# Möbius conjecture for smooth time changes of horocycle flows

Livio Flaminio

Université de Lille

CIRM, Marseille,
May 13-18, 2019

*joint work with G. Forni*

# Randomness of the Möbius function

The Möbius function: $\forall n \in \mathbb{N}^+$

$$\mu(n) = \begin{cases} (-1)^j & \text{if } n \text{ is the product of } j \text{ distinct primes} \\ 0 & \text{otherwise} \end{cases}$$

**Sarnak's conjecture on Möbius orthogonality (or disjointness), 2010**

The Möbius function is uncorrelated with every topological dynamical system of zero entropy: If $(X, T)$ is a such a system, $\forall f \in C(X)$, $\forall x \in X$

$$\lim_n \frac{1}{n} \sum_{0 \le j < n} f(T^j x)\mu(j) = 0.$$

# Randomness of the Möbius function

**Theorem (Bourgain-Sarnak-Ziegler, 2011)**

*The time-1 map of the horocycle flow on the tangent unit bundle of a hyperbolic surface corroborates Sarnak's conjecture on Möbius orthogonality.*

- $G = \mathrm{SL}_2(\mathbb{R})$, $\Gamma < G$ a lattice, $X = \Gamma \backslash G$
- $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
- $T : X \to X$, $T(x) = xu$, for all $x \in X$.
- Then $(X, T)$ satisfies Sarnak's conjecture.

# A fundamental Lemma

## Proposition

*Let* $(c_n)$ *be a sequence of complex numbers satisfying*

$$\limsup_{\substack{p,q\to\infty \\ p\neq q, primes}} \limsup_{N} \left| \frac{1}{N} \sum_{j=1}^{N} c_{pj} \bar{c}_{qj} \right| = 0.$$

*Then, for any multiplicative function* $\nu : \mathbb{N}^+ \to \mathbb{C}$ *such that* $|\nu| \leq 1$, *we have*

$$\sum_{1\leq j\leq N} c_j \nu(j) = o(N)$$

# AOP property

## Definition (El Abdalaoui, Lemańczyk, de la Rue)

A measure theoretical dynamical system $(X, S, m)$ has *Asymptotic Orthogonal Powers* if, for all $f, g \in L^2(X, m)$ of average 0,

$$\lim_{\substack{p,q\to\infty \\ p\neq q,\text{primes}}} \limsup_{\rho\in J(S^p,S^q)} \left| \int_{X^2} f(x)g(y) \, \mathrm{d}\rho(x,y) \right| = 0.$$

## Theorem (El Abdalaoui, Lemańczyk, de la Rue)

*If $(X, S, m)$ has AOP and it is measure theoretical isomorphic to a totally and uniquely ergodic topological system $(Y, T)$, then the conjecture on Möbius orthogonality holds true for $(Y, T)$. In fact, for any multiplicative function $\nu : \mathbb{N} \to \mathbb{C}$ such that $|\nu| \leq 1$, any $f \in C(Y)$ and any $y \in Y$*

$$\sum_{1\leq j\leq N} f(T^j y)\nu(j) = o(N)$$

# Applications

**Theorem (F., Frączek, Kułaga-Przymus, Lemańczyk)**

*A uniquely ergodic topological system, which is measure theoretically isomorphic to an ergodic affine unipotent diffeomorphism of a compact nil-manifold, satisfies the Möbius orthogonlity conjecture.*

# Horocycle flow

- $G = \mathrm{SL}_2(\mathbb{R})$, $\Gamma < G$ co-compact lattice, $X = \Gamma \backslash G$
- geodesic and the horocycle flows on $X = \Gamma \backslash G$

$$g_s(x) = x \begin{pmatrix} e^{s/2} & \\ & e^{-s/2} \end{pmatrix}, \quad h_t(x) = x \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}, \quad \forall x \in X.$$

- commutation relations

$$g_s \circ h_t = h_{e^{-s}t} \circ g_s, \iff (g_s)_* U_x = e^{-s} U_{g_s(x)}, \quad U_x := \frac{d}{dt}\Big|_{t=0} h_t(x)$$

# Horocycle flow

- $L^2(X) = \bigoplus_{\mu \in \text{spec(Cas)}} H_\mu$, $H_\mu$ irreducible unitary $G$-module.
- $W^s(X) = \bigoplus_{\mu \in \text{spec(Cas)}} W^s(H_\mu), \quad \forall s \in \mathbb{R}$.
- $\mathcal{I}^s(X) := \{D \in W^{-s}(X) : D \circ h_t = D\}$

$$\mathcal{I}^s(X) = \bigoplus_\mu \mathcal{I}^s(H_\mu) =: \bigoplus_\mu \mathcal{I}^s(X) \cap W^{-s}(H_\mu).$$

## Proposition

*For all $s \geq s(\mu)$, $\mathcal{I}^s(H_\mu) = \mathcal{I}^{s(\mu)}(H_\mu)$ and $\dim \mathcal{I}^s(H_\mu) = 2$ or $1$. For all $s \geq s(\mu)$, $\mathcal{I}^s(H_\mu)$ is $g_s$-invariant and, unless $\mu = 1/4$,*

$$\exists D_1, (D_2) \in \mathcal{I}^s(H_\mu) : \quad (g_s)_* D_i = e^{\alpha_i s} D_i, \quad \text{Re}\, \alpha_i < 1.$$

# Horocycle flow

## Example

$$H_\mu \approx \{f \in L^2_{\text{loc}}(\mathbb{R}^2) : f(rx) = r^{-1+iu} f(x), f \text{even}, x \in \mathbb{R}^2, r > 0\}$$

$$\|f\| = \|f_{|S^1}\|_{L^2(S^1)}$$

$$h_t f(x, y) = f(x - ty, y), \quad g_s f(x, y) = f(e^{-s/2}x, e^{s/2}y)$$

$$D_1(f) = f(1, 0),$$

$$D_2(f) = \int_{-\infty}^{+\infty} f(t, 1)\, \mathrm{d}t$$

# Time changes of horocycle flows

## Definition

If $\tau > 0$ is a continuous function on $X$ the *time-changed horocycle flow* $h_t^\tau$, $(t \in \mathbb{R})$, is the flow defined by for $x \in SM$

$$h_t^\tau(x) = h_{w_\tau(x,t)}(x), \quad \int_0^{w_\tau(x,t)} \tau\big(h_u(x)\big)\, \mathrm{d}u = t$$

for all $(x, t) \in X \times \mathbb{R}$. Equivalently $\frac{d}{dt}\Big|_{t=0} h_t^\tau = \tau^{-1} U$.

From $(g_s)_* U_x = e^{-s} U_{g_s(x)}$, we get

$$(g_s)_*(\tau^{-1} U) = e^{-s}(\tau^{-1} \circ g_{-s})\, U,$$

or

$$g_s \circ h_t^\tau = h^{e^s\, \tau\circ g_{-s}} \circ g_s$$

# Time changes of horocycle flows

**Theorem (F., Forni)**

*Let $\tau \in W^s(\Gamma\backslash G)$ be a strictly positive function of Sobolev order $s > 2$, not cohomologous to a constant, and let $h_t^\tau$ be the corresponding time change of the horocycle flow $h_t$. Any topological uniquely ergodic dynamical system measurably conjugated to the time-one map $h_1^\tau$ of the flow $h_t^\tau$ satisfies the Möbius orthogonality conjecture.*

**Theorem A (F., Forni)**

*If, for some $0 < p < q$, there exists a non trivial joining of the flows $h_{pt}^\tau$ and $h_{qt}^\tau$ on $\Gamma\backslash G$, then $\tau$ is cohomologous to a constant.*

# Time changes of horocycle flows

**Theorem (Kanigowski, Lemańczyk, Ulcigrai)**

*If $\tau$ has no components on the discrete spectrum of $G$ in $L^2(\Gamma\backslash G)$ and if, for some $0 < p < q$, there exists a non trivial joining of the flows $h^\tau_{pt}$ and $h^\tau_{qt}$ on $\Gamma\backslash G$, then $\tau$ is cohomologous to a constant.*

# Proof of Theorem A

By [F., Forni: 2003] the proof of Theorem A amounts to showing that the time change function $\tau$ belongs to the kernel of every distribution which is invariant under the horocycle flow and distinct from the unique invariant measure:

$$D(\tau) = 0, \quad \text{for all } D \in (C^\infty(\Gamma\backslash G))', \quad \text{s.t. } D \circ h_t = D, D \neq m.$$

If there is a non-trivial joining $\rho$ of the flow $h^\tau_{q_1 t}$ and $h^\tau_{q_2 t}$ then, setting $\sigma_i = \log q_i$, the measure $(g_{\sigma_1}, g_{\sigma_2})_* \rho$ is a non trivial joining of the flows $h^{\tau_1}_t$ and $h^{\tau_2}_t$, where

$$\tau_1 = \tau \circ g_{-\sigma_1}, \quad \tau_2 = \tau \circ g_{-\sigma_2}$$

# Ratner's Theorem

## Theorem (Ratner)

*In the above setting, if* $\tau_1, \tau_2 \in C^1(\Gamma\backslash G)$ *there exists a finite index subgroup* $\widehat{\Gamma} < \Gamma$ *and two finite* $G$*-equivariant covers*

$$p_1 : \widehat{\Gamma}\backslash G \to \Gamma\backslash G \quad \text{and} \quad p_2 : \widehat{\Gamma}\backslash G \to \Gamma\backslash G$$

*such that the function*

$$\tau_1 \circ p_1 - \tau_2 \circ p_2$$

*is cohomologous to* $0$ *for the flow* $h_t$ *on* $\widehat{\Gamma}\backslash G$.

## Proposition

*If* $\tau_1$, $\tau_2 \in W^s(\Gamma\backslash G)$, *with* $s > 2$, *then the primitive of the function* $\tau_1 \circ p_1 - \tau_2 \circ p_2$ *belongs to the space* $\bigcap_{t<1} W^t(\widehat{\Gamma}\backslash G)$.

# Ratner's Theorem

By Ratner

$$\begin{array}{ccc} & \widehat{X} := \widehat{\Gamma}\backslash G & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ X = \Gamma\backslash G & & \Gamma\backslash G \end{array}$$

$G$-equivariant

and

$$\tau \in W^s(X), s > 2 \implies \widehat{D}(\tau_1 \circ p_1 - \tau_2 \circ g_{\sigma_1 - \sigma_2} \circ p_2) = 0, \ \forall \widehat{D} \in \mathcal{I}^{1+\epsilon}(\widehat{X})$$

$$D(\tau) = ?, \quad D \in \mathcal{I}^{1+\epsilon}(\widehat{X})$$

# Pullback

Set $V_i = (p_i)^* W^s(X)$. Define $(p_i)^* : W^{-s}(X) \to W^{-s}(\widehat{X})$, contraction

$$[(p_i)^* D](f) = \begin{cases} D(g), & f \in V_1, f = g \circ p_i \\ 0, & f \in V_1^{\perp} \end{cases}$$

$$\begin{array}{ccc} V_1 & \hookrightarrow & W^s(\widehat{X}) \\ {\scriptstyle p_1^* \approx} \uparrow & \nearrow {\scriptstyle p_1^*} & \\ W^s(X) & & \end{array} \qquad \begin{array}{ccc} V_1' & \hookrightarrow & W^{-s}(\widehat{X}) \\ {\scriptstyle p_1^*} \updownarrow & \nearrow {\scriptstyle p_1^*} & \\ W^{-s}(X) & & \end{array}$$

Summary:

$$[(p_i)^* D](g \circ p_i) = D(g), \ \forall g \in W^s(X)$$

$(p_i)^*$ isometry on its image.

- $(p_i)_* : W^{-s}(\widehat{\Gamma}\backslash G) \to W^{-s}(\Gamma\backslash G)$, contraction

$$[(p_i)_* D](f) = D(f \circ p_i).$$

- composition $P = (p_2)_* \circ (p_1)^*$ is a contraction.

$$P \;\; G\text{-equivariant} \implies P(\mathcal{I}^s(H_\mu)) \subset \mathcal{I}^s(H_\mu)$$

Let $D \in \mathcal{I}^s(H_\mu)$, $P(D) = \lambda D$, $|\lambda| < 1$

$$\begin{aligned} D(\tau) &= [(p_1)^* D](\tau \circ p_1) = [(p_1)^* D](\tau \circ p_2 \circ g_\sigma) \\ &= e^{\alpha\sigma}[(p_1)^* D](\tau \circ p_2) = e^{\alpha\sigma}[P(D)](\tau) = e^{\alpha\sigma}\lambda D(\tau) \end{aligned}$$

$$|e^{\alpha\sigma}| < 1, |\lambda| < 1 \implies D(\tau) = 0. \qquad QED$$

# Bibliography

J. Bourgain, P. Sarnak and T. Ziegler, Disjointness of Moebius from horocycle flows, in *From Fourier analysis and number theory to Radon transforms and geometry*, vol. 28 of Dev. Math., Springer, New York, 2013, 67–83,

A. Bufetov and G. Forni, Limit theorems for horocycle flows, *Ann. Sci. Éc. Norm. Supér. (4)*, **47** (2014), 851–903,

D. Dolgopyat and O. Sarig, Temporal distributional limit theorems for dynamical systems, *J. Stat. Phys.*, **166** (2017), 680–713,

E. H. El Abdalaoui, M. Lemańczyk and T. de la Rue, On spectral disjointness of powers for rank-one transformations and Möbius orthogonality, *J. Funct. Anal.*, **266** (2014), 284–317,

L. Flaminio and G. Forni, Invariant distributions and time averages for horocycle flows, *Duke Math. J.*, **119** (2003), 465–526,

A. Kanigowski, M. Lemańczyk and C. Ulcigrai, On disjointness properties of some parabolic flows, *https://arxiv.org/abs/1810.11576*, Preprint.

M. Ratner, Rigid reparametrizations and cohomology for horocycle flows, *Invent. Math.*, **88** (1987), 341–374,

P. Sarnak, Three lectures on the Möbius function, randomness and dynamics, 2011,