Additive Combinatorics methods in Fractal Geometry I

Pablo Shmerkin

Department of Mathematics and Statistics
T. Di Tella University and CONICET

Dynamics Beyond Uniform Hyperbolicity, CIRM, May 2019

Two theorems

- I will start by stating and briefly discussing two results: one concerning intersections of $\times p, \times q$-invariant Cantor sets (Furstenberg's intersection conjecture) and another one concerning absolute continuity and densities of self-similar measures.
- Other than involving self-similarity, these results may appear rather different. We will see that they both follow from a single theorem on the Frostman exponents of dynamical self-similar measures (all of these terms will be introduced).
- The proof of the main theorem relies heavily on additive combinatorics. I will introduce some of the main tools involved.

Two theorems

- I will start by stating and briefly discussing two results: one concerning intersections of $\times p, \times q$-invariant Cantor sets (Furstenberg's intersection conjecture) and another one concerning absolute continuity and densities of self-similar measures.
- Other than involving self-similarity, these results may appear rather different. We will see that they both follow from a single theorem on the Frostman exponents of dynamical self-similar measures (all of these terms will be introduced).
- The proof of the main theorem relies heavily on additive combinatorics. I will introduce some of the main tools involved.

Two theorems

- I will start by stating and briefly discussing two results: one concerning intersections of $\times p, \times q$-invariant Cantor sets (Furstenberg's intersection conjecture) and another one concerning absolute continuity and densities of self-similar measures.
- Other than involving self-similarity, these results may appear rather different. We will see that they both follow from a single theorem on the Frostman exponents of dynamical self-similar measures (all of these terms will be introduced).
- The proof of the main theorem relies heavily on additive combinatorics. I will introduce some of the main tools involved.

Furstenberg's intersection conjecture

Definition ($\times p$ map)

$$T_p(x) = px \bmod 1.$$

Conjecture (H. Furstenberg 1969)

Let $p, q \geq 2$ be integers such that $\log p / \log q$ is irrational. If $A, B \subset \mathbb{R}/\mathbb{Z}$ are closed-invariant under T_p, T_q respectively, then

$$\dim_H(A \cap f(B)) \leq \max(\dim_H(A) + \dim_H(B) - 1, 0)$$

for all non-constant affine maps $f : \mathbb{R} \to \mathbb{R}$.

Furstenberg's intersection conjecture

Definition ($\times p$ map)

$$T_p(x) = px \bmod 1.$$

Conjecture (H. Furstenberg 1969)

Let $p, q \geq 2$ be integers such that $\log p / \log q$ is irrational. If $A, B \subset \mathbb{R}/\mathbb{Z}$ are closed-invariant under T_p, T_q respectively, then

$$\dim_H(A \cap f(B)) \leq \max(\dim_H(A) + \dim_H(B) - 1, 0)$$

for all non-constant affine maps $f : \mathbb{R} \to \mathbb{R}$.

Heuristics

- This conjecture is part of a series of "$\times 2 \times 3$" conjectures quantifying, in the different ways, the principle that "expansions in incommensurable bases have no common structure".
- If A, B are subsets of $\mathbb{R}^d$ "without common structure" then $\mathrm{codim}(A \cap B) = \mathrm{codim}(A) + \mathrm{codim}(B)$ (this holds for example for linear subspaces in general position).
- Marstrand's slice theorem says that for any compact A, B, one has

$$\dim_{\mathrm{H}}(A \cap f(B)) \leq \max(\dim_{\mathrm{H}}(A) + \dim_{\mathrm{H}}(B) - 1, 0)$$

  for almost all f, and this fails for any smaller number on the RHS. The intersection conjecture says that there are no exceptional f (and in particular the identity is not exceptional).

Heuristics

- This conjecture is part of a series of "$\times 2 \times 3$" conjectures quantifying, in the different ways, the principle that "expansions in incommensurable bases have no common structure".
- If A, B are subsets of $\mathbb{R}^d$ "without common structure" then $\mathsf{codim}(A \cap B) = \mathsf{codim}(A) + \mathsf{codim}(B)$ (this holds for example for linear subspaces in general position).
- Marstrand's slice theorem says that for any compact A, B, one has

$$\dim_{\mathrm{H}}(A \cap f(B)) \leq \max(\dim_{\mathrm{H}}(A) + \dim_{\mathrm{H}}(B) - 1, 0)$$

for almost all f, and this fails for any smaller number on the RHS. The intersection conjecture says that there are no exceptional f (and in particular the identity is not exceptional).

Heuristics

- This conjecture is part of a series of "$\times 2 \times 3$" conjectures quantifying, in the different ways, the principle that "expansions in incommensurable bases have no common structure".
- If A, B are subsets of $\mathbb{R}^d$ "without common structure" then $\mathrm{codim}(A \cap B) = \mathrm{codim}(A) + \mathrm{codim}(B)$ (this holds for example for linear subspaces in general position).
- Marstrand's slice theorem says that for any compact A, B, one has

$$\dim_{\mathrm{H}}(A \cap f(B)) \leq \max(\dim_{\mathrm{H}}(A) + \dim_{\mathrm{H}}(B) - 1, 0)$$

  for almost all f, and this fails for any smaller number on the RHS. The intersection conjecture says that there are no exceptional f (and in particular the identity is not exceptional).

Proof of the intersection conjecture

Theorem (P.S. / Meng Wu (independently) 2016)

Furstenberg's intersection conjecture holds.

Bernoulli convolutions

Definition

Let $\lambda \in (0, 1)$. The Bernoulli convolution ν_λ is the distribution of the random sum

$$X_\lambda = \sum_{n=0}^{\infty} \pm\lambda^n,$$

with the choice of signs IID and equiprobable. In other words,

$$\nu_\lambda(A) = \mathbf{P}(X_\lambda \in A).$$

Bernoulli convolutions: basic properties

- If $\lambda \in (0, 1/2)$, the BC ν_λ is the Cantor-Lebesgue measure on a self-similar Cantor set of dimension $\log 2/\log(1/\lambda) < 1$.
- If $\lambda = 1/2$, the BC ν_λ is Lebesgue measure on the interval $[-2, 2]$.
- For $\lambda \in (1/2, 1)$, the topological support of ν_λ is an interval $I_\lambda = [-1/(1-\lambda), 1/(1-\lambda)]$.

Question

What are the properties of ν_λ for $\lambda \in (1/2, 1)$?

Bernoulli convolutions: basic properties

- If $\lambda \in (0, 1/2)$, the BC ν_λ is the Cantor-Lebesgue measure on a self-similar Cantor set of dimension $\log 2/\log(1/\lambda) < 1$.
- If $\lambda = 1/2$, the BC ν_λ is Lebesgue measure on the interval $[-2, 2]$.
- For $\lambda \in (1/2, 1)$, the topological support of ν_λ is an interval $I_\lambda = [-1/(1-\lambda), 1/(1-\lambda)]$.

Question

What are the properties of ν_λ for $\lambda \in (1/2, 1)$?

Bernoulli convolutions: basic properties

- If $\lambda \in (0, 1/2)$, the BC ν_λ is the Cantor-Lebesgue measure on a self-similar Cantor set of dimension $\log 2/\log(1/\lambda) < 1$.
- If $\lambda = 1/2$, the BC ν_λ is Lebesgue measure on the interval $[-2, 2]$.
- For $\lambda \in (1/2, 1)$, the topological support of ν_λ is an interval $I_\lambda = [-1/(1-\lambda), 1/(1-\lambda)]$.

Question

What are the properties of ν_λ for $\lambda \in (1/2, 1)$?

Bernoulli convolutions: basic properties

- If $\lambda \in (0, 1/2)$, the BC ν_λ is the Cantor-Lebesgue measure on a self-similar Cantor set of dimension $\log 2/\log(1/\lambda) < 1$.
- If $\lambda = 1/2$, the BC ν_λ is Lebesgue measure on the interval $[-2, 2]$.
- For $\lambda \in (1/2, 1)$, the topological support of ν_λ is an interval $I_\lambda = [-1/(1-\lambda), 1/(1-\lambda)]$.

Question

What are the properties of ν_λ for $\lambda \in (1/2, 1)$?

Regularity of Bernoulli convolutions: early history

Jessen-Wintner 1935 ν_λ is either purely singular or absolutely continuous.

Erdős 1939 If $1/\lambda$ is a Pisot number (algebraic integer > 1 all of whose Galois conjugates are < 1 in modulus), then ν_λ is purely singular.

Erdős 1940 The BC ν_λ has a density in C^k for **almost all** $\lambda \in (1 - \varepsilon_k, 1)$.

Garsia 1962 If $1/\lambda$ is Pisot, then $\dim_H(\nu_\lambda) < 1$, where

$$\dim_H(\nu) = \inf\{\dim_H(A) : \nu(A) > 0\}.$$

Solomyak 1995 ν_λ is absolutely continuous with an L^2 density for **almost all** $\lambda \in (1/2, 1)$.

Regularity of Bernoulli convolutions: early history

Jessen-Wintner 1935 ν_λ is either purely singular or absolutely continuous.

Erdős 1939 If $1/\lambda$ is a Pisot number (algebraic integer > 1 all of whose Galois conjugates are < 1 in modulus), then ν_λ is purely singular.

Erdős 1940 The *BC* ν_λ has a density in C^k for **almost all** $\lambda \in (1-\varepsilon_k, 1)$.

Garsia 1962 If $1/\lambda$ is Pisot, then $\dim_H(\nu_\lambda) < 1$, where

$$\dim_H(\nu) = \inf\{\dim_H(A) : \nu(A) > 0\}.$$

Solomyak 1995 ν_λ is absolutely continuous with an L^2 density for **almost all** $\lambda \in (1/2, 1)$.

Regularity of Bernoulli convolutions: early history

Jessen-Wintner 1935 ν_λ is either purely singular or absolutely continuous.

Erdős 1939 If $1/\lambda$ is a Pisot number (algebraic integer > 1 all of whose Galois conjugates are < 1 in modulus), then ν_λ is purely singular.

Erdős 1940 The BC ν_λ has a density in C^k for **almost all** $\lambda \in (1-\varepsilon_k, 1)$.

Garsia 1962 If $1/\lambda$ is Pisot, then $\dim_H(\nu_\lambda) < 1$, where

$$\dim_H(\nu) = \inf\{\dim_H(A) : \nu(A) > 0\}.$$

Solomyak 1995 ν_λ is absolutely continuous with an L^2 density for **almost all** $\lambda \in (1/2, 1)$.

Regularity of Bernoulli convolutions: early history

Jessen-Wintner 1935 ν_λ is either purely singular or absolutely continuous.

Erdős 1939 If $1/\lambda$ is a Pisot number (algebraic integer > 1 all of whose Galois conjugates are < 1 in modulus), then ν_λ is purely singular.

Erdős 1940 The *BC* ν_λ has a density in C^k for **almost all** $\lambda \in (1-\varepsilon_k, 1)$.

Garsia 1962 If $1/\lambda$ is Pisot, then $\dim_H(\nu_\lambda) < 1$, where

$$\dim_H(\nu) = \inf\{\dim_H(A) : \nu(A) > 0\}.$$

Solomyak 1995 ν_λ is absolutely continuous with an L^2 density for **almost all** $\lambda \in (1/2, 1)$.

Regularity of Bernoulli convolutions: early history

Jessen-Wintner 1935 ν_λ is either purely singular or absolutely continuous.

Erdős 1939 If $1/\lambda$ is a Pisot number (algebraic integer > 1 all of whose Galois conjugates are < 1 in modulus), then ν_λ is purely singular.

Erdős 1940 The *BC* ν_λ has a density in C^k for **almost all** $\lambda \in (1 - \varepsilon_k, 1)$.

Garsia 1962 If $1/\lambda$ is Pisot, then $\dim_H(\nu_\lambda) < 1$, where

$$\dim_H(\nu) = \inf\{\dim_H(A) : \nu(A) > 0\}.$$

Solomyak 1995 ν_λ is absolutely continuous with an L^2 density for **almost all** $\lambda \in (1/2, 1)$.

Regularity of Bernoulli convolutions: modern history

Solomyak 1995 ν_λ is absolutely continuous with an L^2 density for **almost all** $\lambda \in (1/2, 1)$.

Hochman 2014 There is a set $\mathcal{E}$ of zero-Hausdorff dimension (we will introduce it later) such that if $\lambda \in (1/2, 1) \setminus \mathcal{E}$ then $\dim_H(\nu_\lambda) = 1$.

P.S. 2014 There is a (different) set $\mathcal{E}$ of zero-Hausdorff dimension such that if $\lambda \in (1/2, 1) \setminus \mathcal{E}$ then ν_λ is absolutely continuous.

Varjú 2019 $\dim_H(\nu_\lambda) = 1$ for all transcendental $\lambda \in (1/2, 1)$

Regularity of Bernoulli convolutions: modern history

Solomyak 1995 ν_λ is absolutely continuous with an L^2 density for **almost all** $\lambda \in (1/2, 1)$.

Hochman 2014 There is a set $\mathcal{E}$ of zero-Hausdorff dimension (we will introduce it later) such that if $\lambda \in (1/2, 1) \setminus \mathcal{E}$ then $\dim_H(\nu_\lambda) = 1$.

P.S. 2014 There is a (different) set $\mathcal{E}$ of zero-Hausdorff dimension such that if $\lambda \in (1/2, 1) \setminus \mathcal{E}$ then ν_λ is absolutely continuous.

Varjú 2019 $\dim_H(\nu_\lambda) = 1$ for all transcendental $\lambda \in (1/2, 1)$

Regularity of Bernoulli convolutions: modern history

Solomyak 1995 ν_λ is absolutely continuous with an L^2 density for **almost all** $\lambda \in (1/2, 1)$.

Hochman 2014 There is a set $\mathcal{E}$ of zero-Hausdorff dimension (we will introduce it later) such that if $\lambda \in (1/2, 1) \setminus \mathcal{E}$ then $\dim_H(\nu_\lambda) = 1$.

P.S. 2014 There is a (different) set $\mathcal{E}$ of zero-Hausdorff dimension such that if $\lambda \in (1/2, 1) \setminus \mathcal{E}$ then ν_λ is absolutely continuous.

Varjú 2019 $\dim_H(\nu_\lambda) = 1$ for all transcendental $\lambda \in (1/2, 1)$

Regularity of Bernoulli convolutions: modern history

Solomyak 1995 ν_λ is absolutely continuous with an L^2 density for **almost all** $\lambda \in (1/2, 1)$.

Hochman 2014 There is a set $\mathcal{E}$ of zero-Hausdorff dimension (we will introduce it later) such that if $\lambda \in (1/2, 1) \setminus \mathcal{E}$ then $\dim_H(\nu_\lambda) = 1$.

P.S. 2014 There is a (different) set $\mathcal{E}$ of zero-Hausdorff dimension such that if $\lambda \in (1/2, 1) \setminus \mathcal{E}$ then ν_λ is absolutely continuous.

Varjú 2019 $\dim_H(\nu_\lambda) = 1$ for all transcendental $\lambda \in (1/2, 1)$

Densities of Bernoulli convolutions

Theorem (P.S. 2016)

There is a set $\mathcal{E}$ of zero-Hausdorff dimension such that ν_λ is absolutely continuous and has a density in L^q for all finite q for all $\lambda \in (1/2, 1) \setminus \mathcal{E}$.

Remarks

- *It follows easily from Solomyak's Theorem that ν_λ has a continuous density for a.e. $\lambda \in (1/\sqrt{2}, 1)$. For $\lambda \in (1/2, 1)$, it was not known even known that typically ν_λ has a density in $L^{2+\varepsilon}$ for any $\varepsilon > 0$.*
- *Unfortunately the exceptional set is completely ineffective so it provides no explicit parameters of absolute continuity. The problem of finding explicit parameters will be discussed by P. Varjú in the second part of the course.*

Densities of Bernoulli convolutions

Theorem (P.S. 2016)

There is a set $\mathcal{E}$ of zero-Hausdorff dimension such that ν_λ is absolutely continuous and has a density in L^q for all finite q for all $\lambda \in (1/2, 1) \setminus \mathcal{E}$.

Remarks

- *It follows easily from Solomyak's Theorem that ν_λ has a continuous density for a.e. $\lambda \in (1/\sqrt{2}, 1)$. For $\lambda \in (1/2, 1)$, it was not known even known that typically ν_λ has a density in $L^{2+\varepsilon}$ for any $\varepsilon > 0$.*
- *Unfortunately the exceptional set is completely ineffective so it provides no explicit parameters of absolute continuity. The problem of finding explicit parameters will be discussed by P. Varjú in the second part of the course.*

Densities of Bernoulli convolutions

Theorem (P.S. 2016)

There is a set $\mathcal{E}$ of zero-Hausdorff dimension such that ν_λ is absolutely continuous and has a density in L^q for all finite q for all $\lambda \in (1/2, 1) \setminus \mathcal{E}$.

Remarks

- *It follows easily from Solomyak's Theorem that ν_λ has a continuous density for a.e. $\lambda \in (1/\sqrt{2}, 1)$. For $\lambda \in (1/2, 1)$, it was not known even known that typically ν_λ has a density in $L^{2+\varepsilon}$ for any $\varepsilon > 0$.*
- *Unfortunately the exceptional set is completely ineffective so it provides no explicit parameters of absolute continuity. The problem of finding explicit parameters will be discussed by P. Varjú in the second part of the course.*

Densities of Bernoulli convolutions

Theorem (P.S. 2016)

There is a set $\mathcal{E}$ of zero-Hausdorff dimension such that ν_λ is absolutely continuous and has a density in L^q for all finite q for all $\lambda \in (1/2, 1) \setminus \mathcal{E}$.

Remarks

- *It follows easily from Solomyak's Theorem that ν_λ has a continuous density for a.e. $\lambda \in (1/\sqrt{2}, 1)$. For $\lambda \in (1/2, 1)$, it was not known even known that typically ν_λ has a density in $L^{2+\varepsilon}$ for any $\varepsilon > 0$.*
- *Unfortunately the exceptional set is completely ineffective so it provides no explicit parameters of absolute continuity. The problem of finding explicit parameters will be discussed by P. Varjú in the second part of the course.*

Convolutions of measures

Definition

Given two measures μ, ν on $\mathbb{R}^d$, their convolution is

$$S(\mu \times \nu), \quad S(x, y) = x + y.$$

In other words,

$$\int f \, d(\mu * \nu) = \int f(x + y) \, d\mu(x) d\nu(y).$$

Example

If $\mu = \sum_j p_j \delta_{a_j}$, then

$$\mu * \nu = \sum_j p_j T_{a_j} \nu, \quad T_a(x) = x + a.$$

Convolutions of measures

Definition

Given two measures μ, ν on $\mathbb{R}^d$, their convolution is

$$S(\mu \times \nu), \quad S(x, y) = x + y.$$

In other words,

$$\int f \, d(\mu * \nu) = \int f(x + y) \, d\mu(x) d\nu(y).$$

Example

If $\mu = \sum_j p_j \delta_{a_j}$, then

$$\mu * \nu = \sum_j p_j T_{a_j} \nu, \quad T_a(x) = x + a.$$

Infinite convolutions

- Suppose $(\mu_n)_{n=1}^{\infty}$ are measures supported on $[-a_n, a_n]$, with $\sum_n a_n < \infty$. Then we can likewise define their infinite convolution:

$$\int f \, d(*_{n=1}^{\infty}\mu_n) = \int f\left(\sum_{n=1}^{\infty} x_n\right) d\mu_1(x_1) d\mu_2(x_2) \ldots .$$

- $$*_{n=1}^{N}\mu_n \to *_{n=1}^{\infty}\mu_n$$

  weakly as $N \to \infty$.

Infinite convolutions

- Suppose $(\mu_n)_{n=1}^{\infty}$ are measures supported on $[-a_n, a_n]$, with $\sum_n a_n < \infty$. Then we can likewise define their infinite convolution:

$$\int f\, d(*_{n=1}^{\infty}\mu_n) = \int f\left(\sum_{n=1}^{\infty} x_n\right) d\mu_1(x_1) d\mu_2(x_2) \ldots .$$

- $$*_{n=1}^{N}\mu_n \to *_{n=1}^{\infty}\mu_n$$

  weakly as $N \to \infty$.

Another way of looking at BCs

$$\nu_\lambda \sim \sum_{n=0}^{\infty} \pm \lambda^n.$$

Let

$$\Delta = \tfrac{1}{2}(\delta_{-1} + \delta_1),$$

$$S_a(x) = ax.$$

Then

$$\nu_\lambda = *_{n=0}^{\infty} S_{\lambda^n} \Delta.$$

This holds because the distribution of a sum of independent RVs is the convolution of the distributions.

Homogeneous self-similar measures

We generalize Bernoulli convolutions. Given a finitely supported measure Δ and $\lambda \in (0,1)$, we define

$$\nu_{\Delta,\lambda} = *_{n=0}^{\infty} S_{\lambda^n}\Delta.$$

Remark

Suppose $\Delta = \sum_{j=1}^{m} p_j \delta_{a_j}$. Then $\nu_{\Delta,\lambda}$ can also be defined via the self-similarity relation

$$\nu_{\Delta,\lambda} = \sum_{j=1}^{m} p_j\, S_\lambda(T_{a_j}\nu_{\Delta,\lambda}).$$

Indeed, this follows from

$$\nu_{\Delta,\lambda} = \Delta * (*_{n=1}^{\infty} S_{\lambda^n}\Delta) = \Delta * S_\lambda \nu_{\Delta,\lambda}.$$

Dynamic self-similarity: idea

- $$\nu_{\Delta,\lambda} = *_{n=0}^{\infty} S_{\lambda^n}\Delta.$$
- In order to prove Furstenberg's intersection conjecture, we need to introduce a wider class of infinite convolutions in which Δ depends on n through iteration of a dynamical system.

Definition

Let (X, T) be a dynamical system and suppose we have a map $\Delta : X \to \mathcal{A}$. We define a family of dynamical self-similar measures $(\mu_x)_{x\in X}$ via

$$\mu_x = *_{n=0}^{\infty} S_{\lambda^n}\Delta(T^n x).$$

Dynamic self-similarity: idea

- $$\nu_{\Delta,\lambda} = *_{n=0}^{\infty} S_{\lambda^n}\Delta.$$
- In order to prove Furstenberg's intersection conjecture, we need to introduce a wider class of infinite convolutions in which Δ depends on n through iteration of a dynamical system.

Definition

Let (X, T) be a dynamical system and suppose we have a map $\Delta : X \to \mathcal{A}$. We define a family of dynamical self-similar measures $(\mu_x)_{x\in X}$ via

$$\mu_x = *_{n=0}^{\infty} S_{\lambda^n}\Delta(T^n x).$$

Dynamic self-similarity: idea

- $$\nu_{\Delta,\lambda} = *_{n=0}^{\infty} S_{\lambda^n}\Delta.$$
- In order to prove Furstenberg's intersection conjecture, we need to introduce a wider class of infinite convolutions in which Δ depends on n through iteration of a dynamical system.

Definition

Let (X, T) be a dynamical system and suppose we have a map $\Delta : X \to \mathcal{A}$. We define a family of dynamical self-similar measures $(\mu_x)_{x\in X}$ via

$$\mu_x = *_{n=0}^{\infty} S_{\lambda^n}\Delta(T^n x).$$

Dynamical self-similarity: setting

In our applications, (X, T) turns out to be a transitive group rotation.

Definition

- G is a compact Abelian group, and $h \in G$ is such that the orbit $\{nh : n \in \mathbb{Z}\}$ is dense. We let $T(g) = g + h$.
- $\lambda \in (0, 1)$ is a contraction parameter.
- $\Delta(x) : G \to \mathcal{A}_C^d$ is a map taking values in purely atomic measures in $\mathbb{R}^d$ with at most C atoms.

We call (G, T, λ, Δ) a model. The measures

$$\mu_x = *_{n=1}^{\infty} S_{\lambda^n} \Delta(T^n x)$$

are called dynamical self-similar measures generated by the model.

Dynamical self-similarity: setting

In our applications, (X, T) turns out to be a transitive group rotation.

Definition

- G is a compact Abelian group, and $h \in G$ is such that the orbit $\{nh : n \in \mathbb{Z}\}$ is dense. We let $T(g) = g + h$.
- $\lambda \in (0, 1)$ is a contraction parameter.
- $\Delta(x) : G \to \mathcal{A}_C^d$ is a map taking values in purely atomic measures in $\mathbb{R}^d$ with at most C atoms.

We call (G, T, λ, Δ) a model. The measures

$$\mu_x = *_{n=1}^{\infty} S_{\lambda^n} \Delta(T^n x)$$

are called dynamical self-similar measures generated by the model.

Dynamical self-similarity: setting

In our applications, (X, T) turns out to be a transitive group rotation.

Definition

- G is a compact Abelian group, and $h \in G$ is such that the orbit $\{nh : n \in \mathbb{Z}\}$ is dense. We let $T(g) = g + h$.
- $\lambda \in (0, 1)$ is a contraction parameter.
- $\Delta(x) : G \to \mathcal{A}_C^d$ is a map taking values in purely atomic measures in $\mathbb{R}^d$ with at most C atoms.

We call (G, T, λ, Δ) a model. The measures

$$\mu_x = *_{n=1}^{\infty} S_{\lambda^n} \Delta(T^n x)$$

are called dynamical self-similar measures generated by the model.

Dynamical self-similarity: setting

In our applications, (X, T) turns out to be a transitive group rotation.

Definition

- G is a compact Abelian group, and $h \in G$ is such that the orbit $\{nh : n \in \mathbb{Z}\}$ is dense. We let $T(g) = g + h$.
- $\lambda \in (0, 1)$ is a contraction parameter.
- $\Delta(x) : G \to \mathcal{A}_C^d$ is a map taking values in purely atomic measures in $\mathbb{R}^d$ with at most C atoms.

We call (G, T, λ, Δ) a model. The measures

$$\mu_x = *_{n=1}^{\infty} S_{\lambda^n} \Delta(T^n x)$$

are called dynamical self-similar measures generated by the model.

Dynamical self-similarity: setting

In our applications, (X, T) turns out to be a transitive group rotation.

Definition

- G is a compact Abelian group, and $h \in G$ is such that the orbit $\{nh : n \in \mathbb{Z}\}$ is dense. We let $T(g) = g + h$.
- $\lambda \in (0, 1)$ is a contraction parameter.
- $\Delta(x) : G \to \mathcal{A}_C^d$ is a map taking values in purely atomic measures in $\mathbb{R}^d$ with at most C atoms.

We call (G, T, λ, Δ) a model. The measures

$$\mu_x = *_{n=1}^{\infty} S_{\lambda^n} \Delta(T^n x)$$

are called dynamical self-similar measures generated by the model.

Dynamical self-similarity: setting

In our applications, (X, T) turns out to be a transitive group rotation.

Definition

- G is a compact Abelian group, and $h \in G$ is such that the orbit $\{nh : n \in \mathbb{Z}\}$ is dense. We let $T(g) = g + h$.
- $\lambda \in (0, 1)$ is a contraction parameter.
- $\Delta(x) : G \to \mathcal{A}_C^d$ is a map taking values in purely atomic measures in $\mathbb{R}^d$ with at most C atoms.

We call (G, T, λ, Δ) a model. The measures

$$\mu_x = *_{n=1}^{\infty} S_{\lambda^n} \Delta(T^n x)$$

are called dynamical self-similar measures generated by the model.

p-Cantor sets

Definition

Let $p \geq 2$ and let $D \subset \{0, 1, \ldots, p-1\}$. We define the Cantor set

$$A_{p,D} = \left\{ \sum_{n=1}^{\infty} a_n p^{-n} : a_n \in D \right\}$$

We say that $A_{p,D}$ is a p-Cantor set.

Example

The middle-third Cantor set is a 3-Cantor set corresponding to $D = \{0, 2\}$.

p-Cantor sets

Definition

Let $p \geq 2$ and let $D \subset \{0, 1, \ldots, p-1\}$. We define the Cantor set

$$A_{p,D} = \left\{ \sum_{n=1}^{\infty} a_n p^{-n} : a_n \in D \right\}$$

We say that $A_{p,D}$ is a *p-Cantor set*.

Example

The middle-third Cantor set is a 3-Cantor set corresponding to $D = \{0, 2\}$.

Measures on p-Cantor sets

- Given a p-Cantor set $A = A_{p,D}$ there is a natural Cantor-Lebesgue measure $\mu = \mu_{p,D}$ supported on it. It is the Hausdorff measure (in its dimension $\log |D| / \log p$), the measure of maximal entropy for T_p, and also a self-similar measure:

$$\mu = *_{n=1}^{\infty} S_{p^{-n}} \Delta, \quad \text{where } \Delta = \frac{1}{|D|} \sum_{a \in D} \delta_a.$$

- Our proof of Furstenberg's intersection conjecture goes via the study of convolutions

$$\mu_{p,D} * \mu_{q,D'}$$

for rationally independent p, q. This convolution is not self-similar, but we will now see it fits into the framework of dynamical self-similarity.

Measures on p-Cantor sets

- Given a p-Cantor set $A = A_{p,D}$ there is a natural Cantor-Lebesgue measure $\mu = \mu_{p,D}$ supported on it. It is the Hausdorff measure (in its dimension $\log|D|/\log p$), the measure of maximal entropy for T_p, and also a self-similar measure:

$$\mu = *_{n=1}^{\infty} S_{p^{-n}}\Delta, \quad \text{where } \Delta = \frac{1}{|D|}\sum_{a\in D}\delta_a.$$

- Our proof of Furstenberg's intersection conjecture goes via the study of convolutions

$$\mu_{p,D} * \mu_{q,D'}$$

for rationally independent p, q. This convolution is not self-similar, but we will now see it fits into the framework of dynamical self-similarity.

Dynamical self-similarity: an example I

Let $p < q$. Let $T : [0, \log q) \to [0, \log q)$,

$$T(x) = x + \log p \bmod \log q.$$

Lemma

Let

$$n'(x) = |\{j \in [1, n] : T^j(x) \in [0, \log p)\}|.$$

Then

$$e^{T^n(x)} p^{-n} = e^x q^{-n'(x)}.$$

Proof.

$$T^n(x) = x + n \log p - n'(x) \log q.$$

Dynamical self-similarity: an example I

Let $p < q$. Let $T : [0, \log q) \to [0, \log q)$,

$$T(x) = x + \log p \bmod \log q.$$

Lemma

Let

$$n'(x) = |\{j \in [1, n] : T^j(x) \in [0, \log p)\}|.$$

Then

$$e^{T^n(x)} p^{-n} = e^x q^{-n'(x)}.$$

Proof.

$$T^n(x) = x + n \log p - n'(x) \log q.$$

□

Dynamical self-similarity: an example II

Let $A = A_{p,\Delta_A}$, $B = A_{q,\Delta_B}$ be p, q-Cantor sets.

$$\mu = *_{n=1}^{\infty} S_{p^{-n}} \Delta_A,$$
$$\nu = *_{k=1}^{\infty} S_{q^{-k}} \Delta_B.$$

Recall

$$e^{T^n(x)} p^{-n} = e^x q^{-n'(x)}, \quad n'(x) = |\{j \in [1,n] : T^j(x) \in [0, \log p)\}|.$$

So if we let

$$\Delta(x) = \begin{cases} \Delta_A * S_{e^x} \Delta_B & \text{if} \quad x \in [0, \log p) \\ \Delta_A & \text{if} \quad x \in [\log p, \log q) \end{cases}.$$

then

$$\mu * S_{e^x} \nu = *_{n=1}^{\infty} S_{p^{-n}} \Delta(T^n(x))$$

Frostman exponent

Our main theorem establishes that dynamical self-similar measures are smooth (under certain assumptions). We measure smoothness by Frostman exponents:

Definition

Let μ be a measure on $\mathbb{R}^d$. The Frostman exponent $\dim_\infty(\mu)$ is the supremum of all s such that

$$\mu(B(x,r)) \le C_s\, r^s \quad \text{for all } r \in (0,1] \text{ and all } x.$$

Remark

- *If $\dim_H(\mu) > s$, then $\mu(B(x,r)) \le r^s$ for μ-almost all x.*
- *If $\dim_\infty(\mu) > s$, then $\mu(B(x,r)) \le r^s$ for all x.*

Frostman exponent

Our main theorem establishes that dynamical self-similar measures are smooth (under certain assumptions). We measure smoothness by Frostman exponents:

Definition

Let μ be a measure on $\mathbb{R}^d$. The Frostman exponent $\dim_\infty(\mu)$ is the supremum of all s such that

$$\mu(B(x,r)) \leq C_s\, r^s \quad \text{for all } r \in (0,1] \text{ and all } x.$$

Remark

- *If* $\dim_{\mathrm{H}}(\mu) > s$, *then* $\mu(B(x,r)) \leq r^s$ *for* μ*-almost all* x.
- *If* $\dim_\infty(\mu) > s$, *then* $\mu(B(x,r)) \leq r^s$ *for* ***all*** x.

Frostman exponent

Our main theorem establishes that dynamical self-similar measures are smooth (under certain assumptions). We measure smoothness by Frostman exponents:

Definition

Let μ be a measure on $\mathbb{R}^d$. The Frostman exponent $\dim_\infty(\mu)$ is the supremum of all s such that

$$\mu(B(x,r)) \leq C_s\, r^s \quad \text{for all } r \in (0,1] \text{ and all } x.$$

Remark

- *If* $\dim_{\mathrm{H}}(\mu) > s$*, then* $\mu(B(x,r)) \leq r^s$ *for* μ*-almost all* x.
- *If* $\dim_\infty(\mu) > s$*, then* $\mu(B(x,r)) \leq r^s$ *for* ***all*** x.

Frostman exponent

Our main theorem establishes that dynamical self-similar measures are smooth (under certain assumptions). We measure smoothness by Frostman exponents:

Definition

Let μ be a measure on $\mathbb{R}^d$. The Frostman exponent $\dim_\infty(\mu)$ is the supremum of all s such that

$$\mu(B(x,r)) \leq C_s\, r^s \quad \text{for all } r \in (0,1] \text{ and all } x.$$

Remark

- *If* $\dim_{\mathrm{H}}(\mu) > s$*, then* $\mu(B(x,r)) \leq r^s$ *for* μ*-almost all* x.
- *If* $\dim_\infty(\mu) > s$*, then* $\mu(B(x,r)) \leq r^s$ *for* ***all*** x.

Exponential separation for DSSM

Definition

We say that a model (G, T, λ, Δ) has exponential separation if for Haar-**almost all** $x \in G$ there is $R > 0$ such that following holds for infinitely many n: the atoms of the discrete approximation

$$\mu_{n,x} = *_{j=1}^{n} S_{\lambda^j} \Delta(T^j x)$$

are (distinct and) e^{-Rn}-separated.

Remarks on exponential separation

Definition (Exponential Separation)

The atoms of $\mu_{n,x}$ are e^{-Rn} separated i.o., for almost all x.

- $\mu_{n,x}$ has $\leq C^n$ atoms. Exponential separation requires that they are e^{-Rn}-separated for infinitely many n. So the separation we require is exponentially small compared to the average separation. This is a very weak condition!
- For self-similar measures ($G = \{0\}$) this notion reduces back to the notion of exponential separation introduced by M. Hochman (we'll come back to this). It is hard to check for specific examples.
- On the other hand, if G is infinite then exponential separation is usually almost trivial to check.

Remarks on exponential separation

Definition (Exponential Separation)

The atoms of $\mu_{n,x}$ are e^{-Rn} separated i.o., for almost all x.

- $\mu_{n,x}$ has $\leq C^n$ atoms. Exponential separation requires that they are e^{-Rn}-separated for infinitely many n. So the separation we require is exponentially small compared to the average separation. This is a very weak condition!
- For self-similar measures ($G = \{0\}$) this notion reduces back to the notion of exponential separation introduced by M. Hochman (we'll come back to this). It is hard to check for specific examples.
- On the other hand, if G is infinite then exponential separation is usually almost trivial to check.

Remarks on exponential separation

Definition (Exponential Separation)

The atoms of $\mu_{n,x}$ are e^{-Rn} separated i.o., for almost all x.

- $\mu_{n,x}$ has $\leq C^n$ atoms. Exponential separation requires that they are e^{-Rn}-separated for infinitely many n. So the separation we require is exponentially small compared to the average separation. This is a very weak condition!
- For self-similar measures ($G = \{0\}$) this notion reduces back to the notion of exponential separation introduced by M. Hochman (we'll come back to this). It is hard to check for specific examples.
- On the other hand, if G is infinite then exponential separation is usually almost trivial to check.

Remarks on exponential separation

Definition (Exponential Separation)

The atoms of $\mu_{n,x}$ are e^{-Rn} separated i.o., for almost all x.

- $\mu_{n,x}$ has $\leq C^n$ atoms. Exponential separation requires that they are e^{-Rn}-separated for infinitely many n. So the separation we require is exponentially small compared to the average separation. This is a very weak condition!
- For self-similar measures ($G = \{0\}$) this notion reduces back to the notion of exponential separation introduced by M. Hochman (we'll come back to this). It is hard to check for specific examples.
- On the other hand, if G is infinite then exponential separation is usually almost trivial to check.

Main Theorem: Frostman exponents of DSSM

Theorem (P.S.)

Let (G, T, λ, Δ) *be a model with exponential separation on* $\mathbb{R}$. *We also assume that the maps* $x \mapsto \Delta(x)$ *and* $x \mapsto \mu_x$ *are continuous a.e., and that* μ_x *is supported on* $[0, 1]$. *Let*

$$s = \min\left(\frac{\int \log \|\Delta(x)\|_\infty \, dx}{\log \lambda}, 1\right),$$

where $\|\Delta\|_\infty = \max_y \Delta\{y\}$.

Then $\dim_\infty(\mu_x) = s$ *for **every*** $x \in G$.

Moreover, for every $\varepsilon > 0$ *there is a constant* C_ε *such that, for all* $x \in G$, $y \in \mathbb{R}$ *and* $r \in (0, 1]$,

$$\mu_x(B(y, r)) \leq C_\varepsilon r^{s-\varepsilon}$$