Invariants de Tutte et triangulations avec modèle d'Ising

Marie Albenque (CNRS and LIX)

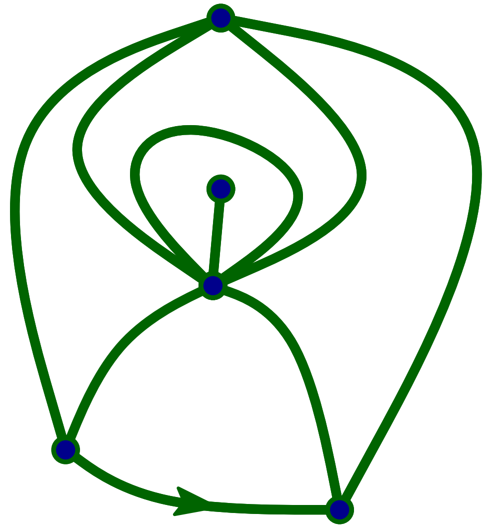
joint work with **Laurent Ménard** (Paris Nanterre)
and **Gilles Schaeffer** (CNRS and LIX)

Journées ALEA, Mars 2019

I - Local limit of triangulations without matter

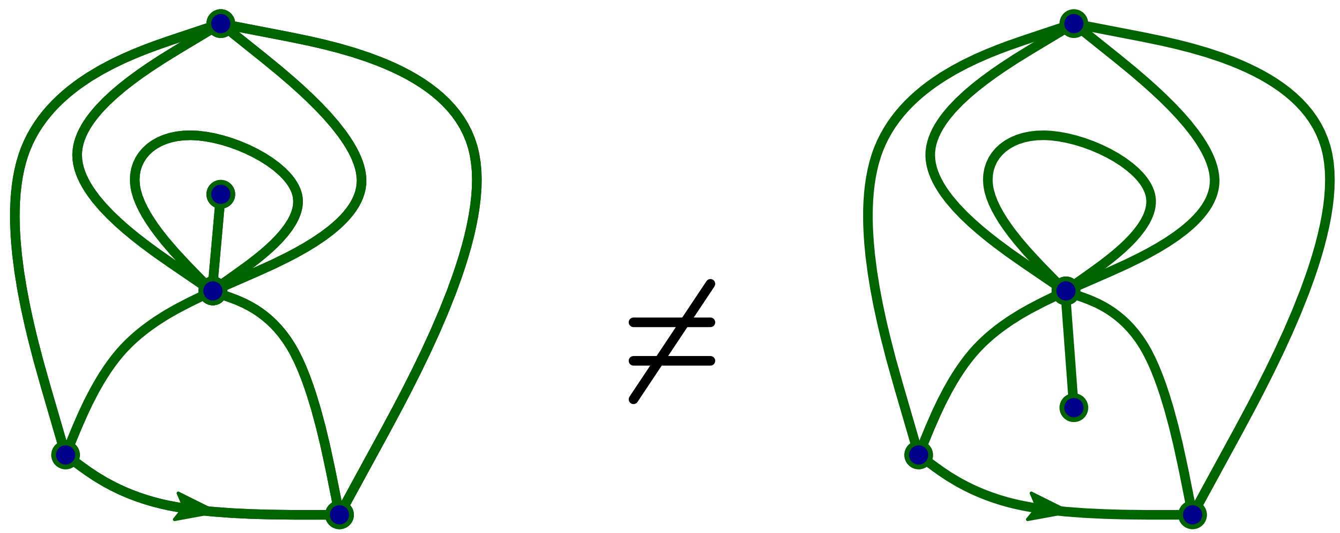
Planar Maps as discrete planar metric spaces

A **triangulation** is the proper embedding of a finite connected graph in the 2-dimensional sphere seen up to continuous deformations, such that all the faces have degree 3.



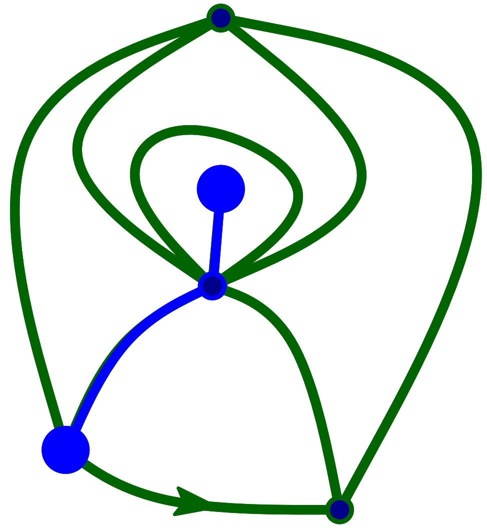
Planar Maps as discrete planar metric spaces

A **triangulation** is the proper embedding of a finite connected graph in the 2-dimensional sphere seen up to continuous deformations, such that all the faces have degree 3.

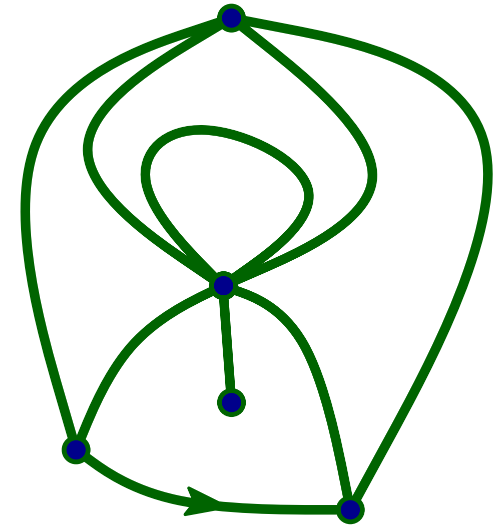


Planar Maps as discrete planar metric spaces

A **triangulation** is the proper embedding of a finite connected graph in the 2-dimensional sphere seen up to continuous deformations, such that all the faces have degree 3.



$\neq$



Plane maps are **rooted** : by orienting an edge.

Distance between two vertices = number of edges between them.
Planar map = **Metric space**

"Classical" large random triangulations

Take a triangulation with n edges uniformly at random. What does it look like if n is large ?

Local point of view : Look at neighborhoods of the root

"Classical" large random triangulations

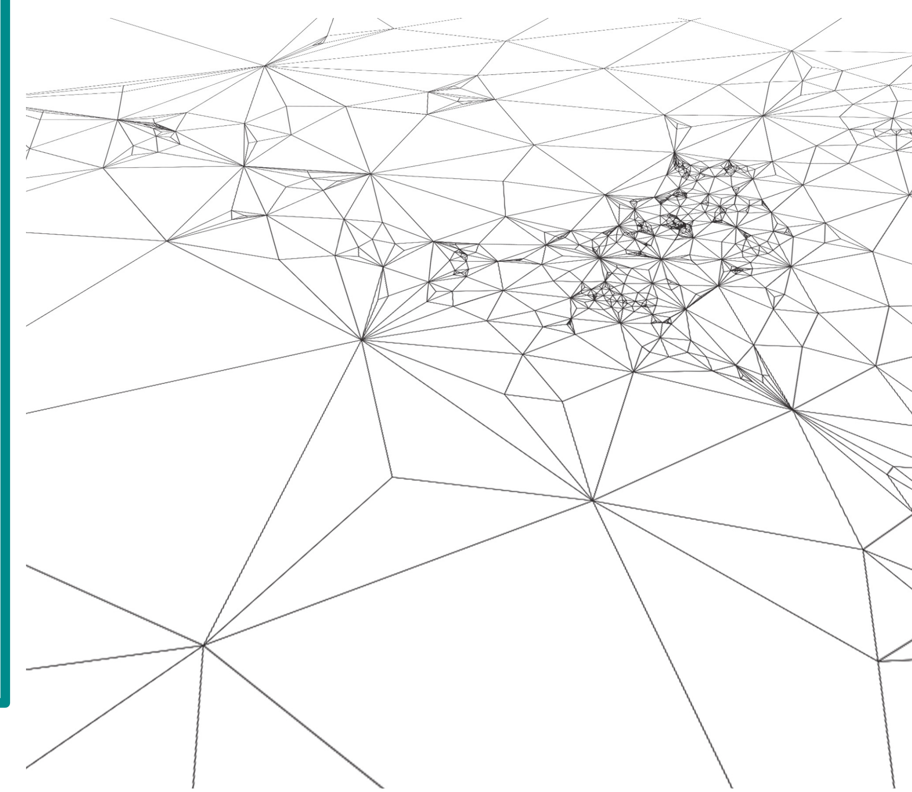
Take a triangulation with n edges uniformly at random. What does it look like if n is large ?

Local point of view : Look at neighborhoods of the root

The **local topology** on finite maps is induced by the distance:

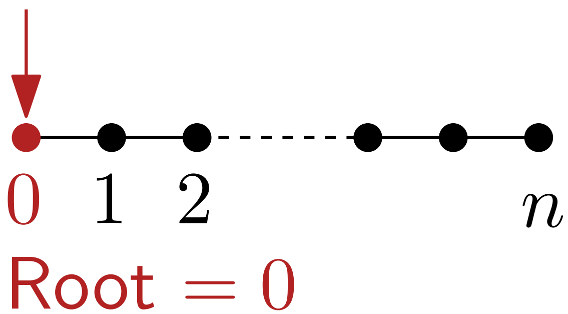
$$d_{loc}(m, m') := \left(1 + \max\{r \geq 0 : B_r(m) = B_r(m')\}\right)^{-1}$$

where $B_r(m)$ is the graph made of all the vertices and edges of m which are within distance r from the root.

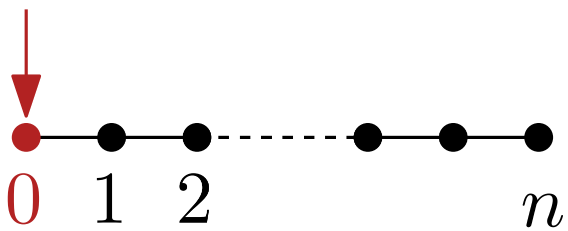


Courtesy of Igor Kortchemski

Local convergence: simple examples



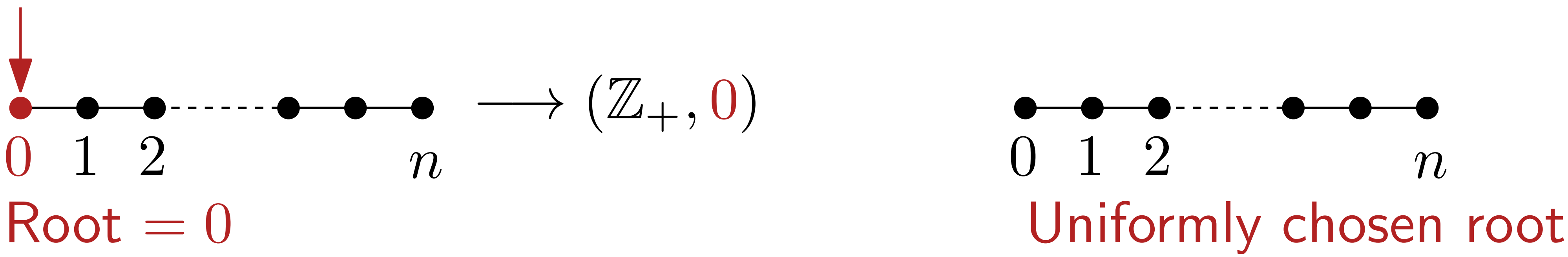
Local convergence: simple examples



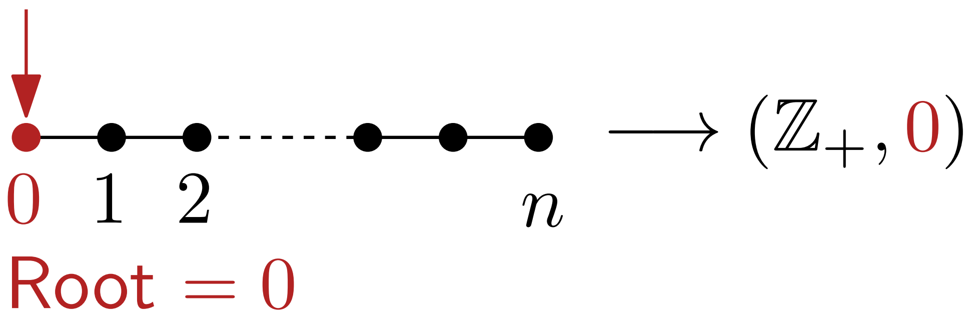
$\longrightarrow (\mathbb{Z}_+, 0)$

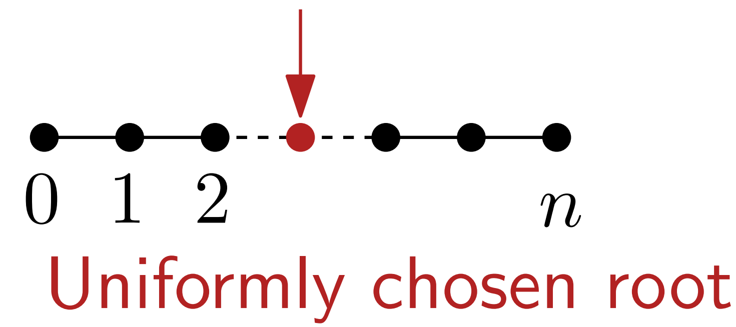
Root $= 0$

Local convergence: simple examples

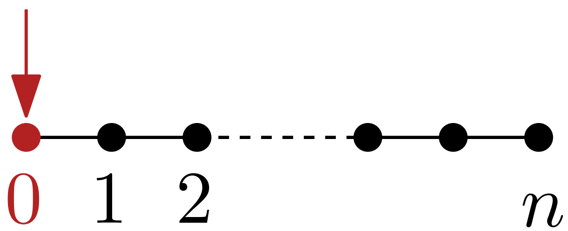


Local convergence: simple examples



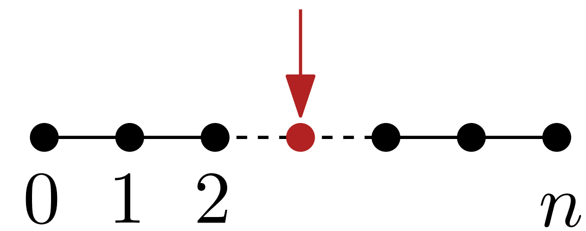


Local convergence: simple examples



$\longrightarrow (\mathbb{Z}_+, 0)$

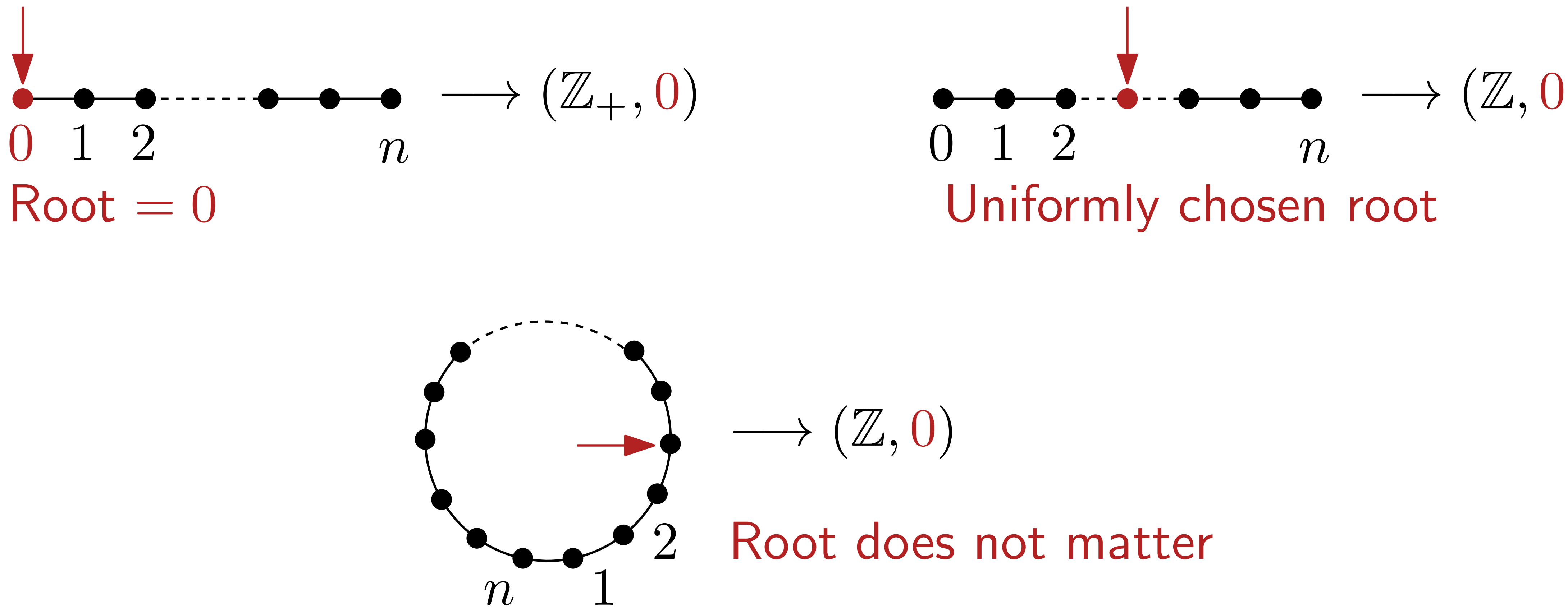
Root $= 0$



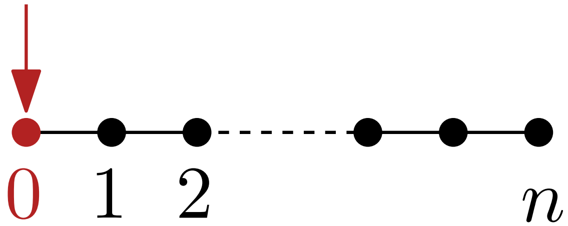
$\longrightarrow (\mathbb{Z}, 0)$

Uniformly chosen root

Local convergence: simple examples

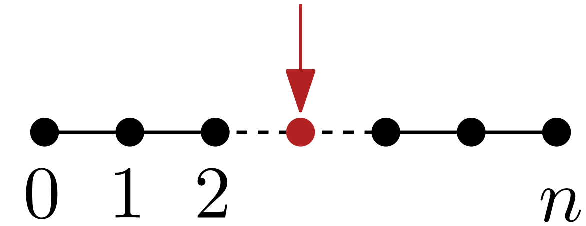


Local convergence: simple examples



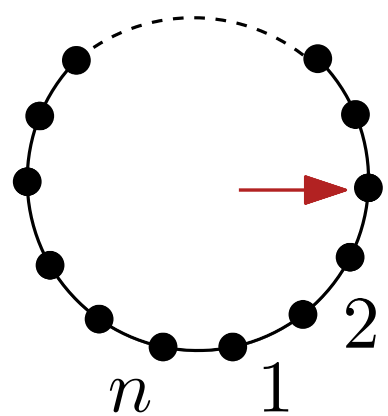
$\longrightarrow (\mathbb{Z}_+, 0)$

Root $= 0$



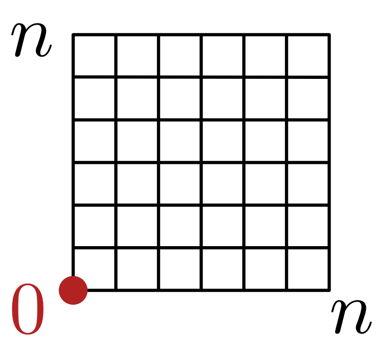
$\longrightarrow (\mathbb{Z}, 0)$

Uniformly chosen root



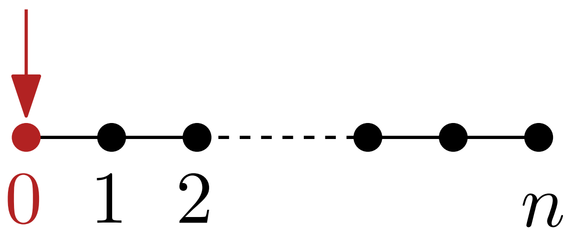
$\longrightarrow (\mathbb{Z}, 0)$

Root does not matter



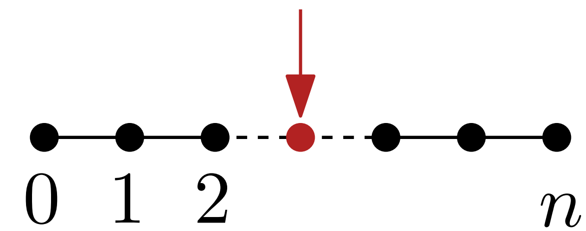
$\longrightarrow (\mathbb{Z}_+^2, 0)$

Local convergence: simple examples



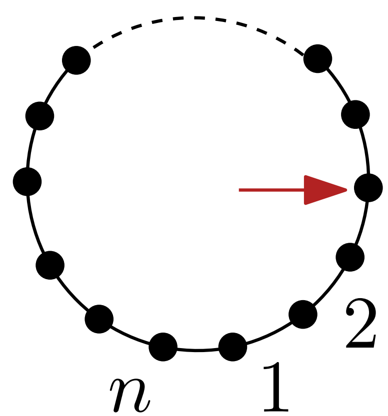
$\longrightarrow (\mathbb{Z}_+, 0)$

Root $= 0$



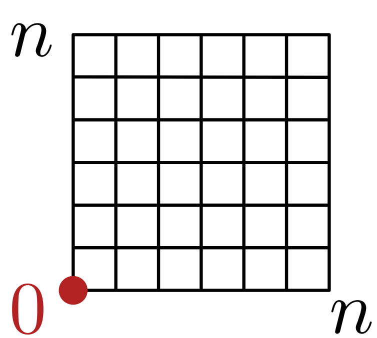
$\longrightarrow (\mathbb{Z}, 0)$

Uniformly chosen root

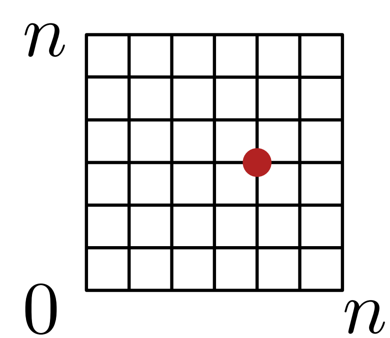


$\longrightarrow (\mathbb{Z}, 0)$

Root does not matter



$\longrightarrow (\mathbb{Z}_+^2, 0)$

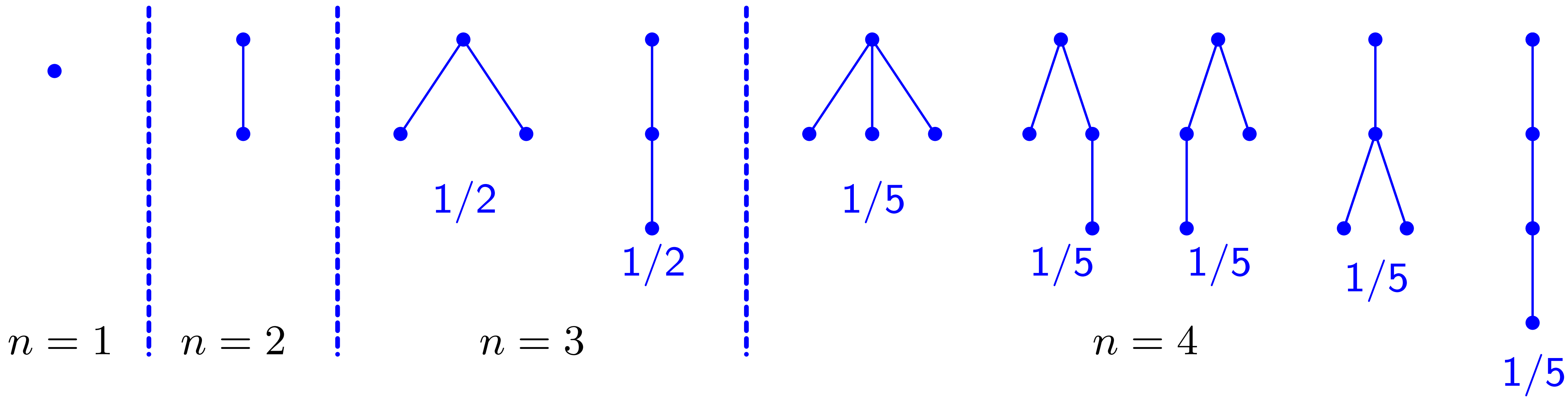


$\longrightarrow (\mathbb{Z}^2, 0)$

Uniformly chosen root

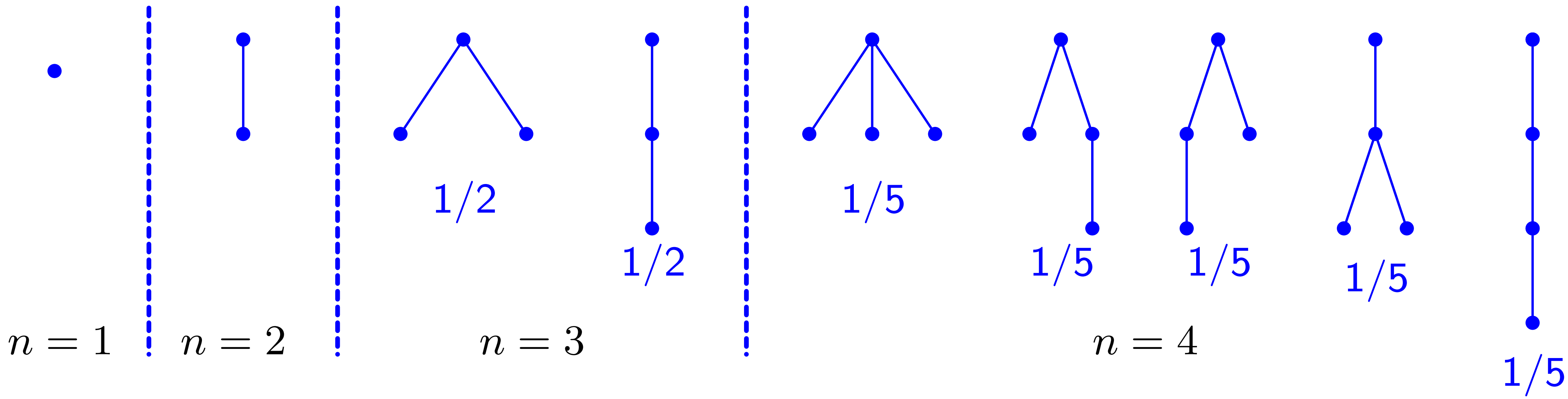
Local convergence: more complicated examples

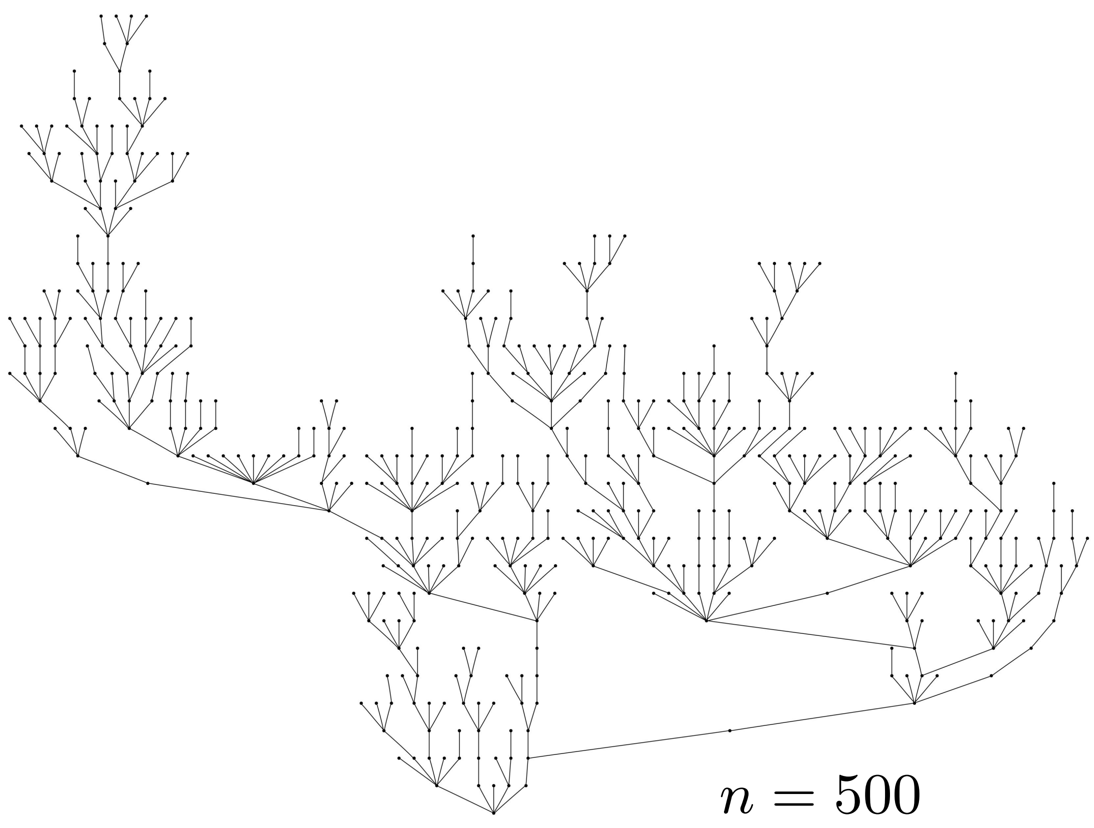
Uniform plane trees with n vertices:

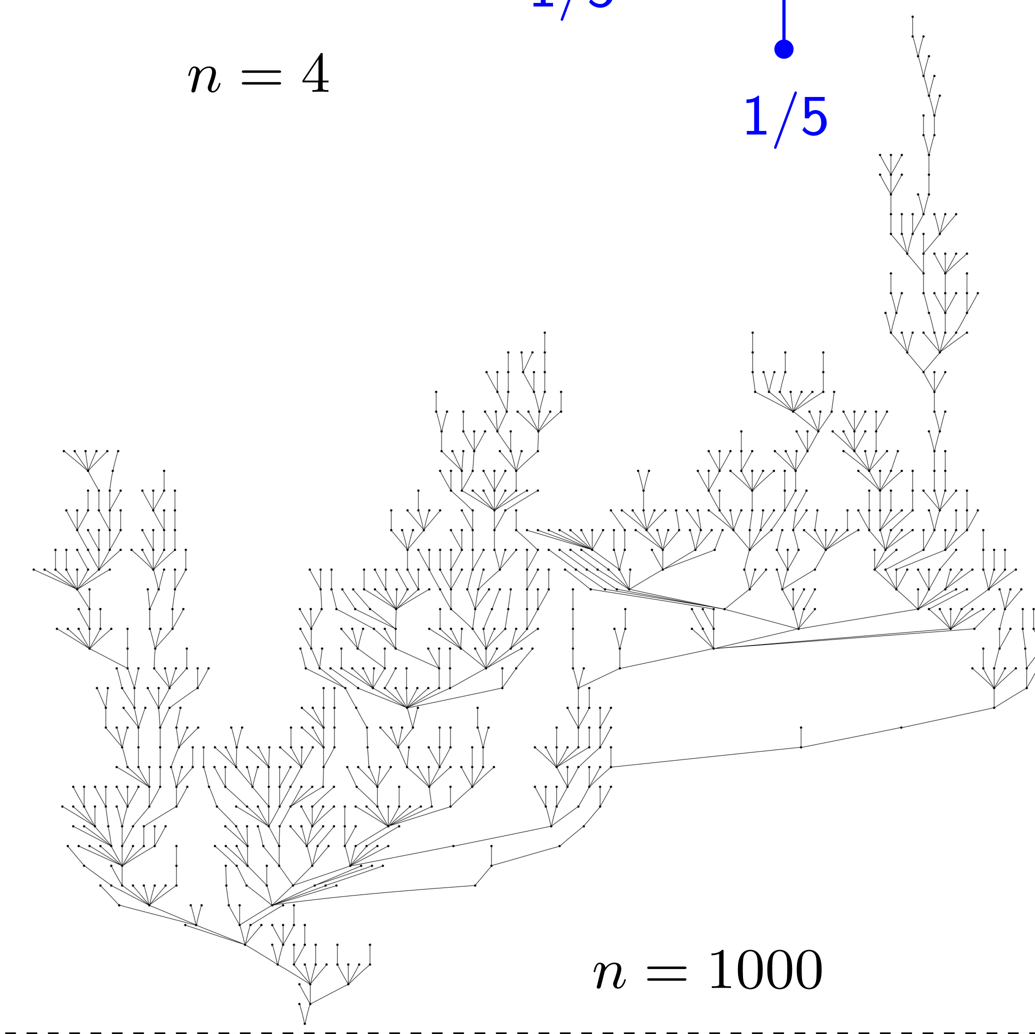


Local convergence: more complicated examples

Uniform plane trees with n vertices:

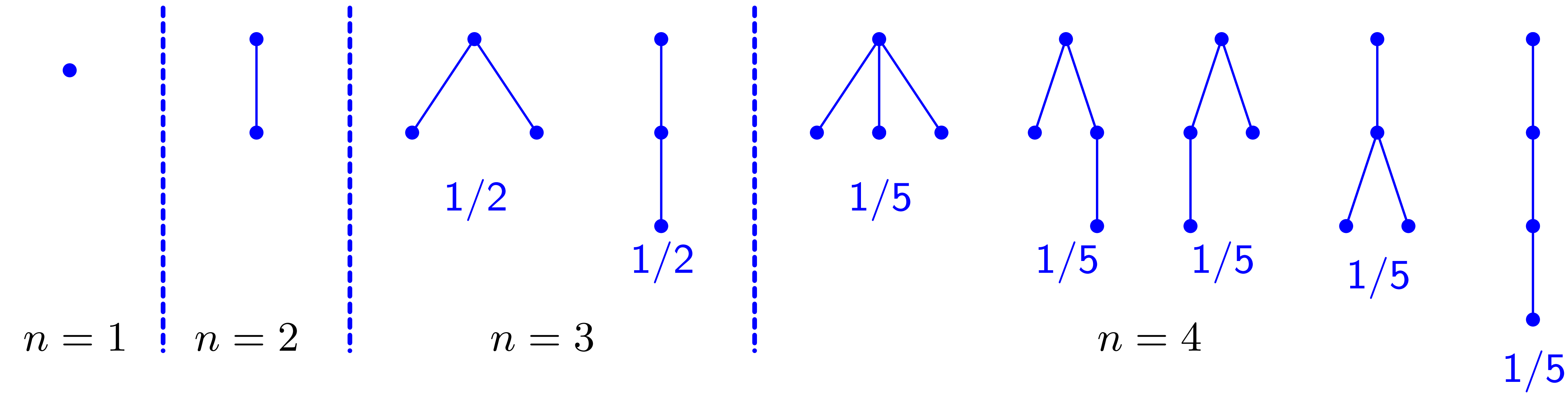




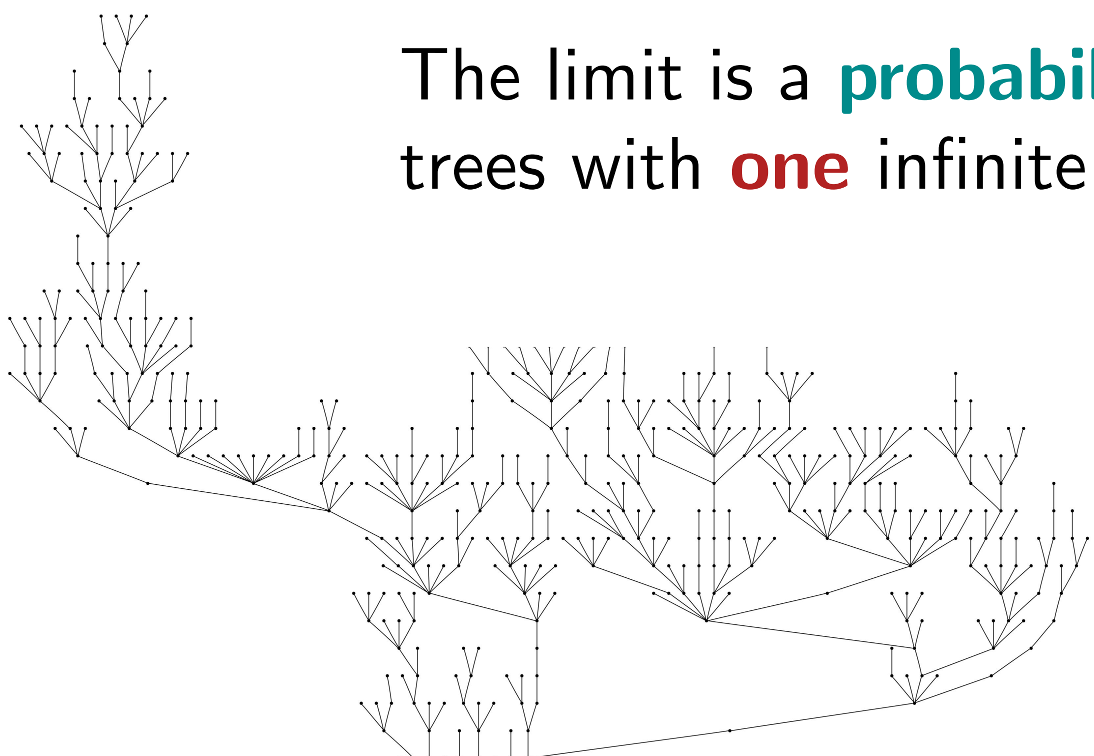


Local convergence: more complicated examples

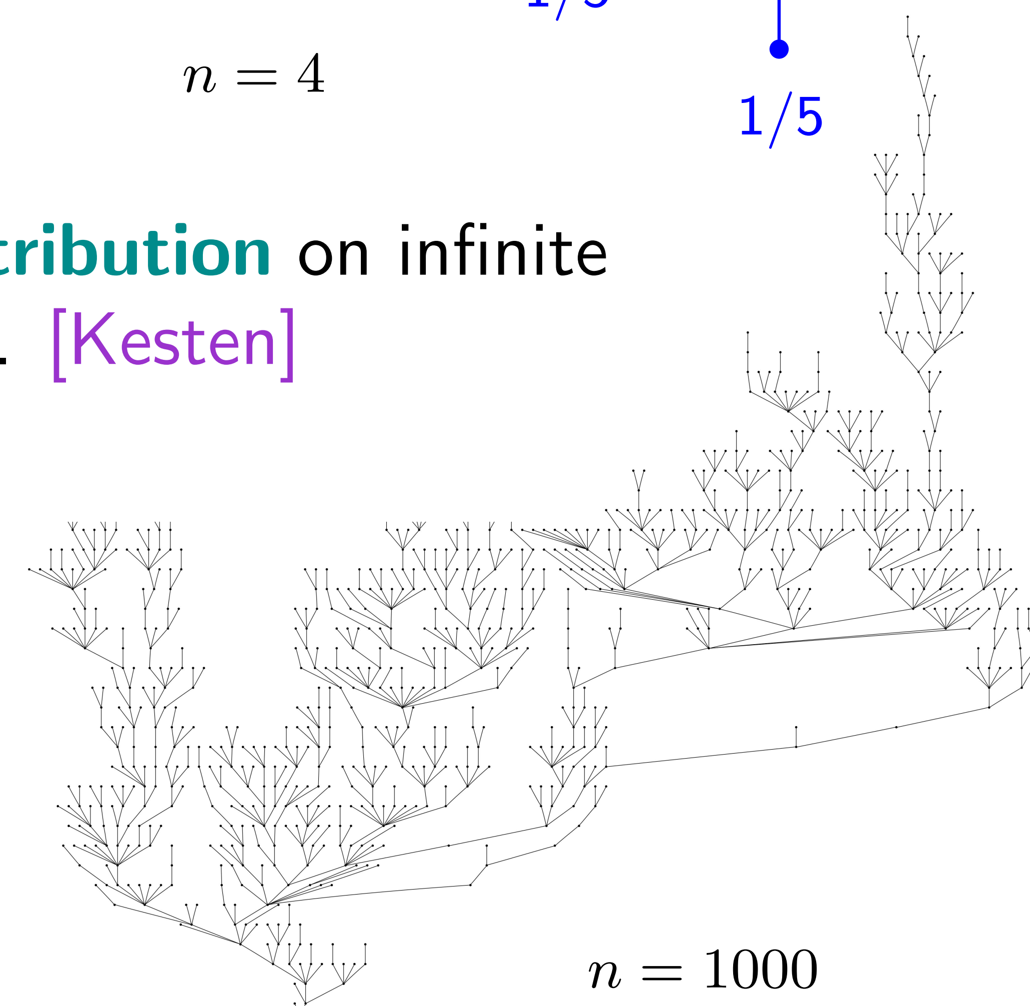
Uniform plane trees with n vertices:



The limit is a **probability distribution** on infinite trees with **one** infinite branch. [Kesten]



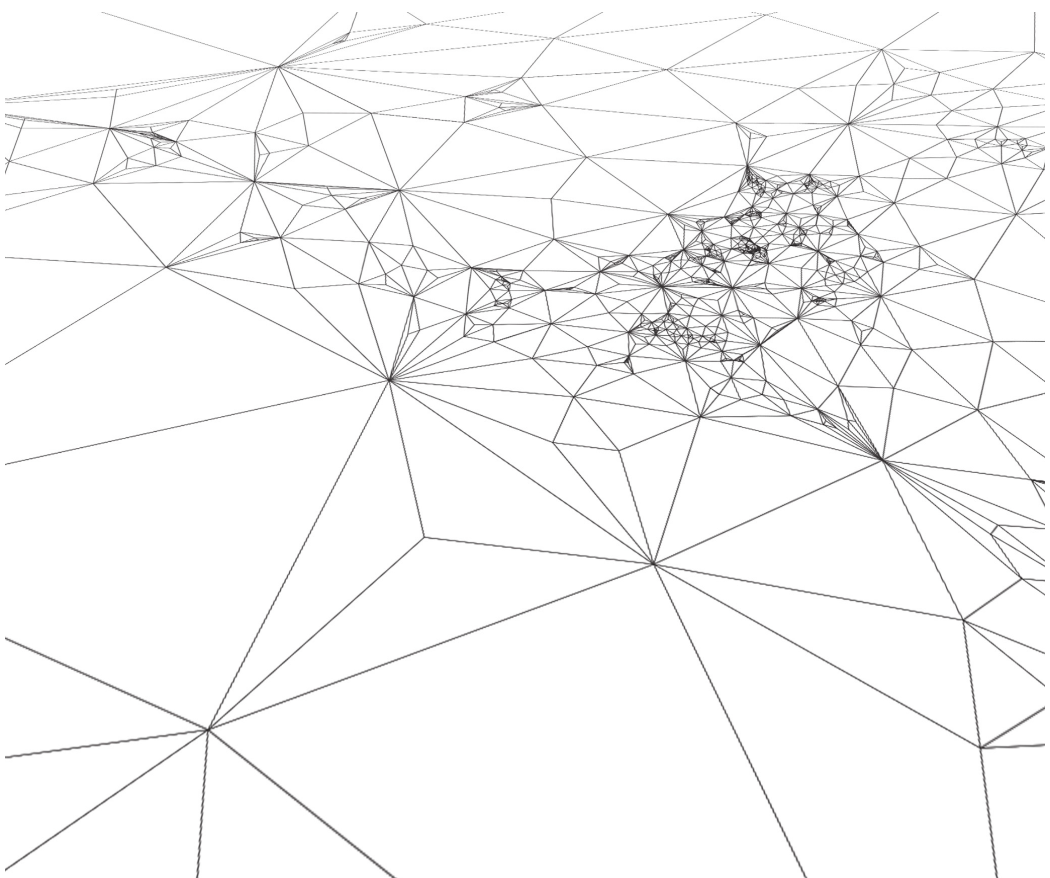
$n = 500$



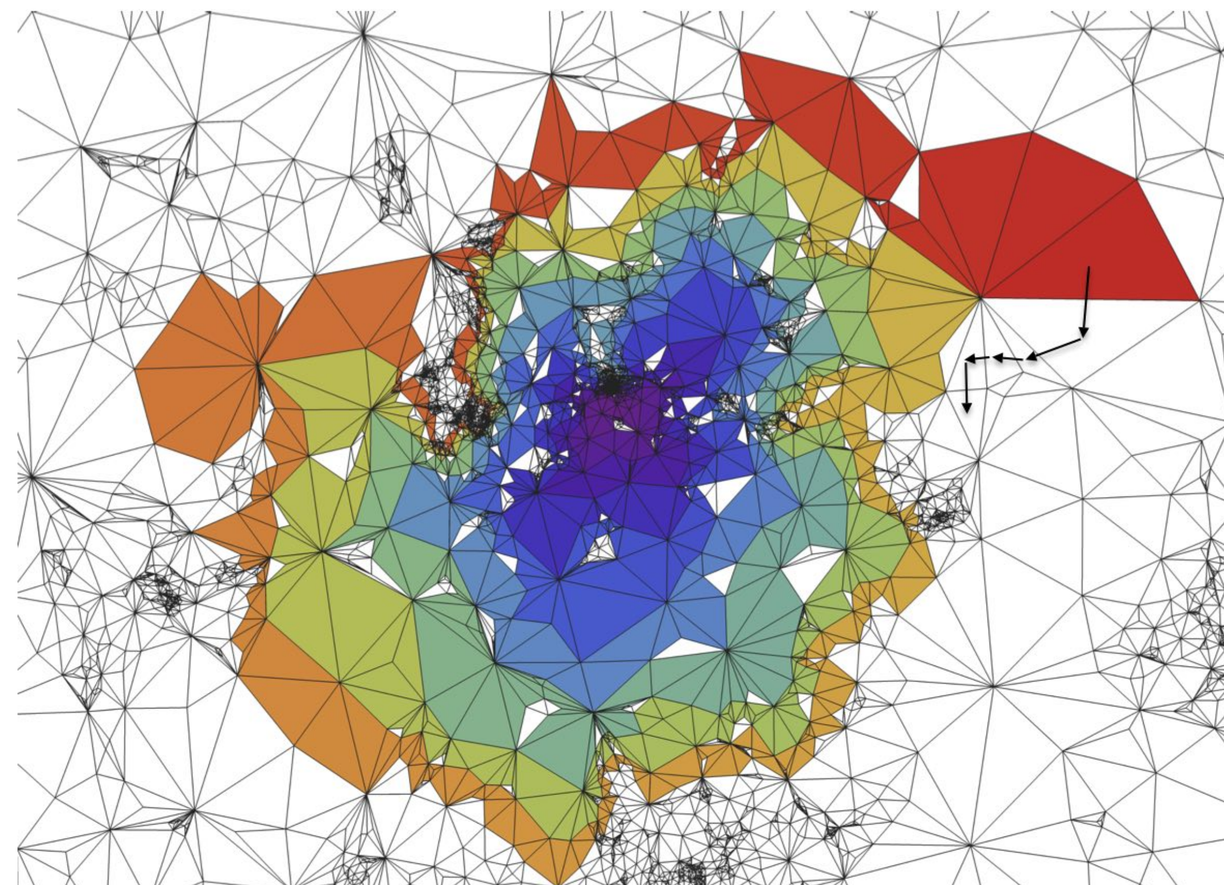
$n = 1000$

Local convergence of uniform triangulations

Theorem [Angel – Schramm, '03]
As $n \to \infty$, the uniform distribution on triangulations of size n converges weakly to a probability measure called the Uniform Infinite Planar Triangulation (or **UIPT**) for the **local topology**.



Courtesy of Igor Kortchemski



Courtesy of Timothy Budd

Local convergence of uniform triangulations

Theorem [Angel – Schramm, '03]
As $n \to \infty$, the uniform distribution on triangulations of size n converges weakly to a probability measure called the Uniform Infinite Planar Triangulation (or **UIPT**) for the **local topology**.

Some properties of the UIPT:

- The UIPT has almost surely one end [Angel – Schramm, '03]
- Volume (nb. of vertices) and perimeters of balls known to some extent.

For example $\mathbb{E}\left[|B_r(\mathbf{T}_\infty)|\right] \sim \frac{2}{7} r^4$ [Angel '04, Curien – Le Gall '12]

- Simple random Walk is recurrent [Gurel-Gurevich and Nachmias '13]

Local convergence of uniform triangulations

Theorem [Angel – Schramm, '03]
As $n \to \infty$, the uniform distribution on triangulations of size n converges weakly to a probability measure called the Uniform Infinite Planar Triangulation (or **UIPT**) for the **local topology**.

Some properties of the UIPT:

- The UIPT has almost surely one end [Angel – Schramm, '03]
- Volume (nb. of vertices) and perimeters of balls known to some extent.

For example $\mathbb{E}\left[|B_r(\mathbf{T}_\infty)|\right] \sim \frac{2}{7}r^4$ [Angel '04, Curien – Le Gall '12]

- Simple random Walk is recurrent [Gurel-Gurevich and Nachmias '13]

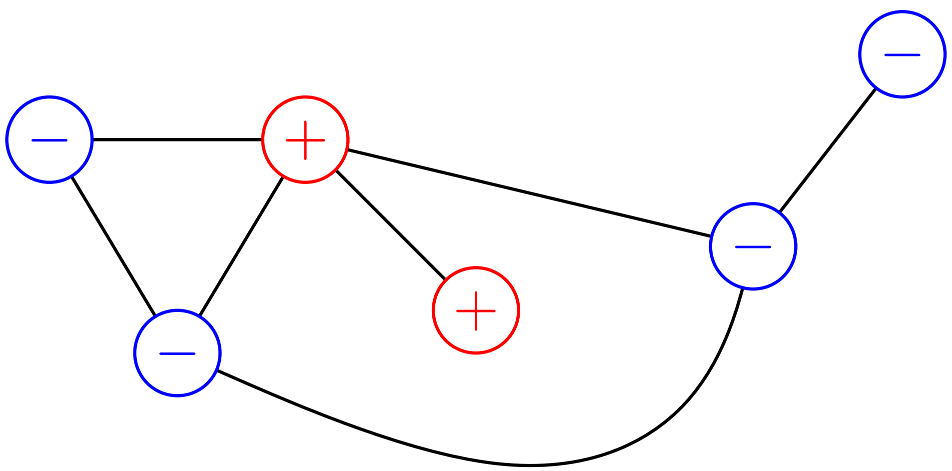
Universality: we expect the **same behavior** for slightly different models (e.g. quadrangulations, triangulations without loops, ...)

II - Ising model on random maps

Adding matter: Ising model on triangulations

First, Ising model on a finite deterministic graph:

$G = (V, E)$ finite graph



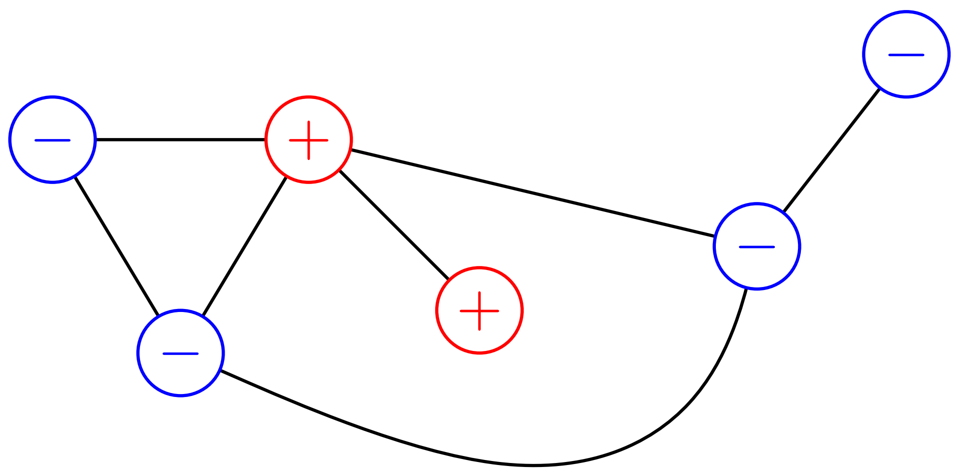
Spin configuration on G:

$$\sigma : V \to \{-1, +1\}.$$

Adding matter: Ising model on triangulations

First, Ising model on a finite deterministic graph:

$G = (V, E)$ finite graph



Spin configuration on G:

$$\sigma : V \to \{-1, +1\}.$$

Ising model on G: take a random spin configuration with probability

$$P(\sigma) \propto e^{-\frac{\beta}{2} \sum_{v \sim v'} \mathbf{1}_{\{\sigma(v) \neq \sigma(v')\}}}$$

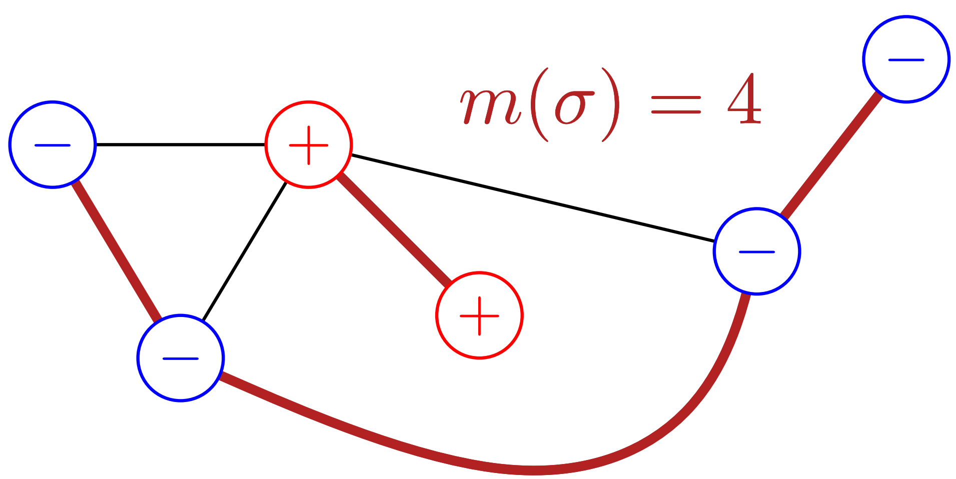
$\beta > 0$: inverse temperature.

$h = 0$: no magnetic field.

Adding matter: Ising model on triangulations

First, Ising model on a finite deterministic graph:

$G = (V, E)$ finite graph



Spin configuration on G:

$$\sigma : V \to \{-1, +1\}.$$

Ising model on G: take a random spin configuration with probability

$$P(\sigma) \propto e^{-\frac{\beta}{2} \sum_{v \sim v'} \mathbf{1}_{\{\sigma(v) \neq \sigma(v')\}}}$$

$\beta > 0$: inverse temperature.
$h = 0$: no magnetic field.

Combinatorial formulation: $P(\sigma) \propto \nu^{m(\sigma)}$
with $m(\sigma) =$ number of monochromatic edges and $\nu = e^{\beta}$.

Adding matter: Ising model on triangulations

$\mathcal{T}_n = \{$rooted planar triangulations with $3n$ edges$\}$.

Random triangulation with spins in $\mathcal{T}_n$ with probability $\propto \nu^{m(T,\sigma)}$?

Adding matter: Ising model on triangulations

$\mathcal{T}_n = \{$rooted planar triangulations with $3n$ edges$\}$.

Random triangulation with spins in $\mathcal{T}_n$ with probability $\propto \nu^{m(T,\sigma)}$?

$$\mathbb{P}_n^{\nu}\Big(\{(T,\sigma)\}\Big) = \frac{\nu^{m(T,\sigma)}\delta_{|e(T)|=3n}}{[t^{3n}]Q(\nu,t)}.$$

where $Q(\nu,t)$ = generating series of **Ising-weighted triangulations**:

$$Q(\nu,t) = \sum_{T\in\mathcal{T}_f} \sum_{\sigma:V(T)\to\{-1,+1\}} \nu^{m(T,\sigma)} t^{e(T)}.$$

Adding matter: Ising model on triangulations

$\mathcal{T}_n = \{$rooted planar triangulations with $3n$ edges$\}$.

Random triangulation with spins in $\mathcal{T}_n$ with probability $\propto \nu^{m(T,\sigma)}$?

$$\mathbb{P}_n^{\nu}\Big(\{(T,\sigma)\}\Big) = \frac{\nu^{m(T,\sigma)}\delta_{|e(T)|=3n}}{[t^{3n}]Q(\nu,t)}.$$

where $Q(\nu,t)$ = generating series of **Ising-weighted triangulations**:

$$Q(\nu,t) = \sum_{T\in\mathcal{T}_f} \sum_{\sigma:V(T)\to\{-1,+1\}} \nu^{m(T,\sigma)} t^{e(T)}.$$

Remark: This is a probability distribution on triangulations **with** spins. But, forgetting the spins gives a probability a distribution on triangulations **without** spins **different from the uniform distribution**.

Adding matter: New asymptotic behavior

Counting exponent for undecorated maps:

coeff $[t^n]$ of generating series of (undecorated) maps
(e.g.: triangulations, quadrangulations, general maps, simple maps,...)
$\sim \kappa \rho^{-n} n^{-\mathbf{5/2}}$

Note : κ and ρ depend on the combinatorics of the model.

Adding matter: New asymptotic behavior

Counting exponent for undecorated maps:

coeff $[t^n]$ of generating series of (undecorated) maps
(e.g.: triangulations, quadrangulations, general maps, simple maps,...)
$\sim \kappa \rho^{-n} n^{-\mathbf{5/2}}$

Note : κ and ρ depend on the combinatorics of the model.

Theorem [Bernardi – Bousquet-Mélou 11]
For every ν the series $Q(\nu, t)$ is algebraic, has $\rho_\nu > 0$ as unique dominant singularity and satisfies

$$[t^{3n}]Q(\nu,t) \underset{n\to\infty}{\sim} \begin{cases} \kappa\, \rho_{\nu_c}^{-n}\, n^{-7/3} & \text{if } \nu = \nu_c = 1 + \frac{1}{\sqrt{7}}, \\ \kappa\, \rho_{\nu}^{-n}\, n^{-5/2} & \text{if } \nu \neq \nu_c. \end{cases}$$

This suggests an unusual behavior of the underlying maps for $\nu = \nu_c$.
See also [Boulatov – Kazakov 1987], [Bousquet-Melou – Schaeffer 03] and [Bouttier – Di Francesco – Guitter 04].

III - Results and idea of proofs

Local convergence of triangulations with spins

Probability measure on triangulations of $\mathcal{T}_n$ with a spin configuration:

$$\mathbb{P}_n^\nu\Big(\{(T,\sigma)\}\Big) = \frac{\nu^{m(T,\sigma)}}{[t^{3n}]Q(\nu,t)}.$$

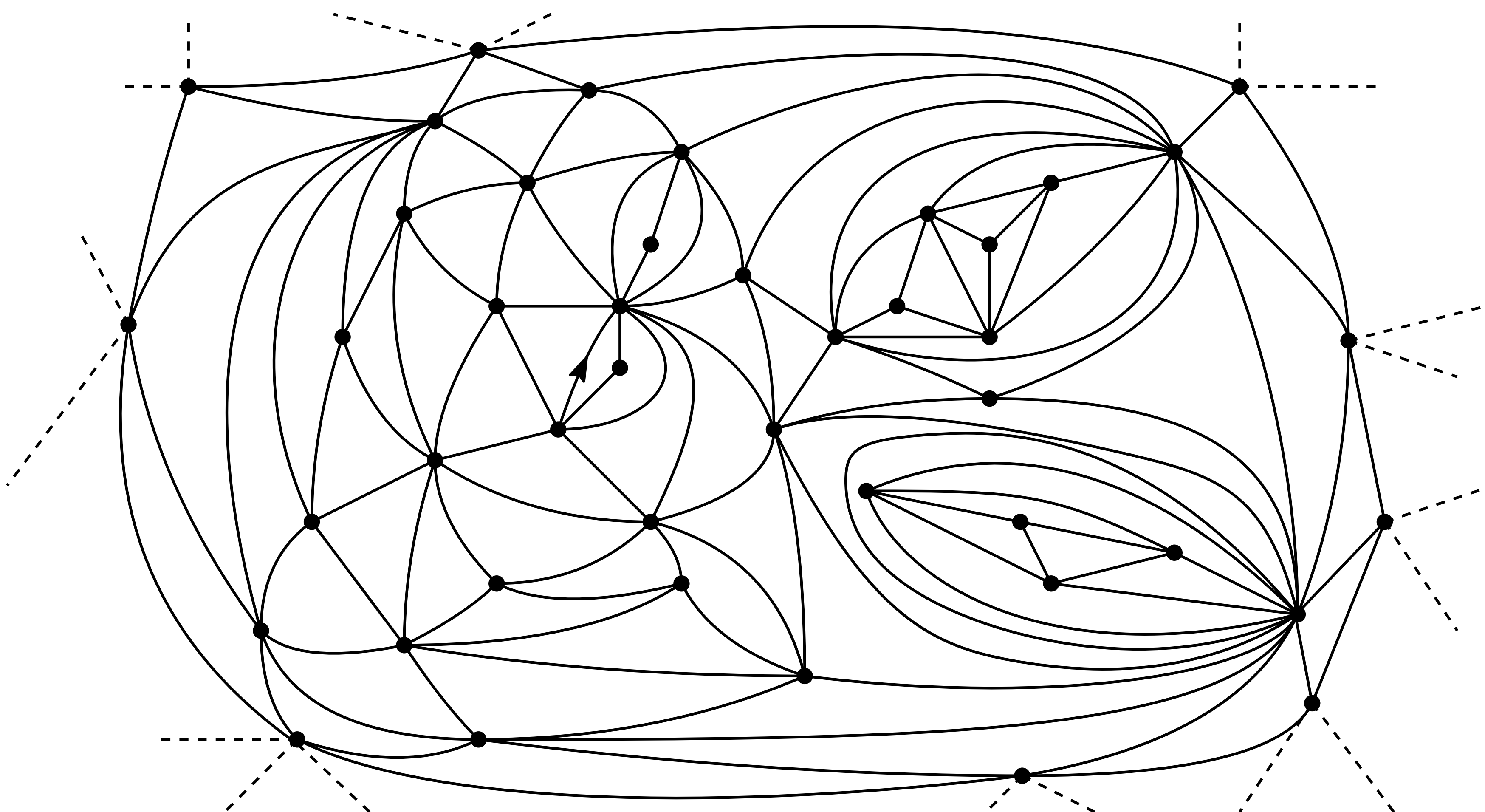
Theorem [AMS]
As $n \to \infty$, the sequence $\mathbb{P}_n^\nu$ converges weakly to a probability measure $\mathbb{P}^\nu$ for the **local topology**.
The measure $\mathbb{P}^\nu$ is supported on infinite triangulations with one end.

Local Topology for planar maps : balls

Definition:
The **local topology** on $\mathcal{M}_f$ is induced by the distance:

$$d_{loc}(m, m') := (1 + \max\{r \geq 0 : B_r(m) = B_r(m')\})^{-1}$$

where $B_r(m)$ is the graph made of all the faces of m with at least one vertex at distance $r-1$ from the root.

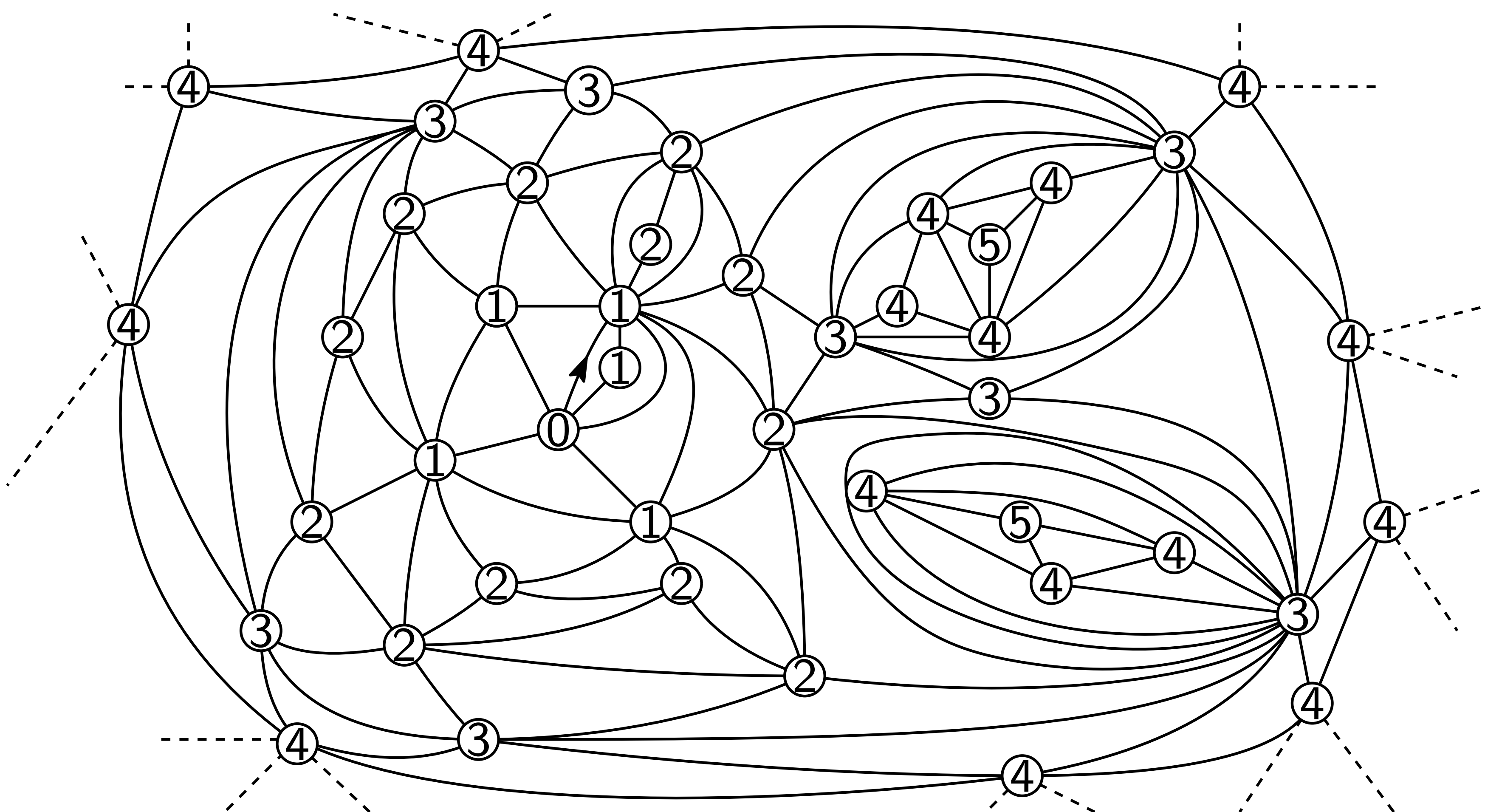


Local Topology for planar maps : balls

Definition:
The **local topology** on $\mathcal{M}_f$ is induced by the distance:

$$d_{loc}(m, m') := \left(1 + \max\{r \geq 0 : B_r(m) = B_r(m')\}\right)^{-1}$$

where $B_r(m)$ is the graph made of all the faces of m with at least one vertex at distance $r - 1$ from the root.



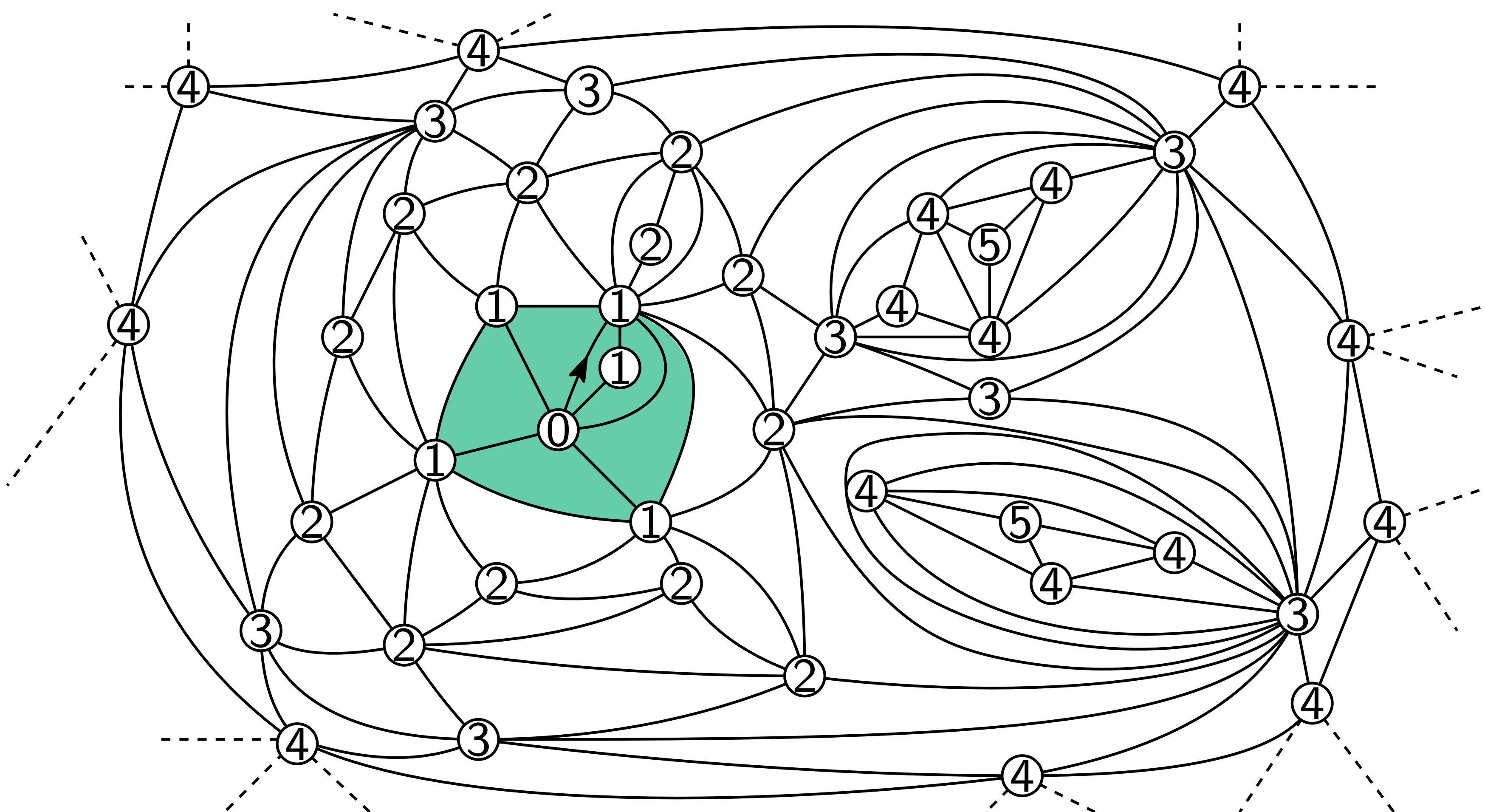
Local Topology for planar maps : balls

Definition:

The **local topology** on $\mathcal{M}_f$ is induced by the distance:

$$d_{loc}(m, m') := \left(1 + \max\{r \geq 0 : B_r(m) = B_r(m')\}\right)^{-1}$$

where $B_r(m)$ is the graph made of all the faces of m with at least one vertex at distance $r-1$ from the root.



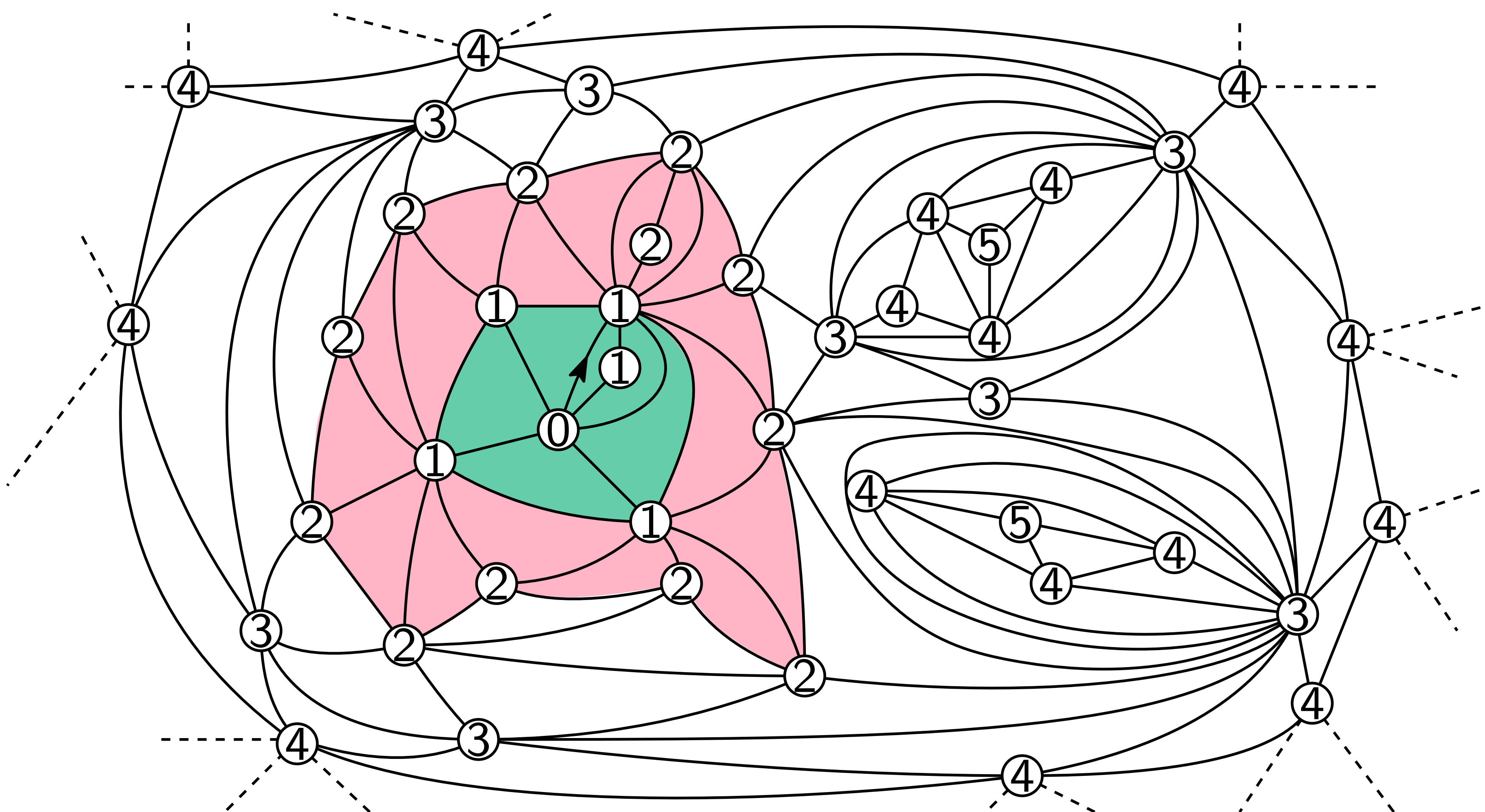
Local Topology for planar maps : balls

Definition:

The **local topology** on $\mathcal{M}_f$ is induced by the distance:

$$d_{loc}(m, m') := \left(1 + \max\{r \geq 0 : B_r(m) = B_r(m')\}\right)^{-1}$$

where $B_r(m)$ is the graph made of all the faces of m with at least one vertex at distance $r-1$ from the root.



Weak convergence for the local topology

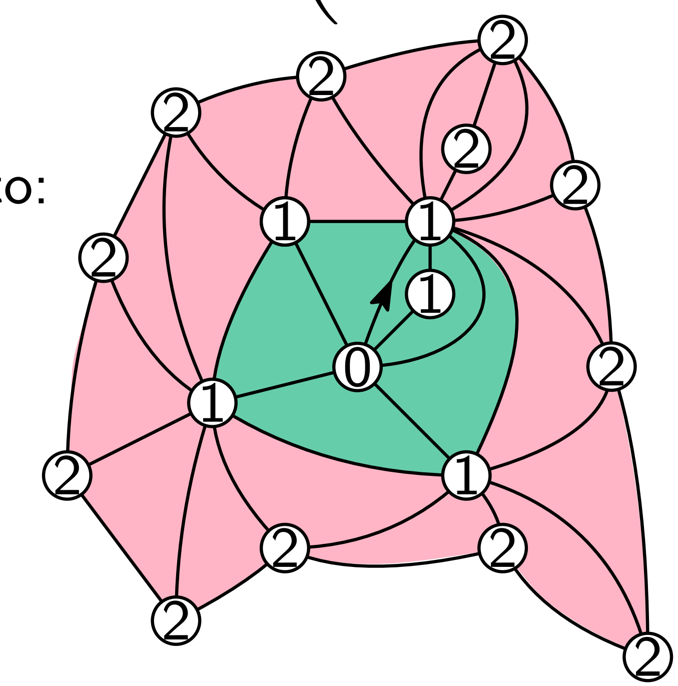
Portemanteau theorem + Levy – Prokhorov metric:

To show that $\mathbb{P}_n^\nu$ converges weakly to $\mathbb{P}^\nu$, prove

1. For every $r > 0$ and every possible ball Δ, show:

$$\mathbb{P}_n^\nu\Big(\{T \in \mathcal{T}_n \,:\, B_r(T) = \Delta\} \Big) \underset{n\to\infty}{\longrightarrow} \mathbb{P}^\nu\Big(\{T \in \mathcal{T}_\infty \,:\, B_r(T) = \Delta\} \Big).$$

For instance for $r = 2$, Δ might be equal to:



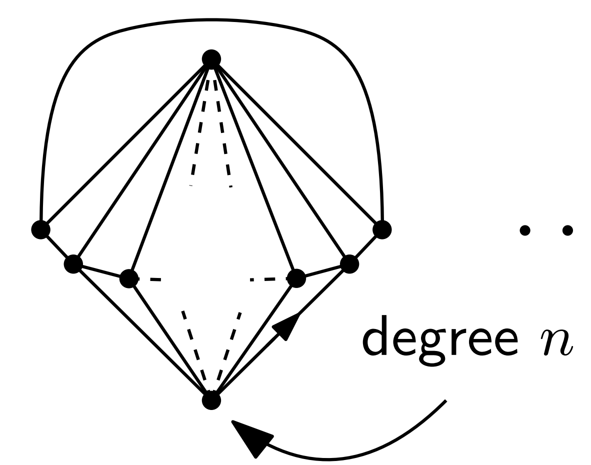
Weak convergence for the local topology

Portemanteau theorem + Levy – Prokhorov metric:
To show that $\mathbb{P}_n^{\nu}$ converges weakly to $\mathbb{P}^{\nu}$, prove

1. For every $r > 0$ and every possible ball Δ, show:

$$\mathbb{P}_n^{\nu}\Big(\{T \in \mathcal{T}_n \, : \, B_r(T) = \Delta\} \Big) \underset{n\to\infty}{\longrightarrow} \mathbb{P}^{\nu}\Big(\{T \in \mathcal{T}_\infty \, : \, B_r(T) = \Delta\} \Big).$$

Problem: the space $(\mathcal{T}, d_{loc})$ is **not compact**! Ex:



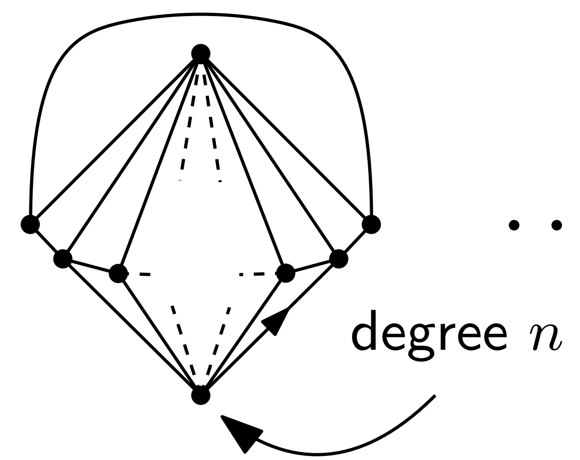
Weak convergence for the local topology

Portemanteau theorem + Levy – Prokhorov metric:
To show that $\mathbb{P}_n^\nu$ converges weakly to $\mathbb{P}^\nu$, prove

1. For every $r > 0$ and every possible ball Δ, show:

$$\mathbb{P}_n^\nu\Big(\{T \in \mathcal{T}_n \,:\, B_r(T) = \Delta\} \Big) \underset{n\to\infty}{\longrightarrow} \mathbb{P}^\nu\Big(\{T \in \mathcal{T}_\infty \,:\, B_r(T) = \Delta\} \Big).$$

Problem: the space $(\mathcal{T}, d_{loc})$ is **not compact**! Ex:



2. No loss of mass at the limit:
the measure $\mathbb{P}^\nu$ defined by the limits in 1. **is a probability measure**.

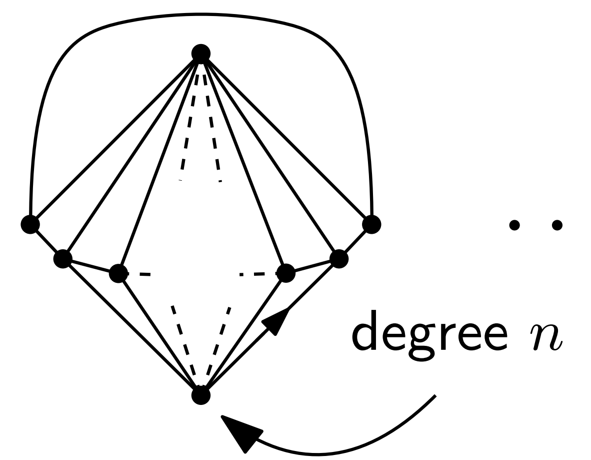
Weak convergence for the local topology

Portemanteau theorem + Levy – Prokhorov metric:
To show that $\mathbb{P}_n^\nu$ converges weakly to $\mathbb{P}^\nu$, prove

1. For every $r > 0$ and every possible ball Δ, show:

$$\mathbb{P}_n^\nu\Big(\{T \in \mathcal{T}_n \,:\, B_r(T) = \Delta\} \Big) \underset{n\to\infty}{\longrightarrow} \mathbb{P}^\nu\Big(\{T \in \mathcal{T}_\infty \,:\, B_r(T) = \Delta\} \Big).$$

Problem: the space $(\mathcal{T}, d_{loc})$ is **not compact**! Ex:



2. No loss of mass at the limit:
the measure $\mathbb{P}^\nu$ defined by the limits in 1. **is a probability measure**.

$$\forall r \geq 0, \quad \sum_{r-\text{balls}\,\Delta} \mathbb{P}^\nu\Big(\{T \in \mathcal{T}_\infty \,:\, B_r(T) = \Delta\} \Big) = 1.$$

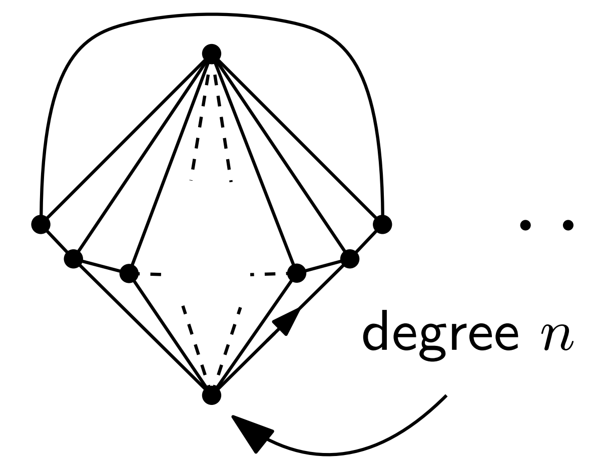
Weak convergence for the local topology

Portemanteau theorem + Levy – Prokhorov metric:
To show that $\mathbb{P}_n^\nu$ converges weakly to $\mathbb{P}^\nu$, prove

1. For every $r > 0$ and every possible ball Δ, show:

$$\mathbb{P}_n^\nu\Big(\{T \in \mathcal{T}_n \,:\, B_r(T) = \Delta\} \Big) \underset{n\to\infty}{\longrightarrow} \mathbb{P}^\nu\Big(\{T \in \mathcal{T}_\infty \,:\, B_r(T) = \Delta\} \Big).$$

Problem: the space $(\mathcal{T}, d_{loc})$ is **not compact**! Ex:

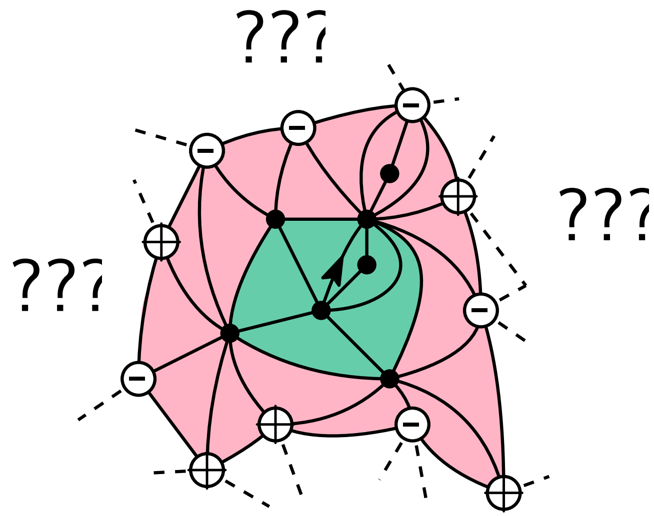


2. No loss of mass at the limit:
the measure $\mathbb{P}^\nu$ defined by the limits in 1. **is a probability measure**.

Enough to prove a **tightness** result, which amounts here to say that $\deg(\text{root})$ is tight.

Local convergence and generating series

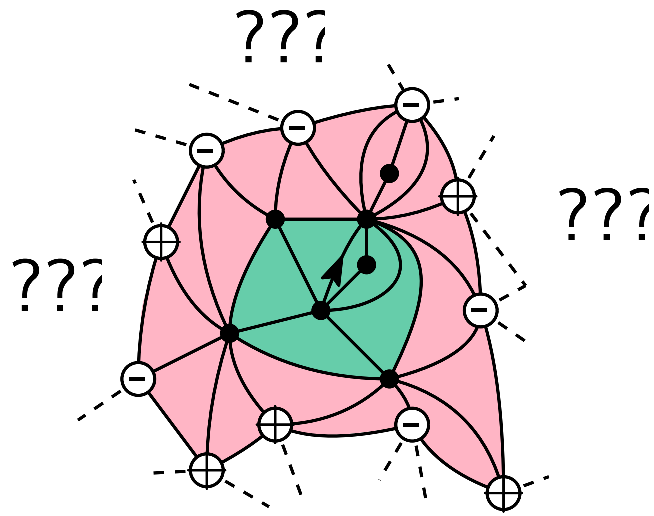
Need to evaluate, for every possible ball Δ

$\mathbb{P}_n\Big($  $\Big)$

Local convergence and generating series

Need to evaluate, for every possible ball Δ

$\mathbb{P}_n\Big($

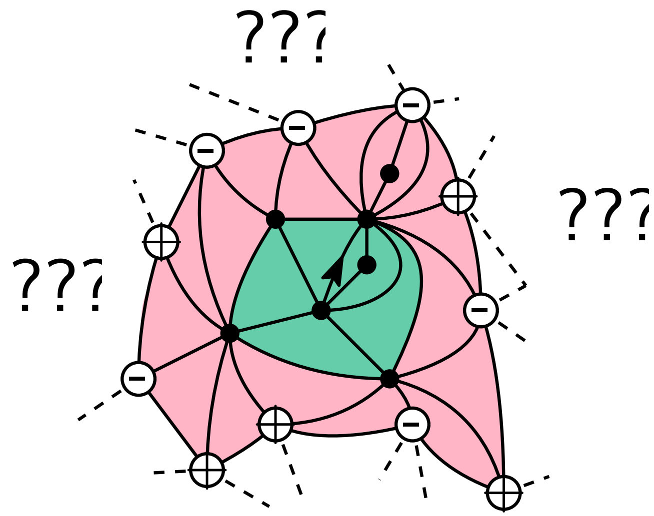


$$\Big) = \frac{\nu^{m(\Delta)-m(\omega)}\,[t^{3n-e(\Delta)+|\omega|}]\mathbf{Z}_\omega(\nu,t)}{[t^{3n}]Q(\nu,t)}$$

Generating series of triangulations with simple boundary and boundary conditions given by ω.
Here $\omega = + - + - - - + - + + -$

Local convergence and generating series

Need to evaluate, for every possible ball Δ

$\mathbb{P}_n\Big($ 

$\Big) = \dfrac{\nu^{m(\Delta)-m(\omega)}\,[t^{3n-e(\Delta)+|\omega|}]\mathbf{Z}_\omega(\nu,t)}{[t^{3n}]Q(\nu,t)}$

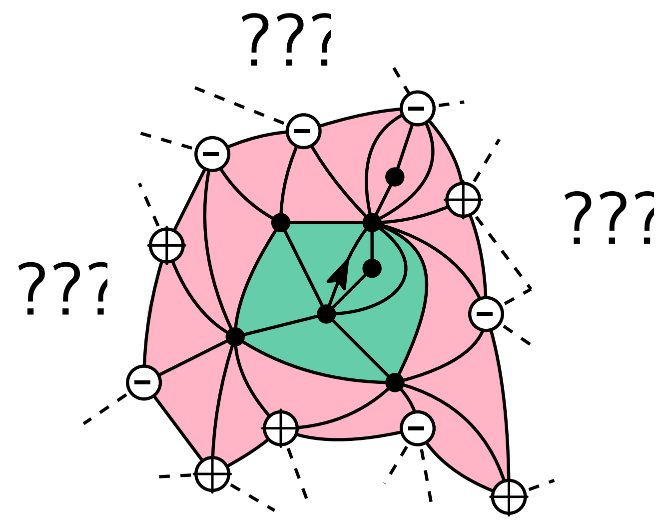
Generating series of triangulations with simple boundary and boundary conditions given by ω.
Here $\omega = + - + - - - + - + + -$

Theorem [AMS]
For every ω, the series $t^{|\omega|}Z_\omega(\nu,t)$ is algebraic, has ρ_ν as unique dominant singularity and satisfies

$$[t^{3n}]t^{|\omega|}Z_\omega(\nu,t) \underset{n\to\infty}{\sim} \begin{cases} \kappa_\omega(\nu_c)\,\rho_{\nu_c}^{-n}\,n^{-7/3} & \text{if } \nu = \nu_c = 1+\frac{1}{\sqrt{7}}, \\ \kappa_\omega(\nu)\,\rho_\nu^{-n}\,n^{-5/2} & \text{if } \nu \neq \nu_c. \end{cases}$$

Local convergence and generating series

Need to evaluate, for every possible ball Δ

$\mathbb{P}_n\Big($  $\Big)$ $= \dfrac{\nu^{m(\Delta)-m(\omega)}\,[t^{3n-e(\Delta)+|\omega|}]\mathbf{Z}_\omega(\nu,t)}{[t^{3n}]Q(\nu,t)}$

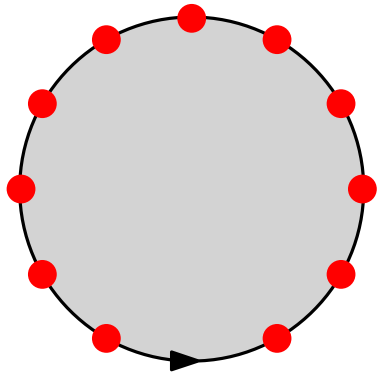
Generating series of triangulations with simple boundary and boundary conditions given by ω.
Here $\omega = + - + - - - + - + + -$

Theorem [AMS]
For every ω, the series $t^{|\omega|}Z_\omega(\nu,t)$ is algebraic, has ρ_ν as unique dominant singularity and satisfies

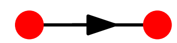
$$[t^{3n}]t^{|\omega|}Z_\omega(\nu,t) \underset{n\to\infty}{\sim} \begin{cases} \kappa_\omega(\nu_c)\,\rho_{\nu_c}^{-n}\,n^{-7/3} & \text{if } \nu=\nu_c=1+\frac{1}{\sqrt{7}}, \\ \kappa_\omega(\nu)\,\rho_\nu^{-n}\,n^{-5/2} & \text{if } \nu\neq\nu_c. \end{cases}$$

Thanks to a "trick", enough to prove the theorem for $\omega = \oplus\ldots\oplus$.

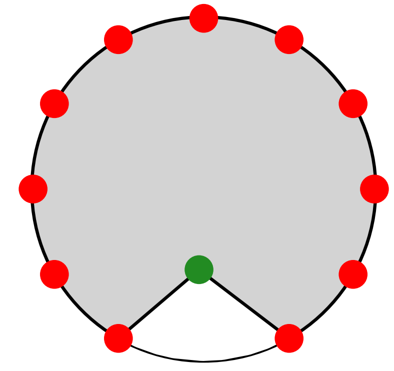
Positive boundary conditions : two catalytic variables



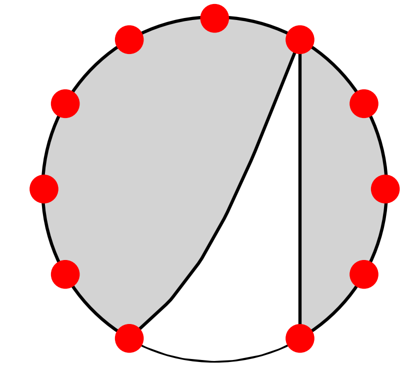
$=$



$+$

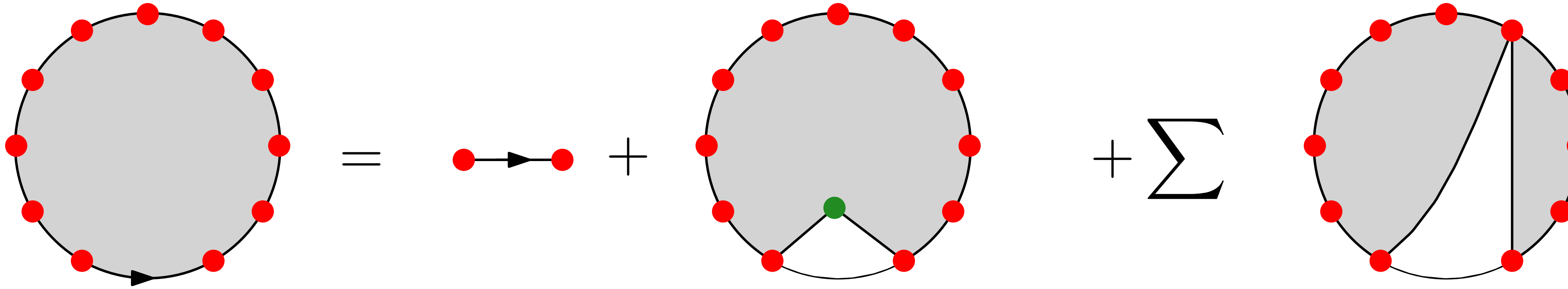


$+\sum$



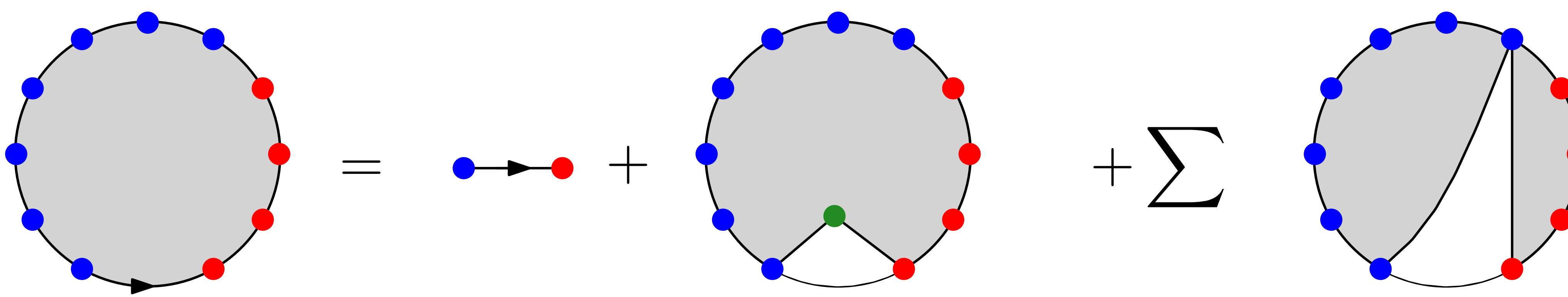
$$A(x) := \sum_{p\geq 1} Z_{\oplus^p} x^p = \ \nu t x^2 + \qquad + \frac{\nu t}{x}(A(x))^2$$

Positive boundary conditions : two catalytic variables



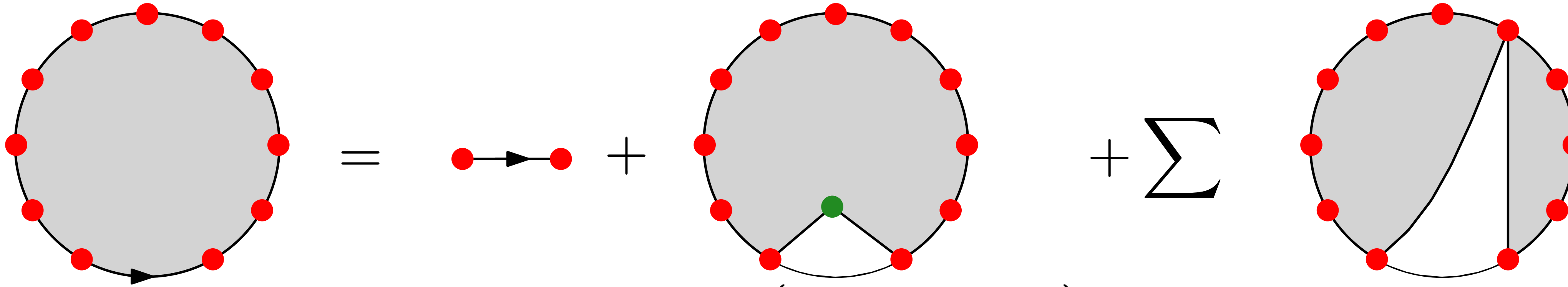
$$A(x) := \sum_{p\geq 1} Z_{\oplus^p} x^p = \nu t x^2 + \qquad + \frac{\nu t}{x}(A(x))^2$$

Peeling equation **at interface** $\ominus$–$\oplus$:



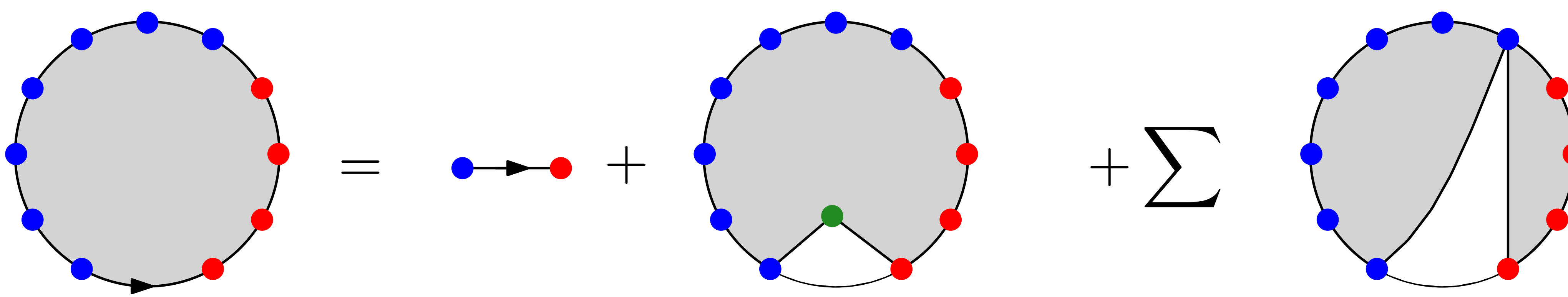
$$S(x,y) := \sum_{p,q\geq 1} Z_{\oplus^p \ominus^q} x^p y^q$$

Positive boundary conditions : two catalytic variables



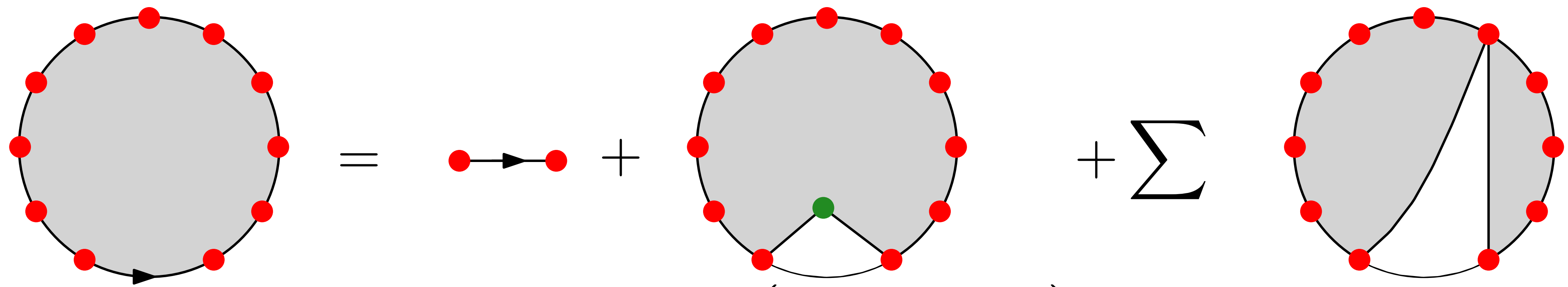
$$A(x) := \sum_{p\geq 1} Z_{\oplus^p} x^p = \nu t x^2 + \frac{\nu t}{x}\left(A(x) - x Z_{\oplus}\right) + \nu t\, [y] S(x,y) + \frac{\nu t}{x}(A(x))^2$$

Peeling equation **at interface** $\ominus$–$\oplus$:



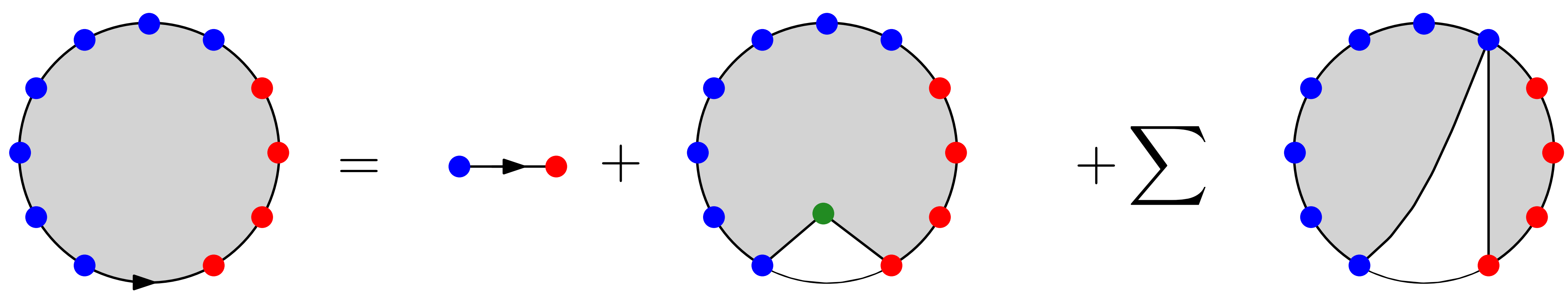
$$S(x,y) := \sum_{p,q\geq 1} Z_{\oplus^p \ominus^q} x^p y^q$$

Positive boundary conditions : two catalytic variables



$$A(x) := \sum_{p\geq 1} Z_{\oplus^p} x^p = \nu t x^2 + \frac{\nu t}{x}\Big(A(x) - x Z_{\oplus}\Big) + \nu t\,[y] S(x,y) \; + \frac{\nu t}{x}(A(x))^2$$

Peeling equation **at interface** $\ominus$–$\oplus$:



$$S(x,y) := \sum_{p,q\geq 1} Z_{\oplus^p \ominus^q} x^p y^q$$

$$= txy + \frac{t}{x}\Big(S(x,y) - x[x]S(x,y)\Big) + \frac{t}{y}\Big(S(x,y) - y[y]S(x,y)\Big)$$

$$+ \frac{t}{x} S(x,y) A(x) + \frac{t}{y} S(x,y) A(y)$$

From two catalytic variables to one: Tutte's invariants

Kernel method: equation for S reads $\quad K(x,y)\cdot S(x,y)=R(x,y)$

where $$K(x,y)=1-\frac{t}{x}-\frac{t}{y}-\frac{t}{x}A(x)-\frac{t}{y}A(y).$$

From two catalytic variables to one: Tutte's invariants

Kernel method: equation for S reads $\quad K(x,y)\cdot S(x,y)=R(x,y)$

where $\quad K(x,y)=1-\frac{t}{x}-\frac{t}{y}-\frac{t}{x}A(x)-\frac{t}{y}A(y).$

1. Find two series Y_1 and Y_2 in $\mathbb{Q}(x)[[t]]$ such that $K(x,Y_i/t)=0$.

From two catalytic variables to one: Tutte's invariants

Kernel method: equation for S reads $\quad K(x,y) \cdot S(x,y) = R(x,y)$

where $$K(x,y) = 1 - \frac{t}{x} - \frac{t}{y} - \frac{t}{x}A(x) - \frac{t}{y}A(y).$$

1. Find two series Y_1 and Y_2 in $\mathbb{Q}(x)[[t]]$ such that $K(x, Y_i/t) = 0$.

   It gives $\frac{1}{Y_1}\left(A(Y_1/t)+1\right) = \frac{1}{Y_2}\left(A(Y_2/t)+1\right)$.

From two catalytic variables to one: Tutte's invariants

Kernel method: equation for S reads $\quad K(x,y)\cdot S(x,y)=R(x,y)$

where $$K(x,y)=1-\frac{t}{x}-\frac{t}{y}-\frac{t}{x}A(x)-\frac{t}{y}A(y).$$

1. Find two series Y_1 and Y_2 in $\mathbb{Q}(x)[[t]]$ such that $K(x,Y_i/t)=0$.

   It gives $\frac{1}{Y_1}\left(A(Y_1/t)+1\right)=\frac{1}{Y_2}\left(A(Y_2/t)+1\right)$.

   $I(y):=\frac{1}{y}\left(A(y/t)+1\right)$ is called an **invariant**.

From two catalytic variables to one: Tutte's invariants

Kernel method: equation for S reads $\quad K(x, y) \cdot S(x, y) = R(x, y)$

where $$K(x, y) = 1 - \frac{t}{x} - \frac{t}{y} - \frac{t}{x}A(x) - \frac{t}{y}A(y).$$

1. Find two series Y_1 and Y_2 in $\mathbb{Q}(x)[[t]]$ such that $K(x, Y_i/t) = 0$.

   It gives $\frac{1}{Y_1}\left(A(Y_1/t) + 1\right) = \frac{1}{Y_2}\left(A(Y_2/t) + 1\right)$.

   $I(y) := \frac{1}{y}\left(A(y/t) + 1\right)$ is called an **invariant**.

2. Work a bit with the help of $R(x, Y_i/t) = 0$ to get a **second invariant** $J(y)$ depending only on $t, \nu, Z_{\oplus}(t), y$ and $A(y/t)$.

From two catalytic variables to one: Tutte's invariants

Kernel method: equation for S reads $\quad K(x,y)\cdot S(x,y)=R(x,y)$

where $$K(x,y)=1-\frac{t}{x}-\frac{t}{y}-\frac{t}{x}A(x)-\frac{t}{y}A(y).$$

1. Find two series Y_1 and Y_2 in $\mathbb{Q}(x)[[t]]$ such that $K(x,Y_i/t)=0$.

   It gives $\frac{1}{Y_1}\left(A(Y_1/t)+1\right)=\frac{1}{Y_2}\left(A(Y_2/t)+1\right)$.

   $I(y):=\frac{1}{y}\left(A(y/t)+1\right)$ is called an **invariant**.

2. Work a bit with the help of $R(x,Y_i/t)=0$ to get a **second invariant** $J(y)$ depending only on $t,\nu,Z_{\oplus}(t),y$ and $A(y/t)$.

3. Prove that $J(y)=C_0(t)+C_1(t)I(y)+C_2(t)I^2(y)$ with C_i's explicit polynomials in $t,Z_{\oplus}(t)$ and $Z_{\oplus^2}(t)$.

Equation with one catalytic variable for $A(y)$!

From two catalytic variables to one: Tutte's invariants

Kernel method: equation for S reads $\quad K(x,y)\cdot S(x,y)=R(x,y)$

where $\quad K(x,y)=1-\frac{t}{x}-\frac{t}{y}-\frac{t}{x}A(x)-\frac{t}{y}A(y).$

1. Find two series Y_1 and Y_2 in $\mathbb{Q}(x)[[t]]$ such that $K(x,Y_i/t)=0$.

   It gives $\frac{1}{Y_1}\left(A(Y_1/t)+1\right)=\frac{1}{Y_2}\left(A(Y_2/t)+1\right).$

   $I(y):=\frac{1}{y}\left(A(y/t)+1\right)$ is called an **invariant**.

2. Work a bit with the help of $R(x,Y_i/t)=0$ to get a **second invariant** $J(y)$ depending only on $t,\nu,Z_{\oplus}(t),y$ and $A(y/t)$.

3. Prove that $J(y)=C_0(t)+C_1(t)I(y)+C_2(t)I^2(y)$ with C_i's explicit polynomials in $t,Z_{\oplus}(t)$ and $Z_{\oplus^2}(t)$.

Equation with one catalytic variable for $A(y)$!

General result of [Bousquet-Mélou,Jehanne, 2006] gives algebraicity of $A(y)$

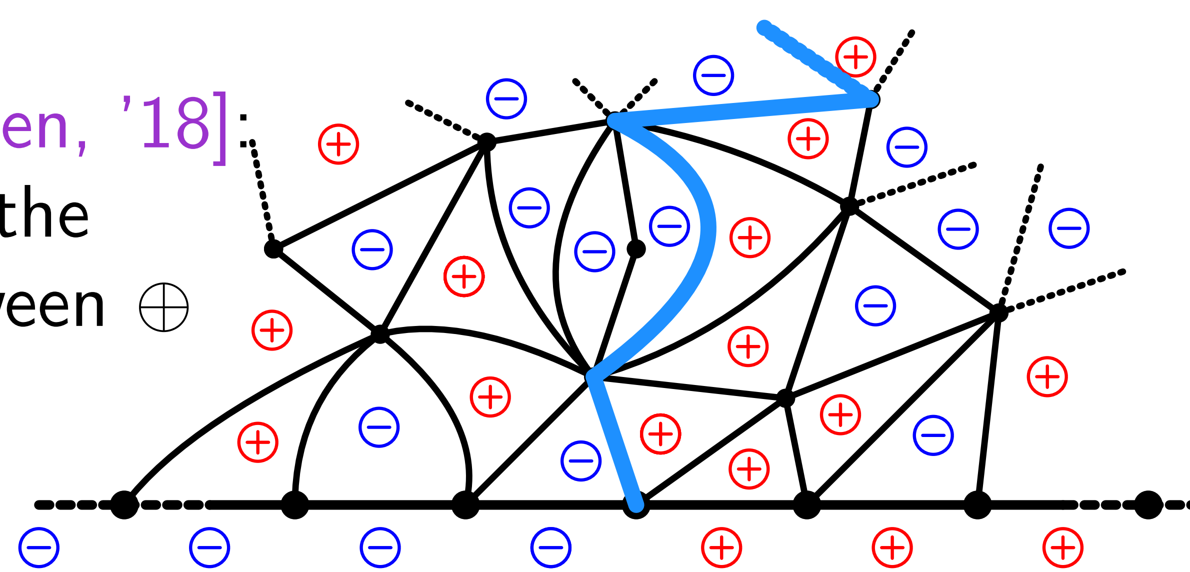
Local convergence of triangulations with spins

Probability measure on triangulations of $\mathcal{T}_n$ with a spin configuration:

$$\mathbb{P}_n^\nu\Big(\{(T,\sigma)\}\Big) = \frac{\nu^{m(T,\sigma)}}{[t^{3n}]Q(\nu,t)}.$$

Theorem [AMS]
As $n \to \infty$, the sequence $\mathbb{P}_n^\nu$ converges weakly to a probability measure $\mathbb{P}^\nu$ for the **local topology**.
The measure $\mathbb{P}^\nu$ is supported on infinite triangulations with one end.

Recent related result by [Chen, Turunen, '18]:
Local convergence for triangulations of the halfplane by studying the interface between $\oplus$ and $\ominus$.



The story so far

What we know:

- Convergence in law for the local toplogy.
- The limiting random triangulation has one end almost surely.

The story so far

What we know:

- Convergence in law for the local toplogy.
- The limiting random triangulation has one end almost surely.
- Recurrence of the random walk

The story so far

What we know:

- Convergence in law for the local toplogy.
- The limiting random triangulation has one end almost surely.
- Recurrence of the random walk

What we should now soon :

- Singularity with respect to the UIPT?
- Some information about the cluster's size.

The story so far

What we know:

- Convergence in law for the local toplogy.
- The limiting random triangulation has one end almost surely.
- Recurrence of the random walk

What we should now soon :

- Singularity with respect to the UIPT?
- Some information about the cluster's size.

What we would like to know :

- Volume growth ?

Adding matter: link with Liouville Quantum Gravity

Maps without matter "converge" to $\sqrt{\frac{8}{3}}$-LQG

[Miermont'13],[Le Gall'13], [Miller,Sheffield '15],

[Holden, Sun '19]

The critical Ising model is *believed* to converge to $\sqrt{3}$-LQG.

Similar statements for other models of decorated maps
(with a spanning subtree ($\gamma = \sqrt{2}$), with a bipolar orientation ($\gamma = \sqrt{4/3}$),...)
but no proofs.

Adding matter: link with Liouville Quantum Gravity

Maps without matter "converge" to $\sqrt{\frac{8}{3}}$-LQG

[Miermont'13],[Le Gall'13], [Miller,Sheffield '15],

[Holden, Sun '19]

The critical Ising model is *believed* to converge to $\sqrt{3}$-LQG.

Similar statements for other models of decorated maps
(with a spanning subtree ($\gamma = \sqrt{2}$), with a bipolar orientation ($\gamma = \sqrt{4/3}$),...)
but no proofs.

For $\gamma \in (0, 2)$, there exists $d_\gamma =$ "fractal dimension of γ-LQG"

$d_\gamma =$ ball volume growth exponent for corresponding maps ??

Adding matter: link with Liouville Quantum Gravity

Maps without matter "converge" to $\sqrt{\frac{8}{3}}$-LQG

[Miermont'13],[Le Gall'13], [Miller,Sheffield '15],

[Holden, Sun '19]

The critical Ising model is *believed* to converge to $\sqrt{3}$-LQG.

Similar statements for other models of decorated maps
(with a spanning subtree ($\gamma = \sqrt{2}$), with a bipolar orientation ($\gamma = \sqrt{4/3}$),...)
but no proofs.

For $\gamma \in (0, 2)$, there exists d_γ = "fractal dimension of γ-LQG"

d_γ = ball volume growth exponent for corresponding maps ??

YES, in some cases [Gwynne, Holden, Sun '17], [Ding, Gwynne '18]

Unknown for Ising, but $d_{\sqrt{3}}$ is a good candidate for the volume growth exponent.

What is $d_{\sqrt{3}}$?

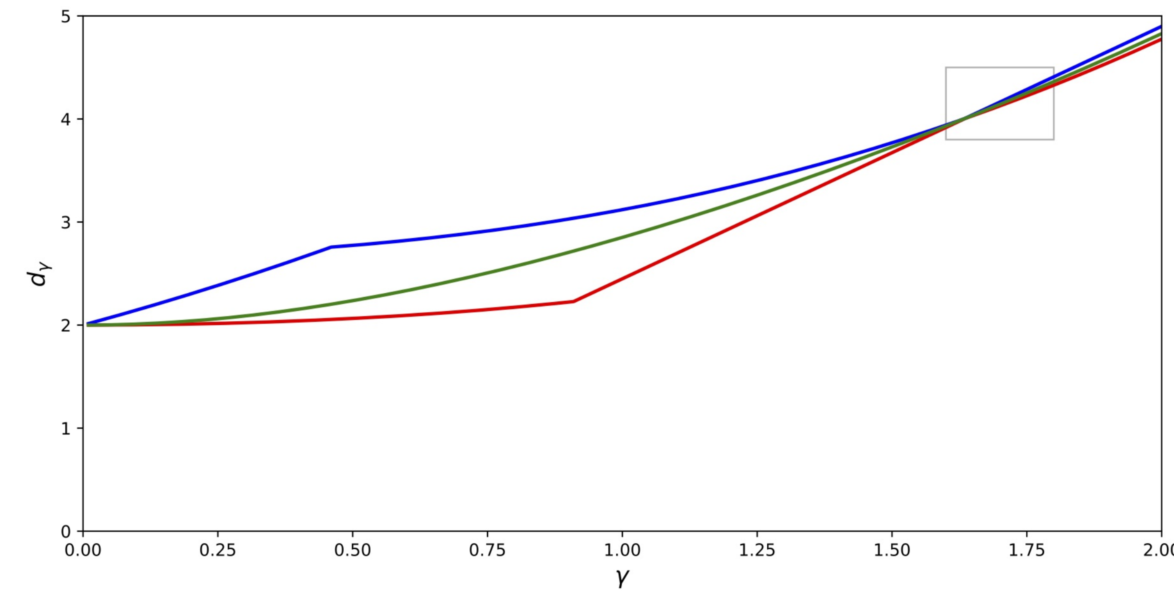
Adding matter: link with Liouville Quantum Gravity

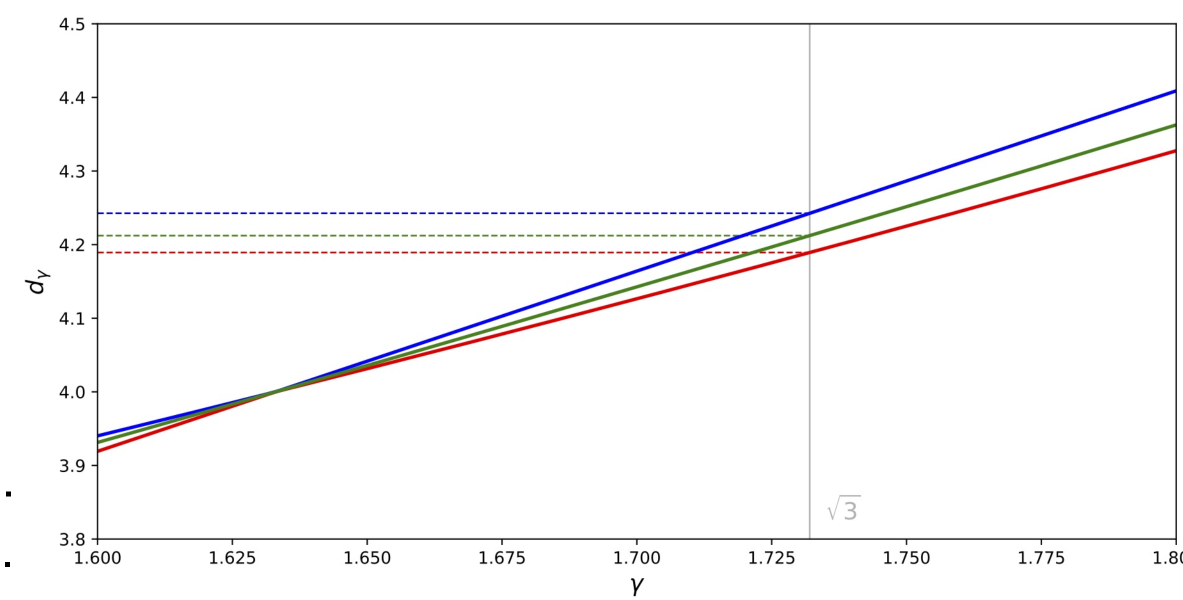
Watabiki's prediction:

$$d_\gamma = 1 + \frac{\gamma^2}{4} + \frac{1}{4}\sqrt{(4+\gamma^2)^2 + 16\gamma^2}$$ gives $d_{\sqrt{3}} \approx 4.21...$

[Ding, Gwynne '18]
Bounds for d_γ which give:
$4.18 \leq d_{\sqrt{3}} \leq 4.25$.

In particular $d_{\sqrt{3}} \neq 4$ and growth volume would then be different than the uniform model.





Green = Watabiki.
Blue and Red = bounds by Ding and Gwynne.

The story so far

What we know:

- Convergence in law for the local toplogy.
- The limiting random triangulation has one end almost surely.
- Recurrence of the random walk

What we should now soon :

- Singularity with respect to the UIPT?
- Some information about the cluster's size.

What we would like to know :

- Volume growth ?
- **At least volume growth $\neq 4$ at ν_c?**
  Mating of trees ? or another approach ?

The story so far

What we know:

- Convergence in law for the local toplogy.
- The limiting random triangulation has one end almost surely.
- Recurrence of the random walk

What we should now soon :

- Singularity with respect to the UIPT?
- Some information about the cluster's size.

What we would like to know :

- Volume growth ?
- **At least volume growth $\neq 4$ at ν_c?**
  Mating of trees ? or another approach ?

Thank you for your attention!

Summer school **Random trees and graphs**
July 1 to 5, 2019 in Marseille France
Org. M. Albenque, J. Bettinelli, J. Rué and L.Menard

Summer school **Random walks and models of complex networks**
July 8 to 19, 2019 in Nice
Org. B. Reed and D. Mitsche

Thank you for your attention!