Around the Plancherel measure on integer partitions

(an introduction to Schur processes without Schur functions)

Jérémie Bouttier

A subject which I learned with Dan Betea, Cédric Boutillier, Guillaume Chapuy, Sylvie Corteel, Sanjay Ramassamy and Mirjana Vuletić

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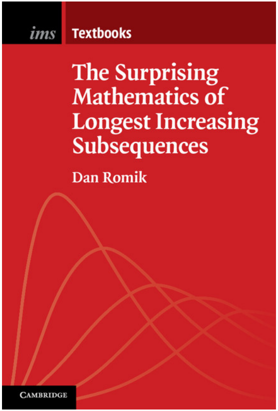

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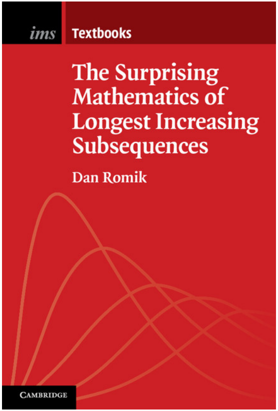

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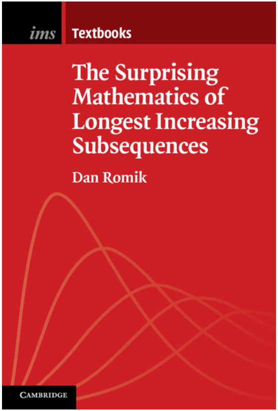

This is the material I would like to present here: fermions because of physics, saddle point computations because, well, we are in Aléa!

Integer partitions and Young diagrams/tableaux

An (integer) partition λ is a finite nonincreasing sequence of positive integers called parts:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0.$$

Its size is $|\lambda| := \sum \lambda_i$ and its length is $\ell(\lambda) := \ell$ (by convention $\lambda_n = 0$ for $n > \ell$).

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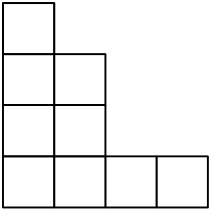

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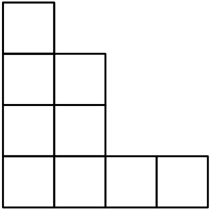

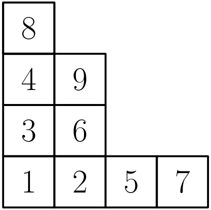

A standard Young tableau (SYT) of shape λ is a filling of the Young diagram of λ by the integers $1, \ldots, |\lambda|$ that is increasing along rows and columns. We denote by d_λ the number of SYTs of shape λ.

Plancherel measure

The Plancherel measure on partitions of size n is the probability measure such that

$$\mathrm{Prob}(\lambda) = \begin{cases} \frac{d_\lambda^2}{n!} & \text{if } \lambda \vdash n, \\ 0 & \text{otherwise.} \end{cases}$$

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which has (at least) two classical proofs:

- representation theory: $n!$ is the dimension of the regular representation of the symmetric group S_n, and d_λ is the dimension of its irreducible representation indexed by λ,
- bijection: the Robinson-Schensted correspondence is a bijection between S_n and the set of triples (λ, P, Q), where $\lambda \vdash n$ and P, Q are two SYTs of shape λ.

Connection with Longest Increasing Subsequences

A property of the Robinson-Schensted correspondence is that if $\sigma \mapsto (\lambda, P, Q)$, then the first part of λ satisfies

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The Longest Increasing Subsequence problem consists in understanding the asymptotic behaviour as $n \to \infty$ of $L_n := L(\sigma_n) = \lambda_1^{(n)}$, where σ_n denotes a uniform random permutation in S_n, and $\lambda^{(n)}$ the random partition to which it maps via the RS correspondence, and whose law is the Plancherel measure.

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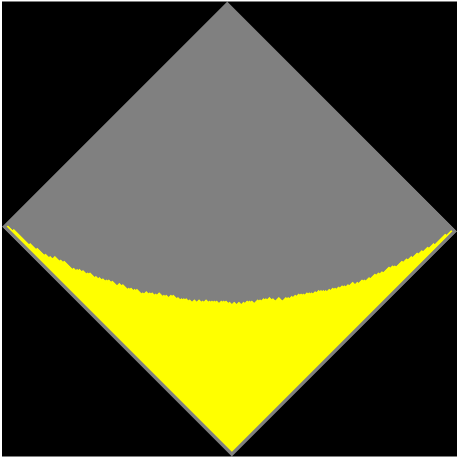
Limit shape



A Plancherel random partition of size 10000 (courtesy of D. Betea)

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- Baik-Deift-Johansson (1999) proved the most precise result

$$\mathbb{P}\left(\frac{L_n - 2\sqrt{n}}{n^{1/6}} \leq s\right) = F_{GUE}(s), \qquad n \to \infty$$

where F_{GUE} is the Tracy-Widow GUE distribution. (See Chapter 2.) The unusual exponent $n^{1/6}$ was previously conjectured by Odlyzko-Rains and Kim based on numerical evidence and bounds.

Topics of the lectures

We will discuss some properties of the Plancherel measure.

1. We will show that the poissonized Plancherel measure (to be defined) is closely related with a determinantal point process (DPP) called the discrete Bessel process. Plan:
   - Some general theory of DPPs
   - Connection with Plancherel measure via fermions
2. We will then investigate asymptotics, in the following regimes:
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These results were obtained indepently in two papers by Borodin, Okounkov and Olshanski (2000) and by Johansson (2001). But we use a different approach developed later by Okounkov *et al.*, which may be generalized to Schur measures and Schur processes. We concentrate on the Plancherel measure for simplicity.

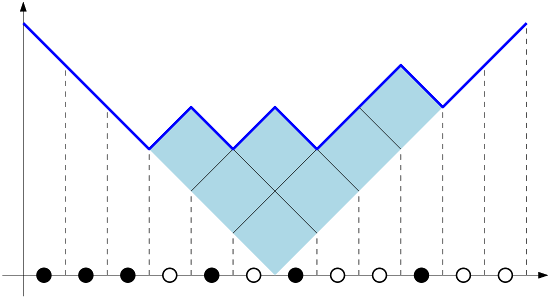
Poissonized Plancherel measure

The poissonized Plancherel measure of parameter θ is the measure

$$\mathrm{Prob}(\lambda) = \frac{d_\lambda^2}{(|\lambda|!)^2} \theta^{|\lambda|} e^{-\theta}.$$

It is a mixture of the Plancherel measures of fixed size, where the size is a Poisson random variable of parameter θ.
We denote by $\lambda^{\langle\theta\rangle}$ a random partition distributed according to the poissonized Plancherel measure, $\lambda^{(n)}$ denoting a Plancherel random partition of size n.

Partitions and particle configurations



To a partition λ, here $(4, 2, 1)$, we associate a set $S(\lambda) \subset \mathbb{Z}' := Z + \frac{1}{2}$ by

$$S(\lambda) = \{\lambda_1 - \frac{1}{2}, \lambda_2 - \frac{3}{2}, \lambda_3 - \frac{5}{2}, \dots\}$$

Here $S(\lambda) = \{\frac{7}{2}, \frac{1}{2}, \frac{-3}{2}, \frac{-7}{2}, \frac{-9}{2}, \dots\}$. Elements of $S(\lambda)$ ("particles" $\bullet$) correspond to the down-steps of the blue curve.

Main result of today

Theorem [Borodin-Okounkov-Olshanski 2000, Johansson 2001]

The particle configuration $S(\lambda^{\langle\theta\rangle})$ associated with the poissonized Plancherel measure is a determinantal point process in the sense that, for any distinct points $\{u_1, \dots, u_n\} \subset \mathbb{Z}'$, we have

$$\mathbb{P}\left(\{u_1, \dots, u_n\} \subset S(\lambda^{\langle\theta\rangle})\right) = \det_{1 \leq i,j \leq n} \mathbf{J}_\theta(u_i, u_j).$$

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The correlation kernel $\mathbf{J}_\theta$ is the discrete Bessel kernel

$$\mathbf{J}_\theta(s,t) = \sum_{\ell \in \mathbb{Z}'_{>0}} J_{s+\ell}(2\sqrt{\theta}) J_{t+\ell}(2\sqrt{\theta}), \qquad s,t \in \mathbb{Z}'$$

where J_n is the Bessel function of order n.

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By the general theory of DPPs, knowing $\mathbf{J}_\theta$ gives all the information on the point process.

Tomorrow

Asymptotics of $\mathbf{J}_\theta$, using saddle point computations. Again this is different from the original techniques of BOO/J, our approach follows Okounkov and Reshetikhin and are robust ("universality").