Skew products, thick, unstable and other unexpected attractors

Yulij Ilyashenko
Lumini

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Including results by

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- Viktor Kleptsyn
- Andrei Negut
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Diffeomorphisms of manifolds with boundary onto themselve

Main features of attractors

- Intermingled basins
- Positive Lebesgue measure of attractors and their complement (thick attractors)

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- Lyapunov unstable Milnor attractors
- Topologically non-invariant Milnor attractors
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Milnor attractor

Milnor attractor is the minimal closed set A_M such that

$$d(f^n(p), A_M) \to 0$$

for a.e. p.

Statistical attractor

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$$\frac{1}{n}\Sigma_1^{n-1} d(f^k(p), A_{stat}) \to 0$$

for a.e. p.

Mimimal attractor

Mimimal attractor is the minimal closed set $A_{\min}$ such that

$$\frac{1}{n}\Sigma_1^{n-1} d(f^k(p), A_{\min}) \to 0$$

in measure.

Attractors with intermingled basins

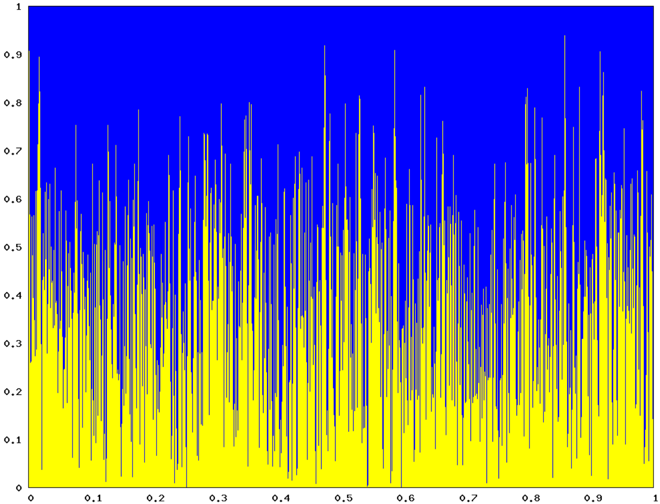
Itai Kan example, 1994

$$X = S^1 \times [0,1],\ F : X \to X$$

$$\partial X = A_0 \cup A_1$$

$$\begin{aligned} F &: (y, x) \mapsto (2y, f_y(x)) \\ f_y(x) &= x + \varepsilon \cos 2\pi y \cdot x(1 - x)) \end{aligned} \tag{1}$$

Intermingled basins



Genericity. Diffeos

Problems

Is it an example or a generic phenomenon? What about diffeomorphisms?

Theorem

Small perturbations of the Itai Kan map still have Milnor attractor

$$A_M = A_0 \cup A_1$$

and basins of A_0 and A_1 are intermingled.

[1] Yu. Ilyashenko, V. Kleptsyn, P. Saltykov, Opennes of the set of boundary preserving maps of an annulus with intermingled attracting basins, JFPTA, electronic publication, 2008

[2] V. Kleptsyn, P. Saltykov, On C^2 persistent intermingled attractors for boundary preserving maps, Trans. Moscow Math. Soc.72:2, pp. 193 – 217, 2011

Theorem

1. There exists a diffeomorphism (skew product over solenoid) with intermingled basins of the Milnor attractor's components [3].
2. This effect persists under small perturbations [2]

[3] Yu. Ilyashenko, Diffeomorphisms with intermingled attracting basins, Functional Analysis and Appl., 42 (2008) no 4

Thick (and fat) attractors

Thick (fat) attractors are Milnor attractors of diffemorphisms (of maps) that have positive, but not full, Lebesgue measure. Example (Ulam-von Newman)

$$x \mapsto 1 - 2x^2$$

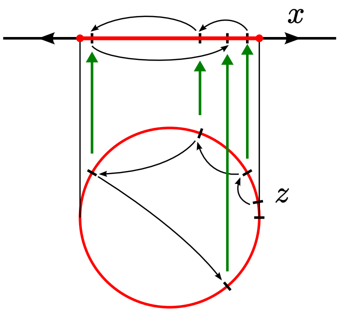


Figure: Projection: $z \mapsto x = \mathrm{Re} z, \; z \mapsto z^2$

Towards thick attractors of diffeomorphisms

A heuristic principle

Effects found for random Dynamic Systems may be observed for diffeomorphisms in higher dimensions.

Random Dynamic Systems

Take two diffeomorphisms $f_j : M \to M$.
Random application of f_0, f_1 is "isomorphic" to a skew product on $X = \Sigma^2 \times M$:

Random Dynamic Systems

$$F : \Sigma^2 \times M \to \Sigma^2 \times M, \\ (\omega, x) \mapsto (\sigma\omega, f_{\omega_0}(x)) \qquad (2)$$

$\Sigma^2 = \{\omega | \omega = \dots \omega_{-n} \dots 0 \dots \omega_n\}$
σ is the Bernoulli shift of ω by one position to the left.

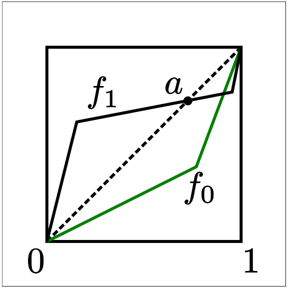
Measure μ on $X : \mu = P \times m$

P is $(\frac{1}{2}, \frac{1}{2})$ Bernoulli measure on Σ^2; m is Lebesgue measure on $I = [0, 1]$.

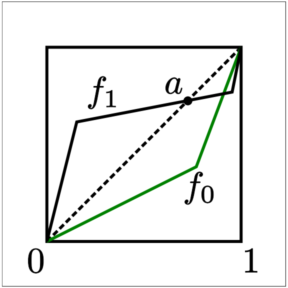
Theorem

$\exists$ a step skew product (2) with Milnor attractor of positive but not full measure.

[4] Yu. Ilyashenko, Thick attractors of step skew products, Regular and Chaotic dynamics, 2010, v. 15, no 2-3, 328-334



- $A_0 = \Sigma^2 \times \{0\}, \ A_1 = \Sigma^2 \times \{1\}$
- A_1 is repelling
- A_0 is repelling in average, because $f_0'(0) \times f_1'(0) > 1$.
- Who is attracting?



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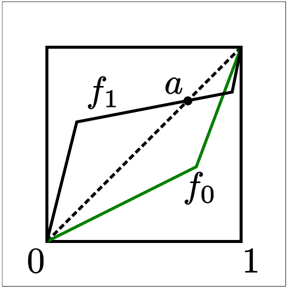

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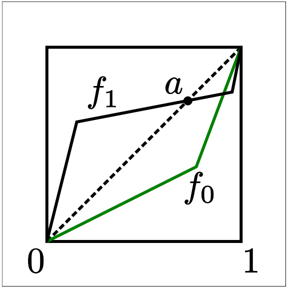

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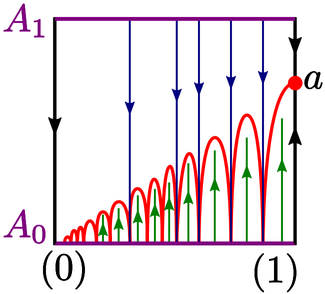
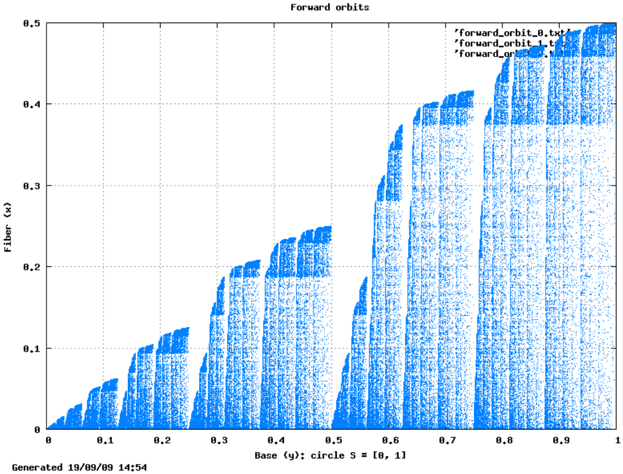


∃ a boundary function

$$\sigma^+ : \Sigma^2 \to [0, a]$$

such that A_M = subgraph of σ^+
$\sigma^+ > 0$ a.e. $\Longrightarrow \mu(A_M) > 0$
$(\sigma^+ = 0)$ is dense in Σ^2.

The domain above $\Gamma = \text{graph } \sigma^+$ lands on $\Gamma \implies \Gamma \subset A_M$.



Case of diffeos

Theorem

$\exists$ a diffeomorphism of $\mathbb{T}^2 \times I$ onto itself with a (similar) thick Milnor attractor. (It is a skew product over an Anosov diffeomorphism $A : \mathbb{T}^2 \to \mathbb{T}^2$)

Main result on thick attractors

An open set with a countable number of hypersurfaces deleted is called quasiopen.

Theorem

$\exists$ a quasiopen set of diffeomorphisms of $\mathbb{T}^2 \times I$ onto itself with a thick Milnor attractor (positive but not full Lebesgue measure).

This quasiopen set belongs to a neighborhood of a diffeo from the previous

Case of diffeos

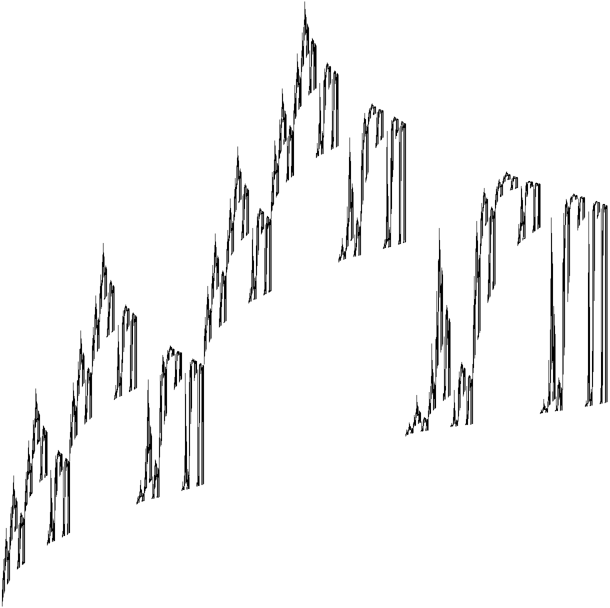
[5] Yu. Ilyashenko, Thick attractors of boundary preservig maps, Indagationes Mathematicae, Volume 22, Issues 3-4, December 2011, Pages 257-314

[6] S. Minkov, A. Okunev, Omega-limit sets of generic points of partially hyperbolic diffeomorphisms, Functional Analysis and Appl.,50:1, pp. 48 – 53, 2016

Bony attractors

Theorem

1. There exist bony attractors of skew products over a Bernoulli shift, with the fiber a segment, that are like the closure of the graph of $\sin \frac{1}{x}$ *[7].*
2. An open set of diffeos with bony attractors exists in the space of maps of a three-torus [8]



Bony attractors

[7] Yu. Kudryashov, Bony attractors, Functional Analysis and Applications, 2010, 44, 3 , 73-76

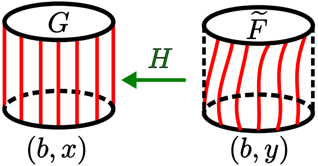
[8] Yu. Kudryashov, Bony attractors and magic billiards, PhD Thesis, 2010-11

HOW TO PASS FROM SKEW PRODUCTS TO DIFFEOS?

Perturbations

A perturbation $\tilde{F}$ of a skew product $F : B \times I \to B \times I$, $(b, x) \mapsto (h(b), f_b(x))$, h hyperbolic is topologically conjugated to a skew product over h:

$$\tilde{F} = H^{-1} \circ G \circ H,$$
$$G : (b, x) \mapsto (h(b), g_b(x))$$
$$H(b, x) = (\beta_b(x), x)$$



Hölder continuity of central foliation

H is not, in general, absolutely continuous.
a.e. for $G \neq$ a.e. for $\tilde{F}$.

Theorem

If h is an Anosov map $\mathbb{T}^2 \to \mathbb{T}^2$ or a solenoid map, then H is Hölder with the exponent close to 1 [?].

[9] Yu. Ilyashenko, A. Negut, Holder properties of perturbed skew products and Fubini regained, Nonlinearity 25 (2012) 2377-2399

[10] Pugh, Shub, Wilkinson, Holder foliations, revisited, JMD 6:1, (2012) 79-120

Hölder and Hausdorff

Hölder maps do not respect the Lebesgue measure, but they do respect the Hausdorff dimension.

Lemma (Falconer)

Let $\varphi : S^1 \to S^1$ be a Hölder homeomorphism with exponent α. Let $K \subset S^1$, $dim_H K = \mathbf{d}$, $\mathbf{d} < \alpha < 1$. Then

$$dim_H \varphi(K) \leq \frac{\mathbf{d}}{\alpha} < 1$$

Special ergodic theorems

Let h be a measure preserving map of B, $\varphi \in C(B)$. Denote

$$\varphi^{-}(b) = \liminf \frac{1}{n}\Sigma_0^{n-1}\varphi(h^k(b))$$

Theorem

[1] Let h be a duplication of a circle, $\varphi \in C(S^1)$, $\int \varphi > 0$. Then $dim_H\{\varphi^- \leq 0\} < 1$.

Special ergodic theorems, contd

Theorem

Let $h : \mathbb{T}^2 \to \mathbb{T}^2$ be an Anosov diffeomorphism. Let $\varphi \in C(\mathbb{T}^2)$, $\int \varphi > 0$. Then $\dim_H\{\varphi^- \leq 0\} < 2$ [9].

D. Ryzhov, S. Minkov and V. Kleptsyn recently improved this result.

[9] P. Saltykov, Special ergodic theorem for Anosov diffeomorphisms on a two-torus, Functional Analysis and Applications, 45, no. 1, 56–63, 2011.

[10] V. Kleptsyn, D. Ryzhov, and S. Minkov. Special ergodic theorems and dynamical large deviations. Nonlinearity, V. 25, N. 11, pp.3189–3196, 2012

Step skew products with interval fibers

Generic skew products named above have a finite number of attracting and repelling sets. Each set is a "bony graph": a graph of a segment-valued function defined on the base, a.e. of these segments is a single point.

Step skew products with interval fibers (contd)

The Milnor attractor A_M of such a skew product is the closure of the "classical graph part" of the surface. The attractor A_M is Lyapunov stable. It is thin (has measure zero).

[11] V. Kleptsyn and D. Volk. Physical measures for nonlinear random walks on interval. Mosc. Math. J., Volume 14, Number 2, pp. 339–365, 2014.

[12] Okunev, I. Shilin, On the attractors of step skew products over the Bernoulli shift, in Proceedings of the Steklov Institute of Mathematics, Volume 297, Issue 1, pp. 235–253, 2017.

Lyapunov unstable Milnor attractors

An attractor A of the map F is Lyapunov stable provided that for any neighborhood V of A there exists a neighborhood W of A such that any point from W never quits V under positive iterates of F.

Theorem

(I. Shilin) There exists an open set U in $Diff^2(M)$ such that for any map from U its Milnor attractor is Lyapunov unstable.

Topologically non-invariant Milnor attractors

There exists a C^1 diffeomorphism of T^2 for which the support of the physical measure is a "thick horseshoe", and has Lebesgue measure zero. This support coincides with the Milnor attractor. Topologically equivalent diffeos of T^2 may have non-homeomorphic Milnor attractors: the "thick horseshoe" and the whole torus.

[13] I. Shilin, Locally topologically generic diffeomorphisms with Lyapunov unstable Milnor attractors. Mosc. Math. J., V. 17, No. 3, pp. 511–553, 2017.

[14] C. Bonatti, S. Minkov, A. Okunev, I. Shilin, Anosov diffeomorphism with a horseshoe that attracts almost any point. arXiv:1802.03977, 2018.

Invisible attractors

[15] Yu. Ilyashenko, A. Negut, Invisible parts of attractors, Nonlinearity, v. 23 (2010) 1199-1219.

This is too much for one talk.

Thanks!