Schauder's estimate for nonlocal kinetic equation and its applications

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A joint work with Mingyan Wu and Xicheng Zhang

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Contents

In elementary physical theory, velocity is the differential of displacement with respect to time, which is

$$\begin{cases} X_t = x + \int_0^t V_s ds \\ V_t = v. \end{cases} \tag{1.1}$$

We add an $\alpha -$ *stable* noise L_t^α on velocity,

$$\begin{cases} X_t = x + \int_0^t V_s ds \\ V_t = v + L_t^\alpha. \end{cases}$$

Define

$$u(t, x, v) = \phi(X_t(x, v), V_t(x, v)) + \int_0^t f(s, X_{t-s}(x, v), V_{t-s}(x, v)).$$

Non-local case

Then u satisfies the following non-local kinetic equation:

$$\begin{cases} \partial_t u(t,x,v) = \Delta_v^{\frac{\alpha}{2}} u(t,x,v) + v \cdot \nabla_x u(t,x,v) + f(t,x,v), \\ u(0,x,v) = \phi(x,v). \end{cases} \tag{1.2}$$

Our work is to consider the more general PDE

$$\begin{aligned} \partial_t u(t,x,v) =& \mathscr{L}^{\alpha}_{\kappa;v} u(t,x,v) + b^1(t,x,v) \cdot \nabla_x u(t,x,v) \\ &+ b^2(t,x,v) \cdot \nabla_v u(t,x,v) + f(t,x,v), \end{aligned}$$

where u, f and κ are functions from $\mathbb{R}^{2d}$ to $\mathbb{R}$, b^1, b^2 are vector fields from $\mathbb{R}^{2d}$ to $\mathbb{R}^d$, and $\mathscr{L}^{\alpha}_{\kappa;v}$ is a symmetric non-local operator given by

$$\mathscr{L}^{\alpha}_{\kappa;v} u(x,v) = p.v. \int_{\mathbb{R}^d} \frac{u(x,v+w) - u(x,v)}{|w|^{d+\alpha}} \kappa(x,v,w) dw,$$

where $\alpha \in (0,2)$.

Local case

When $\alpha = 2$, $\alpha-$stable noise became the BM noise formally,

$$\begin{cases} X_t = x + \int_0^t V_s ds \\ V_t = v + B_t^\alpha. \end{cases}$$

By the same way, we can get a general local PDE,

$$\begin{aligned} \partial_t u(t,x,v) = a^{i,j}(t,x,v)\partial_i\partial_j u(t,x,v) + b^1(t,x,v)\cdot\nabla_x u(t,x,v) \\ + b^2(t,x,v)\cdot\nabla_v u(t,x,v) + f(t,x,v). \end{aligned} \tag{1.3}$$

For this equation, P. E. Chaudru de Raynal, I. Honoré, S. Menozzi has obtained the estimate for more difficult chain case. In particularly,

Proposition 1

If u is a solution of equation (1.3), *then for* $\gamma \in (0,1)$,

$$\|u\|_{C_x^{\frac{2+\gamma}{3}}} + \|u\|_{C_v^{2+\gamma}} \lesssim \|f\|_{C_x^{\frac{\gamma}{3}}} + \|f\|_{C_v^\gamma}.$$

$$\|f\|_{C_x^\gamma} := \|f\|_{L^\infty} + \sup_{v\in\mathbb{R}^d, x\neq y\in\mathbb{R}^d} \frac{f(x,v)-f(y,v)}{|x-y|^\gamma}.$$

Heat Kernel Estimate

Let us go back to the SDE (1.1) We can solve this SDE easily,

$$(X_t, V_t) = (x + tv + \int_0^t L_s^\alpha ds, v + L_t^\alpha).$$

Denote by $p_t(x, v)$ the distribution density of (X_t, V_t), which is also the fundamental solution of PDE (1.2). Basing on the scaling proposition, X. Zhang and Z. Chen proved the following heat kernel estimate

Proposition 2 (Zhang-Chen, 2016)

Assume $\beta + \gamma < \alpha$ and $n, m \in \mathbb{N}_0$, we have

$$\int_{\mathbb{R}^d} |x|^\beta |v|^\gamma |\nabla_x^n \nabla_v^m p_s(x, v)| dxdv \lesssim s^{-\frac{(\alpha+1)(n-\beta)+(m-\gamma)}{\alpha}}. \tag{1.4}$$

References

[1] P. E. Chaudru de Raynal, I. Honoré, and S. Menozzi, *Sharp Schauder Estimates for some Degenerate Kolmogorov Equations*, ArXiv e-prints (2018), available at arXiv:1810.12227.

[2] Z.-Q. Chen and X. Zhang, *L^p-maximal hypoelliptic regularity of nonlocal kinetic Fokker-Planck operators*, J. Math. Pures Appl. (9) **116** (2018), 52–87.

[3] Z.-Q. Chen, X. Zhang, and G. Zhao, *Well-posedness of supercritical SDE driven by Lévy processes with irregular drifts*, ArXiv e-prints (2017), available at arXiv:1709.04632.

[4] F.-Y. Wang and X. Zhang, *Degenerate SDE with Hölder-Dini drift and non-Lipschitz noise coefficient*, SIAM J. Math. Anal. **48** (2016), no. 3, 2189–2226.

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Littlewood-Paley Decomposition

Let $m = (m_1, \cdots, m_n) \in \mathbb{N}^n$ with $m_1 + \cdots + m_n = d$ and $a = (a_1, \cdots, a_n) \in [1, \infty)^n$ be fixed. We introduce the following quasi-distance in $\mathbb{R}^d$ by

$$|x - y|_a := \sum_{i=1}^{n} |x_i - y_i|^{1/a_i}, \ x_i, y_i \in \mathbb{R}^{m_i}.$$

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For $x = (x_1, \cdots, x_n)$, $t > 0$ and $s \in \mathbb{R}$, we denote

$$t^{sa}x := (t^{sa_1}x_1, \cdots, t^{sa_n}x_n) \in \mathbb{R}^d, \ \ B_t^a := \Big\{x \in \mathbb{R}^d : |x|_a \leqslant t\Big\}.$$

Clearly we have

$$|t^a x|_a = t|x|_a, \ \ t > 0.$$

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Clearly we have

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Let ϕ_0^a be a radial C^∞-function on $\mathbb{R}^d$ with

$$\phi_0^a(\xi) = 1 \ \text{ for } \ \xi \in B_1^a \ \text{ and } \ \phi_0^a(\xi) = 0 \ \text{ for } \ \xi \notin B_2^a.$$

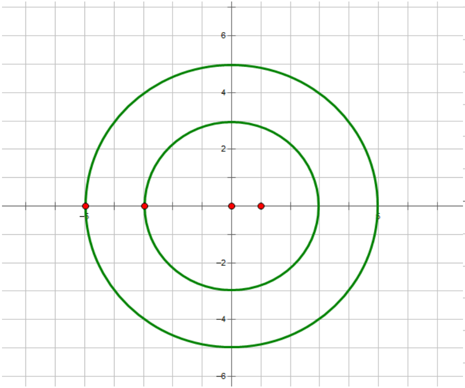
For $\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^d$ and $j \in \mathbb{N}$, define

$$\phi_j^a(\xi) := \phi_0^a(2^{-aj}\xi) - \phi_0^a(2^{-a(j-1)}\xi).$$

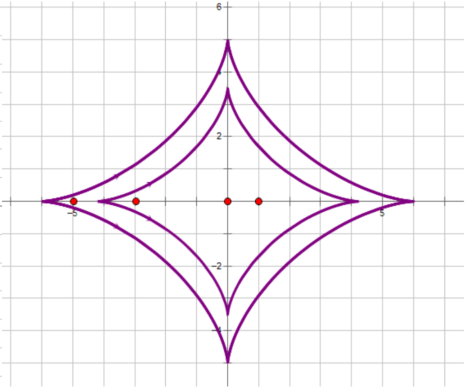
It is easy to see that for $j \in \mathbb{N}$, $\phi_j^a(\xi) = \phi_1^a(2^{-a(j-1)}\xi) \geqslant 0$ and

$$\mathrm{supp}\phi_j^a \subset B_{2^{j+1}}^a \setminus B_{2^{j-1}}^a, \quad \sum_{j=0}^{k} \phi_j^a(\xi) = \phi_0^a(2^{-ak}\xi) \to 1, \quad k \to \infty.$$

When $n = 2$, $m_1 = m_2 = 1$ and $\alpha \in (0, 2)$, the shape of the support set of ϕ_j ($j \neq 0$) is as follow.(The value on the graph is not the accurate value.)



$a = (1, 1)$



$a = (1 + \alpha, 1)$

Anisotropic Besov Space

Definition 3 (Anisotropic Besov and Hölder-Zygmund spaces)

For given $j \in \mathbb{N}_0$, the block operator $\mathcal{R}_j^a$ is defined on $\mathscr{S}'$ by

$$\mathcal{R}_j^a f(x) := (\phi_j^a \hat{f})^{\check{}}(x) = \check{\phi}_j^a * f(x) = 2^{a \cdot m(j-1)} \int_{\mathbb{R}^d} \check{\phi}_1^a (2^{a(j-1)} y) f(x-y) \mathrm{d}y, \qquad (3.1)$$

where $a \cdot m = a_1 m_1 + \cdots + a_n m_n$. For any $s \in \mathbb{R}$, the anisotropic Besov space $\mathbf{B}_{a,\infty}^s$ is defined by

$$\mathbf{B}_{a,\infty}^s := \left\{ f \in \mathscr{S}'(\mathbb{R}^d) : \|f\|_{\mathbf{B}_{a,\infty}^s} := \sup_{j \geqslant 0} \left(2^{sj} \|\mathcal{R}_j^a f\|_\infty \right) < \infty \right\},$$

and the anisotropic Hölder-Zygmund space $\mathbf{C}_a^s$ is defined by

$$\mathbf{C}_a^s := \left\{ f \in \mathbb{R}^d \to \mathbb{R} : \|f\|_{\mathbf{C}_a^s} := \|f\|_\infty + [f]_{\mathbf{C}_a^s} < \infty \right\},$$

where

$$[f]_{\mathbf{C}_a^s} := \sup_h \|\delta_h^{[s]+1} f\|_\infty / |h|_a^s.$$

In particular, if $a = (1, \cdots, 1)$, we shall drop the index a in $\mathbf{B}_{a,\infty}^s$, $\mathcal{R}_j^a$ and $\mathbf{C}_a^s$.

Next we shall only consider the following case of anisotropic Besov spaces:

$$n = 2, \ m_1 = m_2 = d, \ a = (1+\alpha, 1), \ \ \textit{where } \alpha \in (0, 2).$$

The Propositions of Besov Space

Theorem 4 (H. Triebel)

For any $s > 0$*, it holds that*

$$\|f\|_{\mathbf{B}^s_{a,\infty}} \asymp \|f\|_{\mathbf{C}^s_a} \asymp \|f\|_{\mathbf{C}^{s/a_1}_{x_1}} + \cdots + \|f\|_{\mathbf{C}^{s/a_n}_{x_n}},$$

where $\|f\|_{\mathbf{C}^{s/a_i}_{x_i}} := \sup_{j \neq i} \sup_{x_j \in \mathbb{R}^{m_j}} \|f(x_1, \cdots, x_{i-1}, \cdot, x_{i+1}, \cdots, x_n)\|_{\mathbf{C}^{s/a_i}}$.

The proof of theorem can be found in
Serguei Dachkovski, Anisotropic function spaces and related semi-linear hypoelliptic equations, Math. Nachr. 248/249 (2003), 40-61.

For $j \in \mathbb{N}_0$, by definition it is easy to see that

$$\mathcal{R}^a_j = \mathcal{R}^a_j \widetilde{\mathcal{R}}^a_j, \quad \text{where} \quad \widetilde{\mathcal{R}}^a_j := \mathcal{R}^a_{j-1} + \mathcal{R}^a_j + \mathcal{R}^a_{j+1} \text{ with } \mathcal{R}^a_{-1} \equiv 0, \tag{3.2}$$

and $\mathcal{R}^a_j$ is symmetric in the sense that

$$\langle \mathcal{R}^a_j f, g \rangle = \langle f, \mathcal{R}^a_j g \rangle.$$

The cut-off low frequency operator S_k is defined by

$$S_k f := \sum_{j=0}^{k-1} \mathcal{R}^a_j f = 2^{a \cdot mk} \int_{\mathbb{R}^d} \check{\phi}^a_0(2^{ka}(x-y)) f(y) \mathrm{d}y \to f. \tag{3.3}$$

Classical and weak solution

Definition 5 (Classical solution)

Let $\lambda > 0$. We call a bounded continuous function u defined on $\mathbb{R}_+ \times \mathbb{R}^{2d}$ a classical solution if for some $\varepsilon \in (0,1)$,

$$u \in C([0,\infty); \mathbf{C}_v^{\alpha+\varepsilon} \cap \mathbf{C}^{1+\varepsilon}),$$

and for all $t \geqslant 0$ and $x, v \in \mathbb{R}^d$,

$$u(t,x,v) = \int_0^t \Big(\mathscr{L}^\alpha_{\kappa;v} u + b \cdot \nabla u - \lambda u + f \Big)(s,x,v) \mathrm{d}s.$$

Definition 6 (Weak solution)

Let $\lambda > 0$. We call a bounded continuous function u defined on $\mathbb{R}_+ \times \mathbb{R}^{2d}$ and in $L^\infty(\mathbf{C}_v^{\alpha+\varepsilon})$ a weak solution if $b \cdot \nabla u$ is a distribution in some sense and for any $\phi \in C_0^\infty$,

$$\int_0^t \langle u(s), \partial_t \phi(s) \rangle \, \mathrm{d}s = \langle u(0), \phi(0) \rangle + \int_0^t \Big(\langle \mathscr{L}^\alpha_{\kappa;v} u(s) + f(s), \phi(s) \rangle + \langle b \cdot \nabla u(s), \phi(s) \rangle \Big) \mathrm{d}s.$$

Condition $(\mathrm{H}_\gamma^{\beta,\vartheta})$

- For some $c_0 \geqslant 1$ and $\vartheta \in (0,\alpha), \beta \in (0,1)$, it holds that for all $t \geqslant 0$ and $x, v, w \in \mathbb{R}^d$,

$$c_0^{-1} \leqslant \kappa(t,x,v,w) \leqslant c_0, [\kappa(t,\cdot,\cdot,w)]_{\mathbf{C}_v^\beta} + [\![b^{(1)}(t,\cdot)]\!]_{\mathbf{C}_a^{1+\vartheta}} + [b^{(2)}(t,\cdot)]_{\mathbf{C}_v^\beta} \leqslant c_0,$$

where $[\![\,\cdot\,]\!]_{\mathbf{C}_a^{1+\vartheta}}$ is defined by

$$[\![f]\!]_{\mathbf{C}_a^{1+\vartheta}} := [f]_{\mathbf{C}_x^{\frac{1+\vartheta}{1+\alpha}}} + \|\nabla_v f\|_{\mathbf{C}_v^\vartheta}.$$

- For some $\gamma \in [\beta, 1+\alpha)$,

$$[\kappa(t,\cdot,\cdot,w)]_{\mathbf{C}_x^{\gamma/(1+\alpha)}} + [b^{(1)}(t,\cdot)]_{\mathbf{C}_x^{\gamma/(1+\alpha)}} + [b^{(2)}(t,\cdot)]_{\mathbf{C}_x^{\gamma/(1+\alpha)}} \leqslant c_0.$$

- For some closed and convex subset $\mathscr{E} \subset GL_d(\mathbb{R})$, where $GL_d(\mathbb{R})$ is the set of all invertible $d \times d$-matrices,

$$\nabla_v b^{(1)}(t,x,v) \in \mathscr{E}.$$

Theorem 7 (Hao-Wu-Zhang, 2019)

Let $\alpha \in (1,2)$ *and* $\beta \in (0,1), \vartheta \in (0,\alpha)$, $\gamma \in [\beta, 1+\alpha)$. *Under* $(\mathbf{H}_\gamma^{\beta,\vartheta})$, *for any f* $\in \mathbb{L}^\infty_{loc}(\mathbf{C}_x^{\gamma/(1+\alpha)} \cap \mathbf{C}_v^\beta)$, *there is a unique weak solution u satisfying following Schauder estimate,*

$$\|u\|_{\mathbb{L}^\infty_T(\mathbf{C}_x^{(\gamma+\alpha)/(1+\alpha)} \cap \mathbf{C}_v^{\alpha+\beta})} \leqslant C\|f\|_{\mathbb{L}^\infty_T(\mathbf{C}_x^{\gamma/(1+\alpha)} \cap \mathbf{C}_v^\beta)}, \tag{4.1}$$

where the constant C is dependent on α, β, γ *and* ϑ.

Moreover, if $\gamma > 1$ *this solution becomes a classical solution and still has Schauder estimate* (4.1).

Pathwise uniqueness of SDE

($\mathbf{H}^b_{\beta,\gamma}$) For some $c_0 \geqslant 1$, $\gamma \in (0, 1+\alpha)$ and $\beta \in (0,1)$, $\vartheta \in (0, \alpha - 1)$ it holds that for all $t > 0$,

$$[b(t,\cdot)]_{C_x^{\gamma/(1+\alpha)}} + \|\nabla b^{(1)}(t,\cdot)\|_{C_v^\vartheta} + \|b^{(2)}(t,\cdot)\|_{C_v^\beta} \leqslant c_0.$$

($\mathbf{H}^\sigma_1$) σ is Lipschitz continuous and for some $c_0 \geqslant 1$ and all $(t,z) \in \mathbb{R}_+ \times \mathbb{R}^{2d}$,

$$c_0^{-1}|\xi| \leqslant |\sigma(t,z)\xi| \leqslant c_0|\xi|, \quad \xi \in \mathbb{R}^d.$$

Theorem 5.1

Let $\alpha \in (1,2)$, $\gamma \in (1 + \frac{\alpha}{2}, 1+\alpha)$ and $\beta \in (1 - \frac{\alpha}{2}, 1)$. Under **($\mathbf{H}^b_{\beta,\gamma}$)** *and* **($\mathbf{H}^\sigma_1$)***, for each $s \geqslant 0$ and $z \in \mathbb{R}^{2d}$, there exists a unique strong solution $(Z_{s,t})_{t\geqslant s}$ of the following SDE:*

$$\mathrm{d}Z_{s,t} = b_t(Z_{s,t})\mathrm{d}t + (0, \sigma(t, Z_{s,t})\mathrm{d}L_t), \quad Z_{s,s} = z \in \mathbb{R}^{2d}, \ t \geqslant s \geqslant 0. \quad (5.1)$$

Proof of Schauder estimate for heat equation

We consider the heat equation

$$\partial_t u(t,x) = a^{ij}(t,x)\partial_i\partial_j u(t,x) + f(t,x),$$

where $\{a^{ij}\}_{i,j}$ is a symmetric positive matrix, with following condition.

($\mathbf{H}_a^\beta$) For some $c_0 > 0$ and $\beta \in (0,1)$, it holds that for any $t \geqslant 0$ and $x, y, \xi \in \mathbb{R}^d$,

$$c_0^{-1}|\xi|^2 \leqslant a^{ij}(t,x)\xi_i\xi_j \leqslant c_0|\xi|^2, \quad |a(t,x) - a(t,y)| \leqslant c_0|x-y|^\beta.$$

Theorem 6.1

Let $\beta \in (0,1)$. Under ($\mathbf{H}_a^\beta$), there is a constant $C = C(c_0, \beta, d) > 0$ such that for any $T > 0$, and $u \in \mathbb{L}_T^\infty(\mathbf{B}_\infty^{2+\beta})$ with $\partial_t u \in \mathbb{L}_T^\infty(\mathbf{B}_\infty^\beta)$ solving heat equation,

$$\|u\|_{\mathbb{L}_T^\infty(\mathbf{B}_\infty^{2+\beta})} \leqslant C\Big(\|f\|_{\mathbb{L}_T^\infty(\mathbf{B}_\infty^\beta)} + \|u\|_{\mathbb{L}_T^\infty}\Big).$$

Crucial Lemma

In order to prove Theorem 6.1, we need following crucial lemma.

Lemma 6.2

For any $\beta \geqslant 0$, *there is a constant* $C = C(c_0, \beta, d) > 0$ *such that for all* $t \geqslant 0$ *and* $j \in \mathbb{N}$,

$$\int_0^t \int_{\mathbb{R}^d} |x|^\beta |\mathcal{R}_j p_s(x)| dxds \leqslant C 2^{-2j-\beta j}.$$

The main ideal of the proof is the scaling proposition of heat kernel, and notice that

$$0 \notin supp \phi_j, \quad when\ j \neq 0.$$

Besides, we also use the (1.4).

Proof (Translate)

Fix $x_0 \in \mathbb{R}^d$ and define

$$u_{x_0}(t,x) := u(t,x+x_0), \quad \tilde{a}_{x_0}(t,x) := a(t,x+x_0) - a(t,x_0).$$

It is easy to see that

$$\partial_t u_{x_0} = a^{ij}(t,x_0)\partial_j\partial_j u_{x_0} + \tilde{a}_{x_0}\partial_j\partial_j u_{x_0} + f_{x_0}, \quad u_{x_0}(0) = 0.$$

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$$\partial_t u_{x_0} = a^{ij}(t, x_0)\partial_j\partial_j u_{x_0} + \tilde{a}_{x_0}\partial_j\partial_j u_{x_0} + f_{x_0}, \quad u_{x_0}(0) = 0.$$

Let $p^{x_0}_{s,t}$ be the fundamental solution of heat equation with constant coefficient $a(t, x_0)$. For a space-time function f, define

$$P^{x_0}_{s,t} f(s,x) := \int_{\mathbb{R}^d} p^{x_0}_{s,t}(x - y) f(s,y) \mathrm{d}y.$$

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$$P^{x_0}_{s,t} f(s,x) := \int_{\mathbb{R}^d} p^{x_0}_{s,t}(x-y) f(s,y)\mathrm{d}y.$$

By Duhamel's formula we have

$$\begin{aligned} u_{x_0}(t,x) &= \int_0^t P^{x_0}_{s,t}\mathrm{tr}(\tilde{a}_{x_0} \cdot \nabla^2 u_{x_0})(s,x)\mathrm{d}s + \int_0^t P^{x_0}_{s,t} f_{x_0}(s,x)\mathrm{d}s \\ &=: I_1(t,x) + I_2(t,x). \end{aligned}$$

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$$
\begin{aligned}
|\mathcal{R}_j I_1(t,0)| &\leqslant \int_0^t |\mathcal{R}_j P_{s,t}\mathrm{tr}(\tilde{a}\cdot\nabla^2 u)(s,0)|\mathrm{d}s \\
&\lesssim \int_0^t \left(\int_{\mathbb{R}^d} |x|^\beta |\mathcal{R}_j p_{s,t}(x)|\mathrm{d}x\right)\mathrm{d}s\|\nabla^2 u\|_{\mathbb{L}^\infty_T} \lesssim 2^{-2j-\beta j}\|\nabla^2 u\|_{\mathbb{L}^\infty_T} \\
&\leqslant \varepsilon 2^{-2j-\beta j}\|u\|_{\mathbb{L}^\infty_T(\mathbf{B}^{2+\beta}_\infty)} + C_\varepsilon 2^{-2j-\beta j}\|u\|_{\mathbb{L}^\infty_T},
\end{aligned}
$$

where $\varepsilon > 0$ and the last inequality is due to the interpolation and Young's inequalities.

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$$\begin{aligned}|\mathcal{R}_j I_1(t,0)| &\leqslant \int_0^t |\mathcal{R}_j P_{s,t}\mathrm{tr}(\tilde{a}\cdot\nabla^2 u)(s,0)|\mathrm{d}s\\ &\lesssim \int_0^t \left(\int_{\mathbb{R}^d}|x|^\beta|\mathcal{R}_j p_{s,t}(x)|\mathrm{d}x\right)\mathrm{d}s\|\nabla^2 u\|_{\mathbb{L}^\infty_T} \lesssim 2^{-2j-\beta j}\|\nabla^2 u\|_{\mathbb{L}^\infty_T}\\ &\leqslant \varepsilon 2^{-2j-\beta j}\|u\|_{\mathbb{L}^\infty_T(\mathbf{B}^{2+\beta}_\infty)} + C_\varepsilon 2^{-2j-\beta j}\|u\|_{\mathbb{L}^\infty_T},\end{aligned}$$

where $\varepsilon > 0$ and the last inequality is due to the interpolation and Young's inequalities. For $I_2(t,x)$, by (3.2) and Lemma 6.2 again, we have

$$\begin{aligned}|\mathcal{R}_j I_2(t,0)| &\leqslant \int_0^t |\mathcal{R}_j P_{s,t} f(s,0)|\mathrm{d}s \leqslant \int_0^t |\widetilde{\mathcal{R}}_j P^{x_0}_{s,t}\mathcal{R}_j f(s,0)|\mathrm{d}s\\ &\leqslant \int_0^t \left(\int_{\mathbb{R}^d}|\widetilde{\mathcal{R}}_j p_{s,t}(x)|\mathrm{d}x\right)\mathrm{d}s\|\mathcal{R}_j f\|_{\mathbb{L}^\infty_T} \lesssim 2^{-2j-\beta j}\|f\|_{\mathbb{L}^\infty_T(\mathbf{B}^\beta_\infty)}.\end{aligned}$$

Proof (Interpolation)

Combining the above estimates, we obtain that for any $\varepsilon \in (0,1)$ and $j \in \mathbb{N}$,

$$2^{j(2+\beta)}|\mathcal{R}_j u(t,x_0)| = 2^{j(2+\beta)}|\mathcal{R}_j u_{x_0}(t,0)| \lesssim \varepsilon \|u\|_{\mathbb{L}^\infty_T(\mathbf{B}^{2+\beta}_\infty)} + C_\varepsilon \|u\|_{\mathbb{L}^\infty_T} + \|f\|_{\mathbb{L}^\infty_T(\mathbf{B}^\beta_\infty)}.$$

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Moreover, for $j = 0$, it is easy to see that

$$|\mathcal{R}_j u(t,x_0)| \leqslant \|u\|_{\mathbb{L}^\infty_T}.$$

Proof (Interpolation)

Combining the above estimates, we obtain that for any $\varepsilon \in (0,1)$ and $j \in \mathbb{N}$,

$$2^{j(2+\beta)}|\mathcal{R}_j u(t,x_0)| = 2^{j(2+\beta)}|\mathcal{R}_j u_{x_0}(t,0)| \lesssim \varepsilon \|u\|_{\mathbb{L}^\infty_T(\mathbf{B}^{2+\beta}_\infty)} + C_\varepsilon \|u\|_{\mathbb{L}^\infty_T} + \|f\|_{\mathbb{L}^\infty_T(\mathbf{B}^\beta_\infty)}.$$

Moreover, for $j=0$, it is easy to see that

$$|\mathcal{R}_j u(t,x_0)| \leqslant \|u\|_{\mathbb{L}^\infty_T}.$$

Thus by the definition of Besov space, we arrive at

$$\|u\|_{\mathbb{L}^\infty_T(\mathbf{B}^{2+\beta}_\infty)} = \sup_{t\in[0,T]} \sup_{j\in\mathbb{N}_0} 2^{j(2+\beta)}\|\mathcal{R}_j u(t,\cdot)\|_\infty \leqslant \varepsilon \|u\|_{\mathbb{L}^\infty_T(\mathbf{B}^{2+\beta}_\infty)} + C_\varepsilon \|u\|_{\mathbb{L}^\infty_T} + C\|f\|_{\mathbb{L}^\infty_T(\mathbf{B}^\beta_\infty)},$$

which gives the desired estimate by choosing $\varepsilon = 1/2$.

Thank you very much!

Merci!