Hörmander's theorem for semilinear SPDEs

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Joint work with Martin Hairer

Framework

We consider a general semilinear Stratonovich SPDE

$$du_t = Lu_t\,dt + N(u_t)dt + \sum_{i=1}^{d} F_i(u_t) \circ dB_t^i, \quad u_0 \in \mathcal{H}\ , \quad \text{(SPDE)}$$

1. L is a selfadjoint unbounded operator on a Hilbert space $\mathcal{H}$, that induces an analytic semigroup S_t and a family of interpolation spaces $\mathcal{H}_\alpha$ for $\alpha \in \mathbb{R}$.
2. B is d-dimensional Brownian motion.
3. $N \in C^\infty(\mathcal{H}, \mathcal{H}_{-\delta})$ for $0 \le \delta < 1$, and $F_i \in C^\infty(\mathcal{H}, \mathcal{H})$.

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$\mathcal{H} = \mathcal{H}_0$ and for all $\alpha \geq \beta$ and every $\sigma \in [0,1]$, one has

$$\|S_t u\|_{\mathcal{H}_\alpha} \lesssim t^{\beta-\alpha} \|u\|_{\mathcal{H}_\beta}\;, \qquad \|S_t u - u\|_{\mathcal{H}_{\beta-\sigma}} \lesssim t^{\sigma} \|u\|_{\mathcal{H}_\beta}\;.$$

1\. tha… on spa…

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Examples

Navier-Stokes equation in 2 dimensions:

$$dw_t = \Delta w_t dt + \nabla \cdot (w_t \mathcal{K}(w_t))dt + \sum_{j=1}^{d} \big(f_j + g_j w_t\big) \circ dB_t^j.$$

$w_0 \in L^2(\mathbb{T}^2, \mathbb{R})$, $\mathcal{K} = \operatorname{curl} \Delta^{-1},\ f_j, g_j \in C^\infty(\mathbb{T}^2, \mathbb{R})$

Reaction-Diffusion equations:

$$du_t(x) = \Delta u_t dt + p_0(u_t(x))dt + \sum_{j=1}^{d} f_j(x) p_j(u_t(x)) \circ dB_t^j.$$

$u_0 \in H^1(\mathbb{T}, \mathbb{R})$, p_i are polynomials, $f_j \in C^\infty(\mathbb{T}, \mathbb{R})$.

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Malliavin calculus

Let $\Omega = C([0,T], \mathbb{R}^d)$ be a Wiener space for the Brownian motion. Denote by $\mathcal{CM} = H^1([0,T], \mathbb{R}^d)$ a Cameron Martin space of the Wiener measure.

Malliavin derivative $\mathcal{D}u_t : \Omega \to \mathcal{CM} \otimes \mathcal{H}$ of the solution to (SPDE) is defined for every $h \in \mathcal{CM}$ as a limit in probability:

$$\langle \mathcal{D}u_t(\omega), h \rangle_{\mathcal{CM}} = \lim_{\varepsilon \to 0} \varepsilon^{-1} \big(u_t(\omega + \varepsilon h) - u_t(\omega) \big) \, .$$

Malliavin matrix $\mathscr{M}_t : \mathcal{H} \to \mathcal{H}$ of the solution is a positive semi-definite operator defined by

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Smooth Densities

Theorem: Let X be Malliavin smooth $\mathbb{R}^n$ valued random variable, such that $\mathcal{D}^k X$ has moments of all orders for all $k \in \mathbb{N}$. Moreover assume that Malliavin Matrix $\mathscr{M} = \langle \mathcal{D}X, \mathcal{D}X \rangle_{\mathcal{CM}}$ is almost surely invertible and has inverse moments of all orders. Then the law of X has a smooth density with respect to Lebesgue measure on $\mathbb{R}^n$.

Hörmander's theorem

Theorem (G. & Hairer 2018): Let u be a solution to (SPDE) and assume that $\mathbb{E}\|u_t\|^p < \infty$ for all $t > 0$ and $p \geq 1$. Assume in addition that L, N, F_i satisfy Hörmander's condition. Then for every finite dimensional projection $\Pi : \mathcal{H} \to \mathcal{H}$ and for every $t > 0$

$$\mathbb{P}\left(\inf_{\|\Pi\varphi\|=1} \frac{\langle \mathscr{M}_t \varphi, \varphi \rangle}{\|\varphi\|^2} \leq \varepsilon \right) \leq C_p \varepsilon^p.$$

Corollary: Every finite dimensional projection of the solution Πu_t has a smooth density with respect to Lebesgue measure on $\Pi(\mathcal{H})$.

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Duhamel's formula

The Jacobian $J_{t,s}$ of the solution satisfies the following equation

$$\begin{cases} dJ_{t,s} = LJ_{t,s}dt + DN(u_t)J_{t,s}dt + \sum_{i=1}^{d} DF_i(u_t)J_{t,s} \circ dB_t^i, \\ J_{s,s} = \mathrm{id}\,. \end{cases}$$

DN, DF_i are Fréchet derivatives.

One can derive an SPDE for the Malliavin derivative and use Duhamel's formula to obtain for $\varphi \in \mathcal{H}$:

$$\langle \mathscr{M}_t \varphi, \varphi \rangle = \sum_{i=1}^{d} \int_0^t \langle J_{t,s} F_i(u_s), \varphi \rangle^2 ds.$$

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Sketch of the proof

Aim: for all $p \geq 1$, $\varepsilon > 0$: $\mathbb{P}\Big(\inf_{\|\Pi\varphi\|=1} \frac{\langle \mathscr{M}_t\, \varphi, \varphi \rangle}{\|\varphi\|^2} \leq \varepsilon \Big) \leq C_p \varepsilon^p$.

Denote $F_0 = L + N$ and formally differentiate $\langle J_{t,s} F_i(u_s), \varphi \rangle$:

$$d\langle J_{t,s} F_i(u_s), \varphi \rangle = \langle J_{t,s} [F_0, F_i](u_s), \varphi \rangle ds + \langle J_{t,s} [F_j, F_i](u_s), \varphi \rangle \circ dB_s^j \,,$$

where $[A, B](u) = DB(u)A(u) - DA(u)B(u)$ is a Lie bracket.

Define the iterated Lie brackets:

$$\mathscr{A}_0 = \big\{ F_i \,:\, 1 \leq i \leq d \big\} \,,\; \mathscr{A}_{k+1} = \mathscr{A}_k \cup \big\{ [F_i, A] \,:\, A \in \mathscr{A}_k,\, 0 \leq i \leq d \big\}.$$

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"Hörmander's condition": For every $u \in \mathcal{H}$, span of $\bigcup_{k\geq 0} \left\{ A(u) \,:\, A \in \mathscr{A}_k \right\}$ is dense in $\mathcal{H}$.

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$\langle \mathscr{M}_t \varphi, \varphi \rangle$ is small $\implies$ $\langle J_{t,s} F_i(u_s), \varphi \rangle$ is small for $i = 1, \ldots, d$

Norris's lemma + induction $\implies$ $\langle J_{t,s} A(u_s), \varphi \rangle$ is small for every $A \in \bigcup_{k\geq 0} \mathscr{A}_k$

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Itô calculus doesn't work (here)

To apply Norris's lemma in

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one needs $\langle J_{t,s}[F_j,F_i](u_s),\varphi\rangle$ to be a semi martingale. **Which is not** because of the "future" t.

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Rough paths

$\gamma \in (\frac{1}{3}, \frac{1}{2}]$, a pair $(X_t, \mathbb{X}_{t,s}) \in \mathbb{R}^d \times \mathbb{R}^{d\times d}$ is $\mathscr{C}^\gamma$ rough path if

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Theorem (G. & Hairer 18; Gubinelli & Tindel 10): Let $u_0 \in \mathcal{H}$ and $(X, \mathbb{X}) \in \mathscr{C}^\gamma$. Let $N : \mathcal{H} \to \mathcal{H}_{-\delta}$ be Lipschitz and $F_j : \mathcal{H} \to \mathcal{H}$ be C^3. Then there exists $\tau > 0$ and a unique $u \in C([0, \tau), \mathcal{H})$ which is a local in time, mild solution to:

$$u_t = S_t u_0 + \int_0^t S_{t-r} N(u_r) dr + \int_0^t S_{t-r} F_j(u_r) d\mathbf{X}_r^j \quad \text{(RPDE)},$$

for all $t \leq \tau$.

RPDE is well-posed: the solution to the above equation depends continuously on $(X, \mathbb{X})$ and u_0.

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What remains to show?

To proceed as before it remains to show equation

$$d\langle J_{t,s}A(u_s),\varphi\rangle = \langle J_{t,s}[L+N,A](u_s),\varphi\rangle ds + \langle J_{t,s}[F_j,A](u_s),\varphi\rangle d\mathbf{X}_s^j \ ,$$

and use "Rough Norris's Lemma" to proceed same as before.

Problem: only have mild equations for $J_{t,s}$ and u_s and can't "differentiate" the above.

Idea: Rewrite $\langle J_{t,s}A(u_s),\varphi\rangle = \langle A(u_s), J_{t,s}^*\varphi\rangle$, derive mild equation for $A(u_s)$ and $J_{t,s}^*$ and plug thus in.

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Mild Itô's formula for RPDEs

Theorem (G. & Hairer 18): Let $u \in C([0,T],\mathcal{H})$ solve mild (RPDE) and $A \in C^2(\mathcal{H},\mathcal{H})$ then $A(u_t)$ satisfies:

$$\begin{aligned} A(u_t) = S_t A(u_0) &+ \int_0^t S_{t-r}(DA(u_r)N(u_r) + [L,A](u_r))dr \\ &+ \int_0^t S_{t-r} DA(u_r)F_j(u_r)d\mathbf{X}_r^j \\ &+ \frac{1}{2}\int_0^t S_{t-r} D^2 A(u_r)(F_i(u_r),F_k(u_r))d[\mathbf{X}]_r^{i,k} \ , \end{aligned}$$

where $[\mathbf{X}]_t = X_t \otimes X_t - 2\,\mathrm{Sym}(\mathbb{X}_{t,0})$ is a bracket of rough path.

Backwards equations and $J^*_{t,s}$

Proposition (G. & Hairer 18): Let $u \in C([0,T],\mathcal{H})$ solve mild (RPDE) then for all $\varphi \in \mathcal{H}$ there exists a unique solution $K_{t,s}\varphi$ to the backwards rough PDE:

$$\begin{aligned} K_{t,s}\varphi = S_{t-s}\varphi &+ \int_s^t S_{r-s} DN^*(u_r) K_{t,r}\varphi \, dr \\ &+ \int_s^t S_{r-s} DF_i^*(u_r) K_{t,r}\varphi \, d\mathbf{X}_r^i \ , \end{aligned}$$

and such K is an adjoint of the Jacobian: $K_{t,s} = J^*_{t,s}$.

Rough Fubini Theorem

Plugging in the mild equations for $A(u_s)$ and $J^*_{t,s}$ into $\langle A(u_s), J^*_{t,s}\varphi\rangle$ creates double integrals that we want to get rid of.

Theorem (G. & Hairer 18): Let $Y_{t,s}$ be such that both $Y_{\cdot,s}$ and $Y_{t,\cdot}$ are controlled by $(X,\mathbb{X}) \in \mathscr{C}^\gamma$ with the corresponding Gubinelli derivatives being $Y^1_{\cdot,s}$ and $Y^2_{t,\cdot}$. In addition assume that $Y^1_{t,\cdot}$ and $Y^2_{\cdot,s}$ are controlled by $(X,\mathbb{X})$ and have the same Gubinelli derivative $Y^{1,2}_{t,s} = Y^{2,1}_{t,s}$ then the following change of integrals formula holds:

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Proof: approximate X with the smooth functions.

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Final remarks

This finishes the proof (in the case of $[X] \equiv 0$) of the equation

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This equation allows us to use Hörmander's condition and Rough Norris's Lemma (in the case of $(B, \mathbb{B}^{\mathrm{Strat}})$) to carry out the same arguments as in the finite dimensional case.

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Example of the vector fields satisfying Hörmander's condition.

Let $\mathcal{H} = L^2(\mathbb{T}, \mathbb{R})$, $B = (B^1, B^2)$ and

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