State complexity of the multiples of the Thue-Morse set

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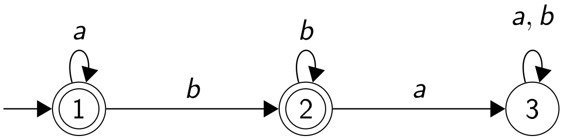
March 7th, 2019

Deterministic finite automaton (DFA) : $\mathscr{A} = (Q, q_0, F, \Sigma, \delta)$



Automata

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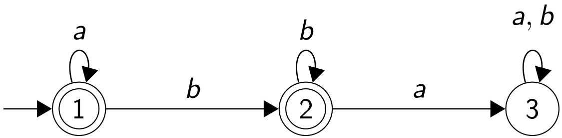

Language – Regular language

Minimal automaton

Theorem

For any regular language L, there exists a unique (up to isomorphism) minimal automaton accepting L.

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An automaton is minimal if and only if it is accessible and reduced.

One algorithm :

1. Eject non accessible states
2. Look for undistinguished states

State complexity

Definition

The *state complexity* of a regular language is the number of states of its minimal automaton.

What do we want to do?

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Let $b \in \mathbb{N}_{\geq 2}$. A subset X of $\mathbb{N}$ is *b-recognizable* if $\text{rep}_b(X)$ is regular.

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Theorem

Let $b \in \mathbb{N}_{\geq 2}$ and $m \in \mathbb{N}$. If $X \subseteq \mathbb{N}$ is b-recognizable, so is mX.

Theorem [Alexeev, 2004]

The state complexity of the language $0^* \operatorname{rep}_b(m\,\mathbb{N})$ is

$$\min_{N \geq 0} \left\{ \frac{m}{\gcd(m, b^N)} + \sum_{n=0}^{N-1} \frac{b^n}{\gcd(b^n, m)} \right\}$$

Thue-Morse

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Thue-Morse

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Thue-Morse

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Thue-Morse

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Thue-Morse

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$$0110100110010110\ldots$$

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The Thue-Morse set is the set

$$\mathscr{T} = \{n \in \mathbb{N} : |\operatorname{rep}_2(n)|_1 \in 2\,\mathbb{N}\}\,.$$

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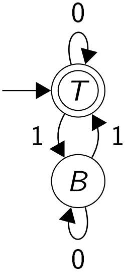

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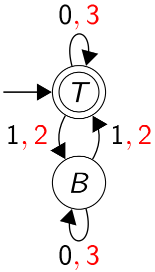

Definition

Let $p, q \in \mathbb{N}_{\geq 2}$. We say that p and q are *multiplicatively independant* if

$$p^a = q^b \Rightarrow a = b = 0.$$

They are said *multiplicativement dependant* otherwise.

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Theorem [Cobham, 1969]

- Let b, b' two multiplicatively independant bases. A subset of $\mathbb{N}$ is both b-recognizable and b'-recognizable iff it is a finite union of arithmetic progressions.
- Let b, b' two multiplicatively dependant bases. A subset of $\mathbb{N}$ is b-recognizable iff it is b'-recognizable.

The result

$$\mathscr{T} = \{n \in \mathbb{N} : |\operatorname{rep}_2(n)|_1 \in 2\,\mathbb{N}\}$$

Theorem

Let $m \in \mathbb{N}$ and $p \in \mathbb{N}_{\geq 1}$.
Then the state complexity of the language $0^* \operatorname{rep}_{2^p}(m\mathscr{T})$ is

$$2k + \left\lceil \frac{z}{p} \right\rceil$$

if $m = k2^z$ with k odd.

The method

Automaton	Language accepted
$\mathscr{A}_{\mathscr{T},2^p}$	$(0,0)^* \operatorname{rep}_{2^p}(\mathscr{T} \times \mathbb{N})$

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$\mathscr{A}_{\mathscr{T},2^p} \times \mathscr{A}_{m,2^p}$	$(0,0)^* \operatorname{rep}_{2^p}(\{(t, mt) : t \in \mathscr{T}\})$
$\pi\left(\mathscr{A}_{\mathscr{T},2^p} \times \mathscr{A}_{m,2^p}\right)$	$0^* \operatorname{rep}_{2^p}(m\mathscr{T})$

The automaton $\mathscr{A}_{\mathscr{T},2^p}$

$$(0,0)^* \{\mathrm{rep}_{2^p}(t,n) : t \in \mathscr{T}, n \in \mathbb{N}\}$$

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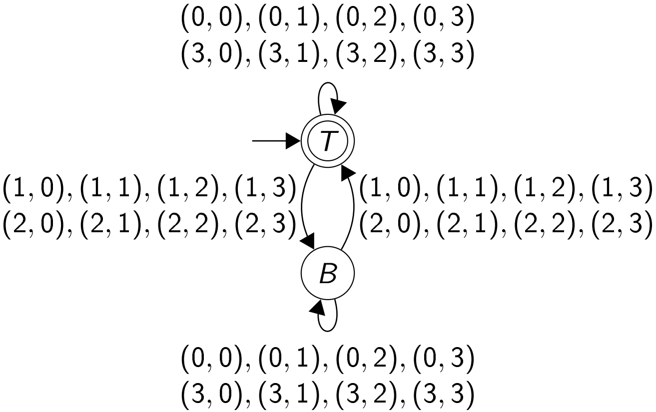
States	T, B
Initial state	T
Final states	T
Alphabet	$\{0,\ldots,2^p-1\}^2$
Transitions	$\delta_{\mathscr{T},2^p}(X,(a,b)) = \begin{cases} X & \text{if } a \in \mathscr{T} \\ \overline{X} & \text{else.} \end{cases}$

The automaton $\mathscr{A}_{\mathscr{T},4}$



The automaton $\mathscr{A}_{m,b}$

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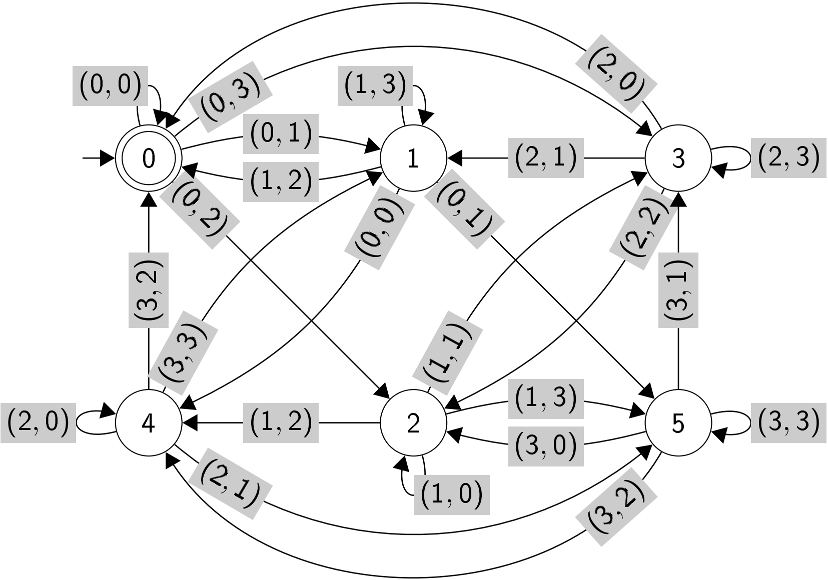
For all $i, j \in \{0, \ldots, m-1\}$, for all $u, v \in \{0, \ldots, b-1\}^*$,

$$\delta_{m,b}(i,(u,v)) = j \Leftrightarrow b^{|(u,v)|} i + \text{val}_b(v) = m\,\text{val}_b(u) + j.$$

The automaton $\mathscr{A}_{6,4}$



The product automaton $\mathscr{A}_{m,2^p} \times \mathscr{A}_{\mathscr{T},2^p}$

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States	$(0, T), \dots, (m-1, T), (0, B), \dots, (m-1, B)$
Initial state	$(0, T)$
Final states	$(0, T)$
Alphabet	$\{0, \dots, 2^p - 1\}^2$
Transitions	$\delta_{\mathscr{T},2^p}((i, X), (u, v)) = (j, Y)$ $\Leftrightarrow 2^{p\lvert(u,v)\rvert} i + \mathrm{val}_{2^p}(v) = m\,\mathrm{val}_{2^p}(u) + j$ and $Y = \begin{cases} X \text{ if } \mathrm{val}_{2^p}(u) \in \mathscr{T} \\ \overline{X} \text{ else.} \end{cases}$

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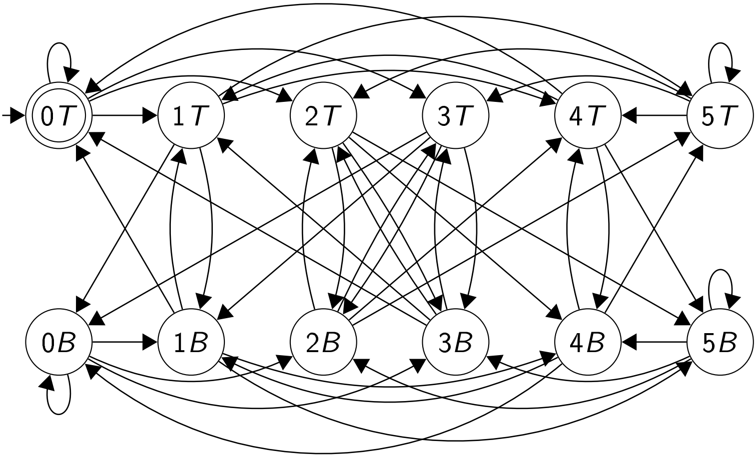
Remark

If i, X, v are fixed, there exist unique j, Y, u such that we have a transition labeled by (u,v) from (i,X) to (j,Y).

The automaton $\mathscr{A}_{6,4} \times \mathscr{A}_{\mathscr{T},4}$



The projected automaton $\pi\left(\mathscr{A}_{m,2^p} \times \mathscr{A}_{\mathscr{T},2^p}\right)$

$0^* \operatorname{rep}_{2^p}\left(m\mathscr{T}\right) = 0^* \left\{\operatorname{rep}_{2^p}(mt) : t \in \mathscr{T}\right\}$

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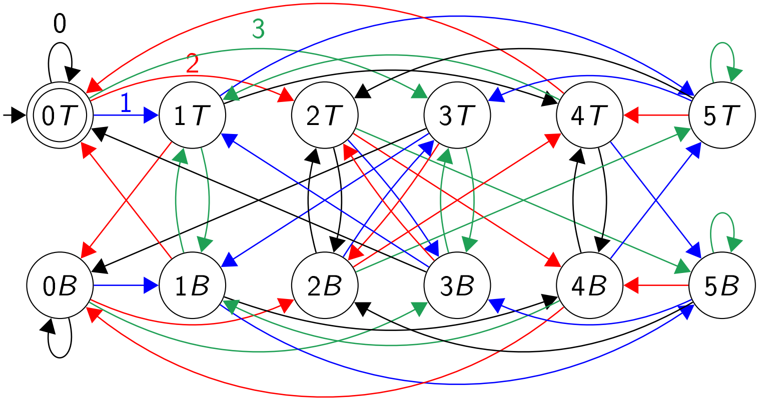

Proposition

The automaton $\pi\left(\mathscr{A}_{m,2^p} \times \mathscr{A}_{\mathscr{T},2^p}\right)$ is

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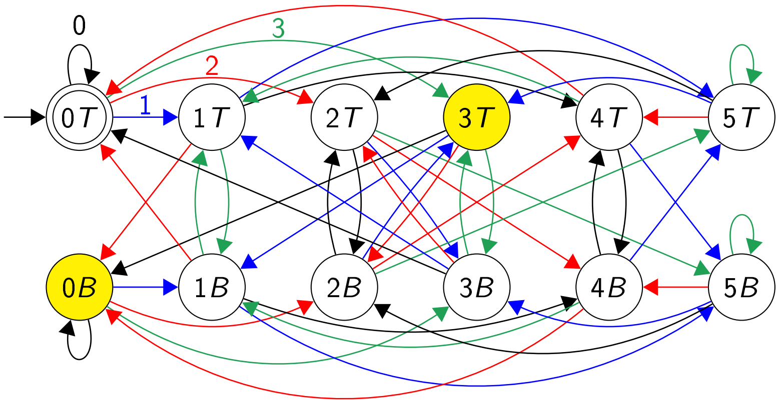
Proposition

In the automaton $\pi\left(\mathscr{A}_{m,2^p} \times \mathscr{A}_{\mathscr{T},2^p}\right)$, the states (i, T) and (i, B) are disjoined for all $i \in \{0, \dots, m-1\}$.

The automaton $\pi(\mathscr{A}_{6,4} \times \mathscr{A}_{\mathscr{T},4})$



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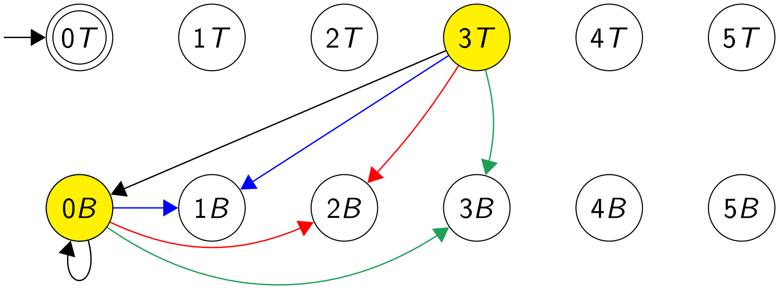
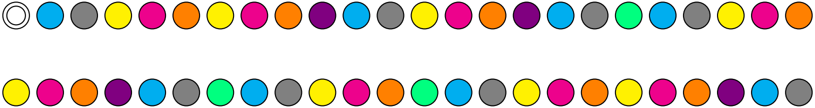


The automaton $\pi(\mathscr{A}_{24,4} \times \mathscr{A}_{\mathscr{T},4})$



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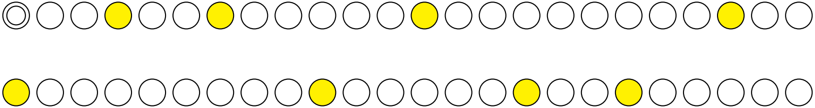

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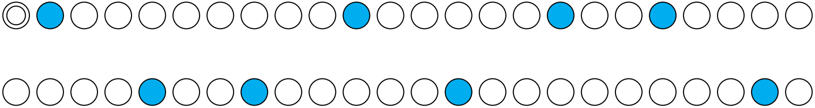

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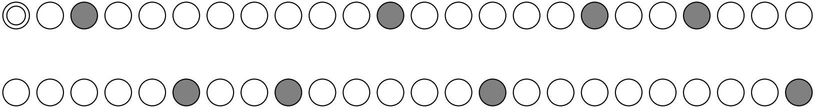

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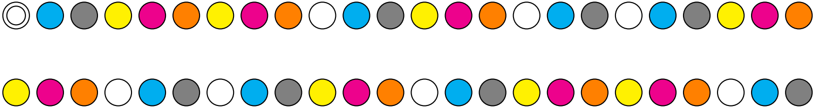

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For $m \in \mathbb{N}, p \in \mathbb{N}_{\geq 1}$, we write $m = k2^z$ with z odd.

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Definition

For all $j \in \{1, \dots, k-1\}$, we set

$$[(j, T)] := \{(j + k\ell, T_\ell) : 0 \leq \ell \leq 2^z - 1\}$$
$$[(j, B)] := \left\{\left(j + k\ell, \overline{T_\ell}\right) : 0 \leq \ell \leq 2^z - 1\right\}.$$

We also set

$$[(0, T)] := \{(0, T)\} \text{ and } [(0, B)] := \left\{\left(k\ell, \overline{T_\ell} : 0 \leq \ell \leq 2^z - 1\right)\right\}.$$

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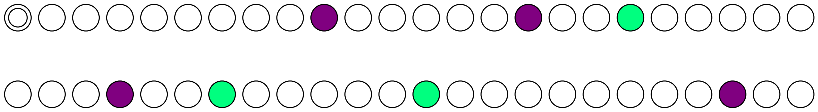

Definition

For all $\alpha \in \{0, \ldots, z-1\}$, we set

$$C_\alpha := \left\{ \left(k2^{z-\alpha-1} + k2^{z-\alpha}\ell, \overline{T_\ell} \right) : 0 \leq \ell \leq 2^\alpha - 1 \right\}.$$

For all $\beta \in \left\{0, \ldots, \left\lceil \frac{z}{p} \right\rceil - 2 \right\}$, we set

$$\Gamma_\beta := \bigcup_{\alpha \in \{\beta p, \ldots, (\beta+1)p-1\}} C_\alpha.$$

We also set

$$\Gamma_{\left\lceil \frac{z}{p} \right\rceil - 1} := \bigcup_{\alpha \in \left\{ \left(\left\lceil \frac{z}{p} \right\rceil - 1 \right) p, \ldots, z-1 \right\}} C_\alpha.$$

We can build a new automaton

We can build a new automaton which is

- accessible

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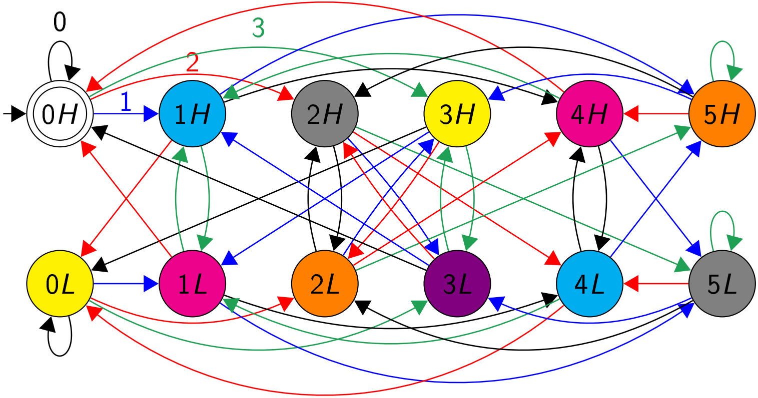
- accessible
- reduced.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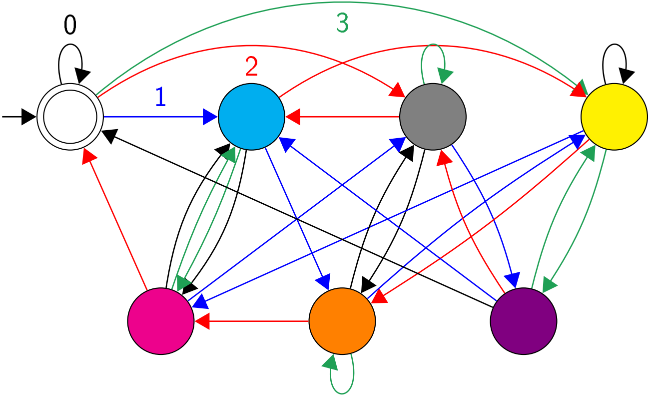
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$2 \times 3 + \left\lceil \frac{1}{2} \right\rceil = 7$

To go further

What about the language

$$0^* \operatorname{rep}_{2^p}(m\mathscr{T} + r)$$

where $r \in \{0, \ldots, m-1\}$?