^5(6) = 6021062260101010$

$\mathbf{w} = 60210622601010602622622602106010106010 1$

$\mathbf{w}[p]$	6	0	2	1	0	6	2	2	6	0	1	0	1	**0**
p	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$\mathbf{par}(p)$	0	0	1	2	2	3	3	4	5	5	6	6	7	7

Where the alphabet matters

Parikh vector

The Parikh vector $\psi(x)$ of a word x is :

$$\psi(x) = \begin{pmatrix} |x|_0 \\ |x|_1 \\ |x|_2 \\ |x|_6 \end{pmatrix}, \text{ example : } \psi(60210) = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Where the alphabet matters

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If b and c are two blocks with same length and same sum then the vector $\mathbf{v} = \psi(b) - \psi(c)$ belongs to the lattice

$$\mathfrak{L} := \{\mathbf{v} \in \mathbb{Z}^4 : (1,1,1,1) \cdot \mathbf{v} = 0 \text{ et } (0,1,2,6) \cdot \mathbf{v} = 0\}.$$

which depends on the chosen alphabet.

Linear algebra

Let p be a position, we define

$$\sigma(p) = \psi(\mathbf{w}[0,p)) = \begin{pmatrix} |\mathbf{w}[0,p)|_0 \\ |\mathbf{w}[0,p)|_1 \\ |\mathbf{w}[0,p)|_2 \\ |\mathbf{w}[0,p)|_6 \end{pmatrix}, \text{ example : } \sigma(9) = \psi(602106226) = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 3 \end{pmatrix}$$

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Lemma (J. Cassaigne *et* al., 2014)

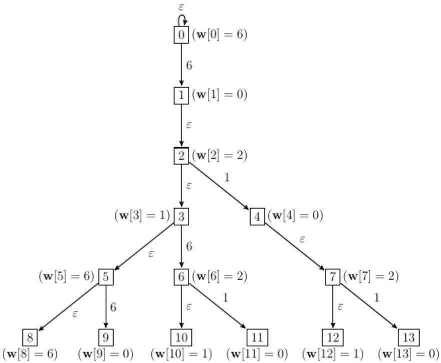
If q is a child of p and a the proper prefix linked to q (via the bijection), we get

$$\sigma(q) = M\sigma(p) + \psi(a)$$

Example :

$$\sigma(9) = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \times \underbrace{\begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}}_{=\sigma(5)=\psi(60210)} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{=\psi(6)}$$

Walk on a tree



Corollary (J. Cassaigne *et* al., 2014)

If $\{p_i\}_{i=0}^{\infty}$ is the ancestral sequence of a position p and denoting a_i the proper prefix used to link p_i to p_{i+1}, we get : $\sigma(p_0) = \sum_{i=0}^{\infty} M^i \psi(a_i)$.

So, how does it work ?

- Using parents and graphs, we get bounds for $\mathbf{v} = \psi(b) - \psi(c)$
- Using the lattice, we get other bounds for $\mathbf{v}$
- $\mathbf{v}$ lies in a ball of fixed radius
- This ball allow us to consider a finite subgraph
- We detect additive cubes by computing

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Using exactly the same arguments but considering two consecutive blocks rather than three, it is possible to detect additive squares

Statistics

Morphisms of size 2

- 32068 morphisms avoiding additive cubes, over 4-letters alphabets included in $\{0, 1, \ldots, 25\}$
- Less than 5% with a fixed point containing additive non-abelian squares
- 23 morphisms avoiding additive cubes over $\{0, 1, 5, 25\}$
- 2 morphisms avoiding additive cubes over $\{0, 2, 5, 11\}$
- At least one morphism for each alphabet included in $\{0, 1, \ldots, 25\}$ except $\{0, 1, 2, 3\}$ and $\{0, 1, 2, 4\}$.
- All morphisms avoiding additive cubes are similar to φ_0

Statistics

Proposition (M. Rao, M. Roseneld, 2018)

The following morphisms avoid additive cubes :

$$\varphi_1 : \begin{cases} 0 \mapsto 001 \\ 1 \mapsto 041 \\ 2 \mapsto 41 \\ 4 \mapsto 442 \end{cases}$$

Morphisms of size 3

- 132 morphisms over 4-letters alphabets $\{0, 1, 2, c\}$ $(4 \leq c \leq 9)$ avoiding additive cubes
- Not all similar to φ_0 : there is an other class
- 9 morphisms avoiding additive cubes over the alphabet $\{0, 1, 2, 4\}$, 5 are similar to φ_0

To be continued

It seems easy to find morphisms whose fixed points avoid additive cube for any 4-letters alphabet, except for $\{0, 1, 2, 3\}$.

Problem 3. *Are additive cubes avoidable over* $\{0, 1, 2, 3\}$*?*

(M. Rao, M. Rosenfeld, 2018)

To be continued

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Problem 3. *Are additive cubes avoidable over* $\{0,1,2,3\}$*?*

(M. Rao, M. Rosenfeld, 2018)

F.M. Dekking, 1979

The following morphism avoids additive-4-powers :

$$\varphi_0 : 0 \mapsto 011, \qquad 1 \mapsto 0001$$

Jamet, L.,Stoll, 2018

The following morphisms avoid additive-4-powers :

$$\varphi_1 : \begin{cases} 0 \mapsto 22 \\ 1 \mapsto 12 \\ 2 \mapsto 10 \end{cases} \quad \text{and} \quad \begin{cases} 0 \mapsto 32 \\ 1 \mapsto 2 \\ 2 \mapsto 01 \\ 3 \mapsto 31 \end{cases}$$

To be continued

Question 1 - M. Rao, M. Rosenfeld, 2018

Are additive cubes avoidable over $\{0, 1, 2, 3\}$?

Question 2

If φ is a morphism avoiding additive cubes on a 4 letter alphabet, do there exist intergers k and n such that :

$$\varphi^k \simeq \varphi_0^n$$

Question 3

Is there an infinite word over a finite alphabet without additive squares ?

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Thank you for your attention

Context and motivations

Uniformly k-repetitive semigroups

A semigroup S is **uniformly-k-repetitive** if for all morphisms $\varphi : \Sigma^+ \to S$ and for all words $w \in \Sigma^+$ long enough, there exists a factor $w_1 \cdots w_k$ in w such that

$$\varphi(w_1) = \cdots = \varphi(w_k) \text{ and } |w_1| = \cdots = |w_k|$$

Context and motivations

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Question of Pirillo and Varricchio (1994)

Is $\mathbb{N}^+$ uniformly k-repetitive for $k \geq 2$?

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Is $\mathbb{N}^+$ uniformly k-repetitive for $k \geq 2$?

Partial answer (J. Cassaigne *et al.*)

$\mathbb{N}^+$ is not uniformly 3-repetitive

J. Justin, 1972
Généralisation du théorème de Van der Waerden sur les semi-groupes répétitifs,
In *Journal of combinatorial theory (A)*, Volume 12, 357-367, 1972.

G. Pirillo, S. Varricchio, 1994
On uniformly repetitive semigroups,
In *Semigroup Forum*, Volume 49, 125-129, 1994.

Context and motivations

The Prouhet-Tarry-Escott problem :

Given $N \geq 1$ and $k \geq 0$ integers, does there exist a partition of $\{0, 1, \ldots, N-1\}$ into two disjoint subsets I and J such that :

$$\sum_{x \in I} x^i = \sum_{y \in J} y^i \quad \forall 0 \leq i \leq k$$

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$$\sum_{x \in I} x^i = \sum_{y \in J} y^i \quad \forall 0 \leq i \leq k$$

If N is a multiple of 2^{k+1}, then a solution is I to be the set of all n for which $t_n = 0$ and J those of all n for which $t_n = 1$ where $t_0 t_1 t_2 t_3 \cdots$ is the Thue-Morse sequence :

$$T = 01101001100101101001011001101001 \cdots$$

E. Prouhet, 1851
Mémoires sur quelques relations entre les puissances et les nombres,
In *C. R. Acad. Sci. Paris Sér. I* , 33, 225, 1851

Our approach

- Want to find other morphic words on other alphabets
- Compute to get some intuition
- In w_0 all additive squares are abelian squares : sufficient to show that w_0 avoids abelian cubes

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Our experimental results

- We find 5% of morphic words with additive and non-abelian squares
- All morphisms are similar to φ_0