Additive Combinatorics methods in Fractal Geometry II

Pablo Shmerkin

Department of Mathematics and Statistics
T. Di Tella University and CONICET

Dynamics Beyond Uniform Hyperbolicity, CIRM, May 2019

Review: dynamical self-similarity

Definition

- G is a compact Abelian group, and $h \in G$ is such that the orbit $\{nh : n \in \mathbb{Z}\}$ is dense. We let $T(g) = g + h$.
- $\lambda \in (0, 1)$ is a contraction parameter.
- $\Delta(x) : G \to \mathcal{A}_C^d$ is a map taking values in purely atomic measures in $\mathbb{R}^d$ with at most C atoms.

We call (G, T, λ, Δ) a model. The measures

$$\mu_x = *_{n=1}^{\infty} S_{\lambda^n} \Delta(T^n x)$$

are called dynamical self-similar measures generated by the model.

Review: dynamical self-similarity

Definition

- G is a compact Abelian group, and $h \in G$ is such that the orbit $\{nh : n \in \mathbb{Z}\}$ is dense. We let $T(g) = g + h$.
- $\lambda \in (0, 1)$ is a contraction parameter.
- $\Delta(x) : G \to \mathcal{A}_C^d$ is a map taking values in purely atomic measures in $\mathbb{R}^d$ with at most C atoms.

We call (G, T, λ, Δ) a model. The measures

$$\mu_x = *_{n=1}^{\infty} S_{\lambda^n} \Delta(T^n x)$$

are called dynamical self-similar measures generated by the model.

Review: dynamical self-similarity

Definition

- G is a compact Abelian group, and $h \in G$ is such that the orbit $\{nh : n \in \mathbb{Z}\}$ is dense. We let $T(g) = g + h$.
- $\lambda \in (0, 1)$ is a contraction parameter.
- $\Delta(x) : G \to \mathcal{A}_C^d$ is a map taking values in purely atomic measures in $\mathbb{R}^d$ with at most C atoms.

We call (G, T, λ, Δ) a model. The measures

$$\mu_x = *_{n=1}^{\infty} S_{\lambda^n} \Delta(T^n x)$$

are called dynamical self-similar measures generated by the model.

Review: dynamical self-similarity

Definition

- G is a compact Abelian group, and $h \in G$ is such that the orbit $\{nh : n \in \mathbb{Z}\}$ is dense. We let $T(g) = g + h$.
- $\lambda \in (0, 1)$ is a contraction parameter.
- $\Delta(x) : G \to \mathcal{A}_C^d$ is a map taking values in purely atomic measures in $\mathbb{R}^d$ with at most C atoms.

We call (G, T, λ, Δ) a model. The measures

$$\mu_x = *_{n=1}^{\infty} S_{\lambda^n} \Delta(T^n x)$$

are called dynamical self-similar measures generated by the model.

Review: dynamical self-similarity

Definition

- G is a compact Abelian group, and $h \in G$ is such that the orbit $\{nh : n \in \mathbb{Z}\}$ is dense. We let $T(g) = g + h$.
- $\lambda \in (0, 1)$ is a contraction parameter.
- $\Delta(x) : G \to \mathcal{A}_C^d$ is a map taking values in purely atomic measures in $\mathbb{R}^d$ with at most C atoms.

We call (G, T, λ, Δ) a model. The measures

$$\mu_x = *_{n=1}^{\infty} S_{\lambda^n} \Delta(T^n x)$$

are called dynamical self-similar measures generated by the model.

Review: dynamical self-similarity

Definition

- G is a compact Abelian group, and $h \in G$ is such that the orbit $\{nh : n \in \mathbb{Z}\}$ is dense. We let $T(g) = g + h$.
- $\lambda \in (0, 1)$ is a contraction parameter.
- $\Delta(x) : G \to \mathcal{A}_C^d$ is a map taking values in purely atomic measures in $\mathbb{R}^d$ with at most C atoms.

We call (G, T, λ, Δ) a model. The measures

$$\mu_x = *_{n=1}^{\infty} S_{\lambda^n} \Delta(T^n x)$$

are called dynamical self-similar measures generated by the model.

Further examples of dynamical self-similar measures

1. Self-homothetic measures on $\mathbb{R}^d$: they correspond to $G = \{0\}$, λ the (common) contraction of maps in the IFS and $\Delta = \sum_i p_i \delta_{t_i}$ (where $t_i \in \mathbb{R}^d$ are translations) is built from the translations and the probabilities of the IFS.
2. If μ, ν are two measures as above with contractions λ_1, λ_2, then $\mu * S_{e^x}\nu$ are DSSM where G is a finite group if $\log \lambda_2 / \log \lambda_1 \in \mathbb{Q}$, and the circle otherwise. This extends to

$$\mu_1 * S_{e^{x_2}} \mu_2 * \cdots * S_{e^{x_m}} \mu_m.$$

3. A homogeneous self-similar measure in dimension d is, by definition, a measure of the form

$$\mu = *_{n=1}^{\infty} S_{\lambda^n} O^n \Delta, \quad O \in \mathbb{O}_d, \lambda \in (0,1), \Delta \in \mathcal{A}.$$

It can be realized as a DSSM where $G = \overline{\langle O \rangle}$, $h = O$, $\Delta(g) = g\Delta$.

Further examples of dynamical self-similar measures

1. Self-homothetic measures on $\mathbb{R}^d$: they correspond to $G = \{0\}$, λ the (common) contraction of maps in the IFS and $\Delta = \sum_i p_i \delta_{t_i}$ (where $t_i \in \mathbb{R}^d$ are translations) is built from the translations and the probabilities of the IFS.
2. If μ, ν are two measures as above with contractions λ_1, λ_2, then $\mu * S_{e^x}\nu$ are DSSM where G is a finite group if $\log \lambda_2 / \log \lambda_1 \in \mathbb{Q}$, and the circle otherwise. This extends to

$$\mu_1 * S_{e^{x_2}}\mu_2 * \cdots * S_{e^{x_m}}\mu_m.$$

3. A homogeneous self-similar measure in dimension d is, by definition, a measure of the form

$$\mu = *_{n=1}^{\infty} S_{\lambda^n} O^n \Delta, \quad O \in \mathbb{O}_d, \lambda \in (0,1), \Delta \in \mathcal{A}.$$

It can be realized as a DSSM where $G = \overline{\langle O \rangle}$, $h = O$, $\Delta(g) = g\Delta$.

Further examples of dynamical self-similar measures

1. Self-homothetic measures on $\mathbb{R}^d$: they correspond to $G = \{0\}$, λ the (common) contraction of maps in the IFS and $\Delta = \sum_i p_i \delta_{t_i}$ (where $t_i \in \mathbb{R}^d$ are translations) is built from the translations and the probabilities of the IFS.
2. If μ, ν are two measures as above with contractions λ_1, λ_2, then $\mu * S_{e^x}\nu$ are DSSM where G is a finite group if $\log\lambda_2 / \log\lambda_1 \in \mathbb{Q}$, and the circle otherwise. This extends to

$$\mu_1 * S_{e^{x_2}}\mu_2 * \cdots * S_{e^{x_m}}\mu_m.$$

3. A homogeneous self-similar measure in dimension d is, by definition, a measure of the form

$$\mu = *_{n=1}^{\infty} S_{\lambda^n} O^n \Delta, \quad O \in \mathbb{O}_d, \lambda \in (0,1), \Delta \in \mathcal{A}.$$

It can be realized as a DSSM where $G = \overline{\langle O \rangle}$, $h = O$, $\Delta(g) = g\Delta$.

Discrete approximations and shifted self-similarity

$$\mu_x = *_{n=0}^{\infty} S_{\lambda^n}\Delta(T^n x).$$

We define the discrete step-n approximations

$$\mu_{x,n} = *_{j=0}^{n-1} S_{\lambda^n}\Delta(T^n x).$$

Note that $\mu_{x,n}$ is purely atomic with $\leq \prod_{j=0}^{n-1} |\text{supp}(\Delta(T^j x))| \leq C^n$ atoms.

We then have the following crucial shifted self-similarity relationship:

$$\mu_x = \mu_{x,n} * S_{\lambda^n}\mu_{T^n x}.$$

This says that μ_x is a convex combination of scaled down (by a factor λ^n) translated copies of $\mu_{T^n x}$.

Review: Frostman exponent

Definition

Let μ be a measure on $\mathbb{R}^d$. The Frostman exponent $\dim_\infty(\mu)$ is the supremum of all s such that

$$\mu(B(x,r)) \leq C_s\, r^s \quad \text{for all } r \in (0,1] \text{ and all } x.$$

Review: exponential separation

Definition

We say that a model (G, T, λ, Δ) has exponential separation if for Haar-**almost all** $x \in G$ there is $R > 0$ such that following holds for infinitely many n: the atoms of the discrete approximation

$$\mu_{n,x} = *_{j=1}^{n} S_{\lambda^j} \Delta(T^j x)$$

are (distinct and) e^{-Rn}-separated.

Main Theorem: Frostman exponents of DSSM

Theorem (P.S.)

Let (G, T, λ, Δ) *be a model with exponential separation on* $\mathbb{R}$. *We also assume that the maps* $x \mapsto \Delta(x)$ *and* $x \mapsto \mu_x$ *are continuous a.e., and that* μ_x *is supported on* $[0,1]$. *Let*

$$s = \min\left(\frac{\int \log \|\Delta(x)\|_\infty \, dx}{\log \lambda}, 1\right),$$

where $\|\Delta\|_\infty = \max_y \Delta\{y\}$.

Then $\dim_\infty(\mu_x) = s$ *for every* $x \in G$.

Moreover, for every $\varepsilon > 0$ *there is a constant* C_ε *such that, for all* $x \in G$, $y \in \mathbb{R}$ *and* $r \in (0,1]$,

$$\mu_x(B(y,r)) \le C_\varepsilon r^{s-\varepsilon}$$

Remarks on main theorem

Remarks

- *In the self-similar case (corresponding to constant Δ) a version of this result was obtained by M. Hochman but his version is for Hausdorff dimension rather than Frostman exponents.*
- *While exponential separation has to be checked for almost all x, the conclusion holds for **all** x*
- *The transitive translation on a compact Abelian group can be replaced by a uniquely ergodic transformation on a compact metric space.*

Remarks on main theorem

Remarks

- *In the self-similar case (corresponding to constant Δ) a version of this result was obtained by M. Hochman but his version is for Hausdorff dimension rather than Frostman exponents.*
- *While exponential separation has to be checked for almost all x, the conclusion holds for **all** x*
- *The transitive translation on a compact Abelian group can be replaced by a uniquely ergodic transformation on a compact metric space.*

Remarks on main theorem

Remarks

- *In the self-similar case (corresponding to constant Δ) a version of this result was obtained by M. Hochman but his version is for Hausdorff dimension rather than Frostman exponents.*
- *While exponential separation has to be checked for almost all x, the conclusion holds for **all** x*
- *The transitive translation on a compact Abelian group can be replaced by a uniquely ergodic transformation on a compact metric space.*

Remarks on main theorem

Remarks

- *In the self-similar case (corresponding to constant Δ) a version of this result was obtained by M. Hochman but his version is for Hausdorff dimension rather than Frostman exponents.*
- *While exponential separation has to be checked for almost all x, the conclusion holds for **all** x*
- *The transitive translation on a compact Abelian group can be replaced by a uniquely ergodic transformation on a compact metric space.*

Application 1: Furstenberg's Intersection Conjecture

Theorem (P.S./Meng Wu 2016)

Suppose $\log p/\log q \notin \mathbb{Q}$. *If* A, B *are closed and* T_p, T_q*-invariant, then*

$$\dim_{\mathrm{H}}(A \cap f(B)) \le \overline{\dim}_{\mathrm{B}}(A \cap f(B)) \le \max(\dim_{\mathrm{H}}(A) + \dim_{\mathrm{H}}(B) - 1, 0)$$

for all affine bijections $f : \mathbb{R} \to \mathbb{R}$.

Remark

A T_p*-invariant set* A *can be embedded in a* T_{p^n}*-Cantor set of dimension* $\dim_{\mathrm{H}}(A) + \varepsilon$ *(the allowed digits are the length-*n *sequences appearing in* A*). So it is enough to prove the theorem under the assumption that* A, B *are* p, q*-Cantor sets.*

Application 1: Furstenberg's Intersection Conjecture

Theorem (P.S./Meng Wu 2016)

Suppose $\log p/\log q \notin \mathbb{Q}$*. If A, B are closed and T_p, T_q-invariant, then*

$$\dim_H(A \cap f(B)) \leq \overline{\dim}_B(A \cap f(B)) \leq \max(\dim_H(A) + \dim_H(B) - 1, 0)$$

for all affine bijections $f : \mathbb{R} \to \mathbb{R}$.

Remark

A T_p-invariant set A can be embedded in a T_{p^n}-Cantor set of dimension $\dim_H(A) + \varepsilon$ (the allowed digits are the length-n sequences appearing in A). So it is enough to prove the theorem under the assumption that A, B are p, q-Cantor sets.

Review: convolutions of Cantor measures as DSSM

Let A, B be p, q-Cantor sets, and let μ, ν be the natural measures on them. We saw yesterday that

$$\mu * S_{e^x}\nu = *_{n=1}^{\infty} S_{p^{-n}}\Delta(T^n x),$$

where

$$T(x) = x + \log p \mod \log q$$

and

$$\Delta(x) = \begin{cases} \Delta_A * S_{e^x}\Delta_B & \text{if} \quad x \in [0, \log p) \\ \Delta_A & \text{if} \quad x \in [\log p, \log q) \end{cases} .$$

A corollary of the main result

Corollary

For all $u \in \mathbb{R} \setminus \{0\}$ it holds that

$$\dim_\infty(\mu * S_u \nu) = \min(\dim_H(A) + \dim_H(B), 1) =: s.$$

Remarks

- *All the assumptions in the main theorem are clear except exponential separation, which is a simple lemma.*
- *A small calculation shows that indeed the value of s given by the main theorem is the RHS above.*
- *The main theorem gives the corollary for $u \in [1, \log q]$. Using self-similarity it is easy to expand this to all non-zero u.*

A corollary of the main result

Corollary

For all $u \in \mathbb{R} \setminus \{0\}$ it holds that

$$\dim_\infty(\mu * S_u\nu) = \min(\dim_H(A) + \dim_H(B), 1) =: s.$$

Remarks

- *All the assumptions in the main theorem are clear except exponential separation, which is a simple lemma.*
- *A small calculation shows that indeed the value of s given by the main theorem is the RHS above.*
- *The main theorem gives the corollary for $u \in [1, \log q]$. Using self-similarity it is easy to expand this to all non-zero u.*

A corollary of the main result

Corollary

For all $u \in \mathbb{R} \setminus \{0\}$ it holds that

$$\dim_\infty(\mu * S_u \nu) = \min(\dim_H(A) + \dim_H(B), 1) =: s.$$

Remarks

- *All the assumptions in the main theorem are clear except exponential separation, which is a simple lemma.*
- *A small calculation shows that indeed the value of s given by the main theorem is the RHS above.*
- *The main theorem gives the corollary for $u \in [1, \log q]$. Using self-similarity it is easy to expand this to all non-zero u.*

A corollary of the main result

Corollary

For all $u \in \mathbb{R} \setminus \{0\}$ it holds that

$$\dim_\infty(\mu * S_u \nu) = \min(\dim_H(A) + \dim_H(B), 1) =: s.$$

Remarks

- *All the assumptions in the main theorem are clear except exponential separation, which is a simple lemma.*
- *A small calculation shows that indeed the value of s given by the main theorem is the RHS above.*
- *The main theorem gives the corollary for $u \in [1, \log q]$. Using self-similarity it is easy to expand this to all non-zero u.*

A corollary of the main result

Corollary

For all $u \in \mathbb{R} \setminus \{0\}$ *it holds that*

$$\dim_\infty(\mu * S_u \nu) = \min(\dim_H(A) + \dim_H(B), 1) =: s.$$

Remarks

- *All the assumptions in the main theorem are clear except exponential separation, which is a simple lemma.*
- *A small calculation shows that indeed the value of* s *given by the main theorem is the RHS above.*
- *The main theorem gives the corollary for* $u \in [1, \log q]$. *Using self-similarity it is easy to expand this to all non-zero* u.

Some additional remarks

- Up to a similarity map, $A \cap (rB + t)$ is the same as $(A \times B) \cap \{y = rx + t\}$.
- $\mu(B(x, r)) \approx r^{\dim_H(A)}$ for $x \in A$, and likewise for B, so

$$(\mu \times \nu)(B(x, r)) \approx r^{\dim_H(A)+\dim_H(B)}$$

  for $(x, y) \in A \times B$.
- The convolution $\mu * S_u\nu$ is the push-forward of $\mu \times \nu$ under the projection $\Pi_u(x, y) = x + uy$.

Some additional remarks

- Up to a similarity map, $A \cap (rB + t)$ is the same as $(A \times B) \cap \{y = rx + t\}$.
- $\mu(B(x,r)) \approx r^{\dim_H(A)}$ for $x \in A$, and likewise for B, so

$$(\mu \times \nu)(B(x,r)) \approx r^{\dim_H(A)+\dim_H(B)}$$

  for $(x,y) \in A \times B$.
- The convolution $\mu * S_u \nu$ is the push-forward of $\mu \times \nu$ under the projection $\Pi_u(x,y) = x + uy$.

Some additional remarks

- Up to a similarity map, $A \cap (rB + t)$ is the same as $(A \times B) \cap \{y = rx + t\}$.
- $\mu(B(x, r)) \approx r^{\dim_H(A)}$ for $x \in A$, and likewise for B, so

$$(\mu \times \nu)(B(x, r)) \approx r^{\dim_H(A)+\dim_H(B)}$$

  for $(x, y) \in A \times B$.
- The convolution $\mu * S_u \nu$ is the push-forward of $\mu \times \nu$ under the projection $\Pi_u(x, y) = x + uy$.

Conclusion of the proof

- Let $\{B(x_j,\delta)\}_{j=1}^M$ be disjoint collection of balls intersecting $(A\times B)\cap y=-x/u+t$. We need to bound M from above.
- $$(\mu\times\nu)\left(\cup_{j=1}^M B(x_j,\delta)\right)\gtrsim M\delta^{\dim_H(A)+\dim_H(B)}.$$
- But (since Π_u is Lipschitz and the line $y=-x/u+t$ is the fiber $\Pi_u^{-1}(tu)$)
$$\Pi_u\left(\cup_{j=1}^M B(x_j,\delta)\right)\subset B(tu,C\delta).$$
- Since $\dim_\infty(\Pi_u(\mu\times\nu))=\min(\dim_H(A)+\dim_H(B),1)$, we conclude
$$\begin{aligned}(\mu\times\nu)\left(\cup_{j=1}^M B(x_j,\delta)\right)&\le\Pi_u(\mu\times\nu)\Pi_u\left(\cup_{j=1}^M B(x_j,\delta)\right)\\&\lesssim_\varepsilon\delta^{\min(\dim_H(A)+\dim_H(B),1)-\varepsilon}.\end{aligned}$$

Conclusion of the proof

- Let $\{B(x_j,\delta)\}_{j=1}^M$ be disjoint collection of balls intersecting $(A\times B)\cap y=-x/u+t$. We need to bound M from above.
-
$$(\mu\times\nu)\left(\cup_{j=1}^M B(x_j,\delta)\right)\gtrsim M\delta^{\dim_H(A)+\dim_H(B)}.$$
- But (since Π_u is Lipschitz and the line $y=-x/u+t$ is the fiber $\Pi_u^{-1}(tu)$)
$$\Pi_u\left(\cup_{j=1}^M B(x_j,\delta)\right)\subset B(tu,C\delta).$$
- Since $\dim_\infty(\Pi_u(\mu\times\nu))=\min(\dim_H(A)+\dim_H(B),1)$, we conclude
$$(\mu\times\nu)\left(\cup_{j=1}^M B(x_j,\delta)\right)\le\Pi_u(\mu\times\nu)\Pi_u\left(\cup_{j=1}^M B(x_j,\delta)\right)$$
$$\lesssim_\varepsilon \delta^{\min(\dim_H(A)+\dim_H(B),1)-\varepsilon}.$$

Conclusion of the proof

- Let $\{B(x_j,\delta)\}_{j=1}^M$ be disjoint collection of balls intersecting $(A\times B)\cap y=-x/u+t$. We need to bound M from above.
- $$(\mu\times\nu)\left(\cup_{j=1}^M B(x_j,\delta)\right)\gtrsim M\delta^{\dim_{\mathrm{H}}(A)+\dim_{\mathrm{H}}(B)}.$$
- But (since Π_u is Lipschitz and the line $y=-x/u+t$ is the fiber $\Pi_u^{-1}(tu)$)
$$\Pi_u\left(\cup_{j=1}^M B(x_j,\delta)\right)\subset B(tu,C\delta).$$
- Since $\dim_\infty(\Pi_u(\mu\times\nu))=\min(\dim_{\mathrm{H}}(A)+\dim_{\mathrm{H}}(B),1)$, we conclude
$$(\mu\times\nu)\left(\cup_{j=1}^M B(x_j,\delta)\right)\le\Pi_u(\mu\times\nu)\Pi_u\left(\cup_{j=1}^M B(x_j,\delta)\right)$$
$$\lesssim_\varepsilon \delta^{\min(\dim_{\mathrm{H}}(A)+\dim_{\mathrm{H}}(B),1)-\varepsilon}.$$

Conclusion of the proof

- Let $\{B(x_j,\delta)\}_{j=1}^M$ be disjoint collection of balls intersecting $(A\times B)\cap y=-x/u+t$. We need to bound M from above.
-
$$(\mu\times\nu)\left(\cup_{j=1}^M B(x_j,\delta)\right)\gtrsim M\delta^{\dim_H(A)+\dim_H(B)}.$$
- But (since Π_u is Lipschitz and the line $y=-x/u+t$ is the fiber $\Pi_u^{-1}(tu)$)
$$\Pi_u\left(\cup_{j=1}^M B(x_j,\delta)\right)\subset B(tu,C\delta).$$
- Since $\dim_\infty(\Pi_u(\mu\times\nu))=\min(\dim_H(A)+\dim_H(B),1)$, we conclude
$$(\mu\times\nu)\left(\cup_{j=1}^M B(x_j,\delta)\right)\le\Pi_u(\mu\times\nu)\Pi_u\left(\cup_{j=1}^M B(x_j,\delta)\right)$$
$$\lesssim_\varepsilon \delta^{\min(\dim_H(A)+\dim_H(B),1)-\varepsilon}.$$

Exponential separation in the deterministic case I

Fix $\Delta \in \mathcal{A}, \lambda \in (0,1)$. Let

$$\nu = \nu_{\Delta,\lambda} = *_{n=0}^{\infty} S_{\lambda^n}\Delta.$$

The atoms of

$$\nu_n = *_{j=0}^{n-1} S_{\lambda^j}\Delta$$

are of the form $P(\lambda)$, for a polynomial of degree < 1 with coefficients in $D := \text{supp}(\Delta)$.

Therefore exponential separation holds if and only if there are $R > 0$ and infinitely many n such that

$$|Q(\lambda)| > e^{-Rn}$$

for all polynomials Q of degree $< n$ with coefficients in $D - D$.

In the Bernoulli convolution setting, $D - D = \{-1, 0, 1\}$.

Exponential separation in the deterministic case II

Lemma (M. Hochman 2014)

- *If $D = \mathrm{supp}\Delta$ is algebraic, then there is exponential separation if and only if λ is not a root of a polynomial with coefficients in $D - D$.*
- *For any fixed Δ, there is exponential separation for all λ outside of a set of zero Hausdorff dimension.*

Corollary

- $\dim_\infty(\nu_\lambda) = 1$ *for all algebraic numbers in* $(1/2, 1)$ *which are not roots of a* $\{-1, 0, 1\}$*-polynomial.*
- $\dim_\infty(\nu_\lambda) = 1$ *for all* $\lambda \in (1/2, 1)$ *outside of a set of zero Hausdorff dimension.*

Exponential separation in the deterministic case II

Lemma (M. Hochman 2014)

- *If $D = \text{supp}\Delta$ is algebraic, then there is exponential separation if and only if λ is not a root of a polynomial with coefficients in $D - D$.*
- *For any fixed Δ, there is exponential separation for all λ outside of a set of zero Hausdorff dimension.*

Corollary

- $\dim_\infty(\nu_\lambda) = 1$ *for all algebraic numbers in* $(1/2, 1)$ *which are not roots of a* $\{-1, 0, 1\}$*-polynomial.*
- $\dim_\infty(\nu_\lambda) = 1$ *for all* $\lambda \in (1/2, 1)$ *outside of a set of zero Hausdorff dimension.*

Exponential separation in the deterministic case II

Lemma (M. Hochman 2014)

- *If $D = \text{supp}\Delta$ is algebraic, then there is exponential separation if and only if λ is not a root of a polynomial with coefficients in $D - D$.*
- *For any fixed Δ, there is exponential separation for all λ outside of a set of zero Hausdorff dimension.*

Corollary

- $\dim_\infty(\nu_\lambda) = 1$ *for all algebraic numbers in* $(1/2, 1)$ *which are not roots of a* $\{-1, 0, 1\}$*-polynomial.*
- $\dim_\infty(\nu_\lambda) = 1$ *for all* $\lambda \in (1/2, 1)$ *outside of a set of zero Hausdorff dimension.*

Exponential separation in the deterministic case II

Lemma (M. Hochman 2014)

- *If $D = \mathrm{supp}\Delta$ is algebraic, then there is exponential separation if and only if λ is not a root of a polynomial with coefficients in $D - D$.*
- *For any fixed Δ, there is exponential separation for all λ outside of a set of zero Hausdorff dimension.*

Corollary

- $\dim_\infty(\nu_\lambda) = 1$ *for all algebraic numbers in* $(1/2, 1)$ *which are not roots of a* $\{-1, 0, 1\}$*-polynomial.*
- $\dim_\infty(\nu_\lambda) = 1$ *for all* $\lambda \in (1/2, 1)$ *outside of a set of zero Hausdorff dimension.*

Exponential separation in the deterministic case II

Lemma (M. Hochman 2014)

- *If $D = \text{supp}\Delta$ is algebraic, then there is exponential separation if and only if λ is not a root of a polynomial with coefficients in $D - D$.*
- *For any fixed Δ, there is exponential separation for all λ outside of a set of zero Hausdorff dimension.*

Corollary

- $\dim_\infty(\nu_\lambda) = 1$ *for all algebraic numbers in* $(1/2, 1)$ *which are not roots of a* $\{-1, 0, 1\}$*-polynomial.*
- $\dim_\infty(\nu_\lambda) = 1$ *for all* $\lambda \in (1/2, 1)$ *outside of a set of zero Hausdorff dimension.*

Exponential separation in the deterministic case II

Lemma (M. Hochman 2014)

- *If $D = \mathrm{supp}\Delta$ is algebraic, then there is exponential separation if and only if λ is not a root of a polynomial with coefficients in $D - D$.*
- *For any fixed Δ, there is exponential separation for all λ outside of a set of zero Hausdorff dimension.*

Corollary

- $\dim_\infty(\nu_\lambda) = 1$ *for all algebraic numbers in* $(1/2, 1)$ *which are not roots of a* $\{-1, 0, 1\}$*-polynomial.*
- $\dim_\infty(\nu_\lambda) = 1$ *for all* $\lambda \in (1/2, 1)$ *outside of a set of zero Hausdorff dimension.*

Application 2: Densities of Bernoulli convolutions

Theorem (P.S. 2016)

The BC ν_λ has a density in every L^q for $\lambda \in (1/2, 1)$ outside of a set of exceptions of zero Hausdorff dimension.

Separating ν_λ into large and pseudorandom parts

$$\nu_\lambda = *_{n=0}^{\infty} S_{\lambda^n}\Delta = \left(*_{k|n}^{\infty} S_{\lambda^n}\Delta\right) * \left(*_{k\nmid n}^{\infty} S_{\lambda^n}\Delta\right) =: \nu_{\lambda^k} * \eta_{\lambda,k}.$$

1. Erdős-Kahane: for all $\lambda \in (0,1)$ outside of a set of zero Hausdorff dimension, $|\widehat{\nu}_\lambda(\xi)| \le C_\lambda |\xi|^{-\delta(\lambda)}$: polynomial Fourier decay.
2. The measures η_λ are also homogeneous self-similar measures: the contraction ratio is λ^k and the atomic measure is

$$*_{j=0}^{k=1} S_{\lambda^j}\Delta.$$

If exponential separation holds for ν_λ then it also holds for $\eta_{\lambda,k}$ (fewer atoms to consider, we are skipping some digits). So from the main theorem we have

$$\dim_\infty(\eta_{\lambda,k}) = \min\left(\frac{(k-1)\log 2}{k\log(1/\lambda)}, 1\right).$$

for all $\lambda \in (1/2, 1)$ outside of a zero-dimensional set of exceptions, provided k is taken large enough.

Separating ν_λ into large and pseudorandom parts

$$\nu_\lambda = *_{n=0}^{\infty} S_{\lambda^n}\Delta = \left(*_{k|n}^{\infty} S_{\lambda^n}\Delta\right) * \left(*_{k\nmid n}^{\infty} S_{\lambda^n}\Delta\right) =: \nu_{\lambda^k} * \eta_{\lambda,k}.$$

1. Erdős-Kahane: for all $\lambda \in (0,1)$ outside of a set of zero Hausdorff dimension, $|\widehat{\nu}_\lambda(\xi)| \leq C_\lambda |\xi|^{-\delta(\lambda)}$: polynomial Fourier decay.
2. The measures η_λ are also homogeneous self-similar measures: the contraction ratio is λ^k and the atomic measure is

$$*_{j=0}^{k=1} S_{\lambda^j}\Delta.$$

If exponential separation holds for ν_λ then it also holds for $\eta_{\lambda,k}$ (fewer atoms to consider, we are skipping some digits). So from the main theorem we have

$$\dim_\infty(\eta_{\lambda,k}) = \min\left(\frac{(k-1)\log 2}{k\log(1/\lambda)}, 1\right).$$

for all $\lambda \in (1/2, 1)$ outside of a zero-dimensional set of exceptions, provided k is taken large enough.

Conclusion of the proof

By taking the union of the exceptional sets over all *k*, we get that for $\lambda \in (1/2, 1)$ outside of a set of zero Hausdorff dimension, we can split

$$\nu_\lambda = \nu'_\lambda * \eta_\lambda,$$

where

- ν'_λ has power Fourier decay,
- $\dim_\infty(\eta_\lambda) = 1$.

Proposition (P.S.-B. Solomyak 2016)

If ρ has power Fourier decay and η has full Frostman exponent, then the convolution $\rho * \eta$ is absolutely continuous and the density is in L^q for all finite q.

Conclusion of the proof

By taking the union of the exceptional sets over all *k*, we get that for $\lambda \in (1/2, 1)$ outside of a set of zero Hausdorff dimension, we can split

$$\nu_\lambda = \nu'_\lambda * \eta_\lambda,$$

where

- ν'_λ has power Fourier decay,
- $\dim_\infty(\eta_\lambda) = 1$.

Proposition (P.S.-B. Solomyak 2016)

If ρ has power Fourier decay and η has full Frostman exponent, then the convolution $\rho * \eta$ is absolutely continuous and the density is in L^q for all finite q.

Conclusion of the proof

By taking the union of the exceptional sets over all *k*, we get that for $\lambda \in (1/2, 1)$ outside of a set of zero Hausdorff dimension, we can split

$$\nu_\lambda = \nu'_\lambda * \eta_\lambda,$$

where

- ν'_λ has power Fourier decay,
- $\dim_\infty(\eta_\lambda) = 1$.

Proposition (P.S.-B. Solomyak 2016)

If ρ has power Fourier decay and η has full Frostman exponent, then the convolution $\rho * \eta$ is absolutely continuous and the density is in L^q for all finite q.

L^q dimensions

Definition

Given a probability μ on $\mathbb{R}^d$ and $q \in (1, \infty)$, we let

$$S_n(\mu, q) = \sum_{I \in \mathcal{D}_n} \mu(I)^q,$$

$$\dim_q(\mu) = \liminf_{n \to \infty} \frac{\log S_n(\mu, q)}{n(1-q)} \in [0, d].$$

- $q \mapsto \dim_q(\mu)$ is non-increasing and $\dim_q(\mu) \to \dim_\infty(\mu)$ as $q \to \infty$.
- The main theorem holds not only for Frostman exponents but also for L^q dimensions.
- In the proof it is crucial that $q < \infty$.

L^q dimensions

Definition

Given a probability μ on $\mathbb{R}^d$ and $q \in (1, \infty)$, we let

$$S_n(\mu, q) = \sum_{I \in \mathcal{D}_n} \mu(I)^q,$$

$$\dim_q(\mu) = \liminf_{n \to \infty} \frac{\log S_n(\mu, q)}{n(1-q)} \in [0, d].$$

- $q \mapsto \dim_q(\mu)$ is non-increasing and $\dim_q(\mu) \to \dim_\infty(\mu)$ as $q \to \infty$.
- The main theorem holds not only for Frostman exponents but also for L^q dimensions.
- In the proof it is crucial that $q < \infty$.

L^q dimensions

Definition

Given a probability μ on $\mathbb{R}^d$ and $q \in (1, \infty)$, we let

$$S_n(\mu, q) = \sum_{I \in \mathcal{D}_n} \mu(I)^q,$$

$$\dim_q(\mu) = \liminf_{n \to \infty} \frac{\log S_n(\mu, q)}{n(1-q)} \in [0, d].$$

- $q \mapsto \dim_q(\mu)$ is non-increasing and $\dim_q(\mu) \to \dim_\infty(\mu)$ as $q \to \infty$.
- The main theorem holds not only for Frostman exponents but also for L^q dimensions.
- In the proof it is crucial that $q < \infty$.

L^q dimensions

Definition

Given a probability μ on $\mathbb{R}^d$ and $q \in (1, \infty)$, we let

$$S_n(\mu, q) = \sum_{I \in \mathcal{D}_n} \mu(I)^q,$$

$$\dim_q(\mu) = \liminf_{n \to \infty} \frac{\log S_n(\mu, q)}{n(1-q)} \in [0, d].$$

- $q \mapsto \dim_q(\mu)$ is non-increasing and $\dim_q(\mu) \to \dim_\infty(\mu)$ as $q \to \infty$.
- The main theorem holds not only for Frostman exponents but also for L^q dimensions.
- In the proof it is crucial that $q < \infty$.

Main Theorem: L^q dimensions of DSSM

Theorem (P.S.)

Let (G, T, λ, Δ) be a model with exponential separation on $\mathbb{R}$. We also assume that the maps $x \mapsto \Delta(x)$ and $x \mapsto \mu_x$ are continuous a.e., and that μ_x is supported on $[0, 1]$. Let

$$s(q) = \min\left(\frac{\int \log \|\Delta(x)\|_q^q \, dx}{(q-1)\log\lambda}, 1\right),$$

where $\|\Delta\|_q^q = \sum_y \Delta(y)^q$.

Then

$$\dim_q(\mu_x) = s(q)$$

for every $x \in G$ and $q > 1$.

Tools involved in the proof

1. Additive combinatorics: an inverse theorem for the L^q norm of the convolution of two finitely supported measures(Balog-Szemerédi-Gowers Theorem, Bourgain's additive part of discretized sum-product results).
2. Ergodic theory: key role played by subadditive cocycle over a uniquely ergodic transformation (cocycle borrowed from Nazarov-Peres-S. 2012, uses the proof of the subadditive ergodic theorem given by Katznelson-Weiss).
3. Multifractal analysis (L^q spectrum, regularity at points of differentiability).
4. General scheme of proof follows Mike Hochman's strategy in his landmark paper on the dimensions of self-similar measures, but there are substantial differences.

Tools involved in the proof

1. Additive combinatorics: an inverse theorem for the L^q norm of the convolution of two finitely supported measures(Balog-Szemerédi-Gowers Theorem, Bourgain's additive part of discretized sum-product results).
2. Ergodic theory: key role played by subadditive cocycle over a uniquely ergodic transformation (cocycle borrowed from Nazarov-Peres-S. 2012, uses the proof of the subadditive ergodic theorem given by Katznelson-Weiss).
3. Multifractal analysis (L^q spectrum, regularity at points of differentiability).
4. General scheme of proof follows Mike Hochman's strategy in his landmark paper on the dimensions of self-similar measures, but there are substantial differences.

Tools involved in the proof

1. Additive combinatorics: an inverse theorem for the L^q norm of the convolution of two finitely supported measures(Balog-Szemerédi-Gowers Theorem, Bourgain's additive part of discretized sum-product results).
2. Ergodic theory: key role played by subadditive cocycle over a uniquely ergodic transformation (cocycle borrowed from Nazarov-Peres-S. 2012, uses the proof of the subadditive ergodic theorem given by Katznelson-Weiss).
3. Multifractal analysis (L^q spectrum, regularity at points of differentiability).
4. General scheme of proof follows Mike Hochman's strategy in his landmark paper on the dimensions of self-similar measures, but there are substantial differences.

Tools involved in the proof

1. Additive combinatorics: an inverse theorem for the L^q norm of the convolution of two finitely supported measures(Balog-Szemerédi-Gowers Theorem, Bourgain's additive part of discretized sum-product results).
2. Ergodic theory: key role played by subadditive cocycle over a uniquely ergodic transformation (cocycle borrowed from Nazarov-Peres-S. 2012, uses the proof of the subadditive ergodic theorem given by Katznelson-Weiss).
3. Multifractal analysis (L^q spectrum, regularity at points of differentiability).
4. General scheme of proof follows Mike Hochman's strategy in his landmark paper on the dimensions of self-similar measures, but there are substantial differences.

A submultiplicative cocycle

$$\dim_q(\mu_x) = \lim_{m\to\infty} \frac{\log\left(\sum_{I\in\mathcal{D}_m \mu(I)^q}\right)}{(1-q)n}.$$

Let

$$\psi_n(x) = \sum_{I\in\mathcal{D}_{m(n)}} \mu(I)^q,$$

where $2^{m(n)} \approx \lambda^n$ (so that $|I| \approx \lambda^n$). Then

$$\dim_q(\mu_x) = \lim_{n\to\infty} \frac{\log\psi_n(x)}{(q-1)(\log\lambda)n}.$$

Lemma

$$\psi_{n+k}(x) \le C_q \psi_n(x)\psi_k(T^n x).$$

Use of unique ergodicity

$$\psi_{n+k}(x) \leq C_q \psi_n(x) \psi_k(T^n x).$$

By the subadditive ergodic theorem, there exists $D(q)$ such that

$$\lim_{n\to\infty} \frac{\log \psi_n(x)}{(q-1)(\log \lambda) n} = D(q) \quad \text{for a.e. } x \in G.$$

Lemma (Furman; follows from the Katznelson-Weiss proof of the subadditive ergodic theorem)

$$\liminf_{n\to\infty} \frac{\log \psi_n(x)}{(q-1)(\log \lambda) n} \geq D(q) \quad \textit{uniformly in } x \in G.$$

Reduction from "everywhere" to "almost everywhere"

We need to show that

$$\dim_q(\mu_x) = \lim_{n\to\infty} \frac{\log \psi_n(x)}{(q-1)(\log\lambda)n} = \min\left(\frac{\int \log \|\Delta(x)\|_q^q \, dx}{(q-1)\log\lambda}, 1 \right) =: s(q).$$

The upper bound $\dim_q(\mu, x) \le s(q)$ for all x is easy. But we saw that

$$\liminf_{n\to\infty} \frac{\log \psi_n(x)}{(q-1)(\log\lambda)n} \ge D(q) \text{ for all } x.$$

So it is enough to show that $D(q) = s(q)$. In other words, it is enough to prove that

$$\lim_{m\to\infty} \frac{\log \sum_{I\in\mathcal{D}_m} \mu(I)^q}{(1-q)m} = s(q) \quad \text{for **almost all** } x \in G.$$

End of part II

Merci beaucoup!