Beyond Uniform Hyperbolicity 2019

Specification, Lecture 3: Examples!!!!!

Vaughn Climenhaga

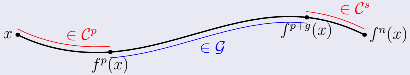
May 17, 2019

Theorem (C.–Thompson 2016)

X compact metric space, $f\colon X \to X$ continuous, $\epsilon = 28\delta > 0$.

Assume: $h^{\perp}_{\exp}(\epsilon) < h_{\text{top}}(f)$, $\quad \sup\{h_\mu : \Gamma_\epsilon(x) \neq \{x\}\ \mu\text{-a.e.}\}$

and $\exists$ a decomposition $\mathcal{C}^p, \mathcal{G}, \mathcal{C}^s$ of $X \times \mathbb{N}$ s.t.



1. $\mathcal{G}^M$ *has specification at scale δ for every $M \in \mathbb{N}$;*
2. $h(\mathcal{C}^p \cup \mathcal{C}^s, \delta) < h_{\text{top}}(f)$. $\quad \overline{\lim}\, \frac{1}{n} \log \#(\mathcal{C}^p_n \cup \mathcal{C}^s_n, \delta)$

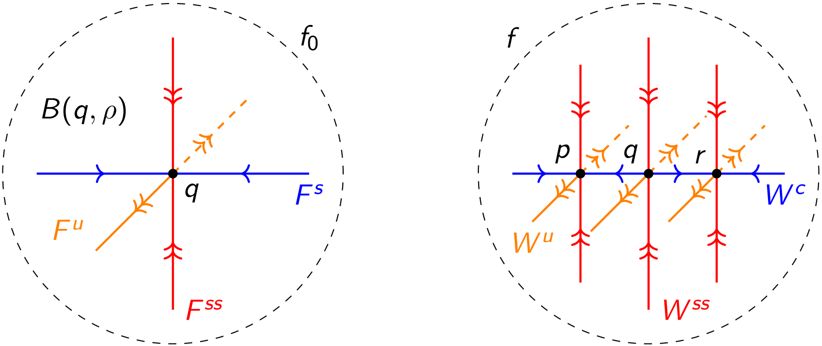
Then (X, f) has a unique measure of maximal entropy.

If you prefer, can use stronger hypotheses:

1. $h^{\perp}_{\exp}(f) = \lim_\epsilon h^{\perp}_{\exp}(\epsilon) < h_{\text{top}}(f)$
2. $\mathcal{G}$ has specification at every scale
3. $h(\mathcal{C}^p \cup \mathcal{C}^s) = \lim_\delta h(\mathcal{C}^p \cup \mathcal{C}^s) < h_{\text{top}}(f)$

Mañé example on $\mathbb{T}^3$

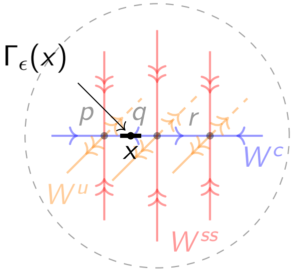
Linear f_0 with $0 < \lambda^{ss} < \lambda^s < 1 < \lambda^u$. Perturb near fixed point q.



Unique MME known: Ures, Buzzi–Fisher–Sambarino–Vásquez

- Let's study anyway! (Our method gives equilibrium states...)

Expansivity for the Mañé example



Any $g \overset{C^1}{\approx} f$ has $E^u \oplus E^c \oplus E^s$.

- All 1-dim, $W^{u,s}$ minimal.
- E^c integrates to W^c.

Bad news: Not expansive!
$\Gamma_\epsilon(x) \neq \{x\}$ when x on $W^c(q)$.

Good news: 'Mostly' expansive...
$\Gamma_\epsilon(x) \subset W^c(x)$ always.

Let $\varphi^c(x) := \log \|Dg|_{E^c(x)}\|$ and $\lambda^c(\mu) = \int \varphi^c \, d\mu$.

Suppose μ ergodic and $\lambda^c(\mu) < -r$. Then for μ-a.e. x:

- average of φ^c is $< -r$ both forward and backward in time;
- average is $< -r/2$ for all $y \in \Gamma_{\epsilon(r)}(x)$, so $\Gamma_\epsilon(x) = \{x\}$.

One-dimensional center: obstructions to expansivity

f **any** partially hyp. diffeo with $\dim E^c = 1$ and E^c integrable

$$\forall r > 0\ \exists \epsilon > 0 \text{ such that } |\lambda^c(\mu)| > r \text{ implies } \mu \text{ almost expansive}$$

$$\Downarrow$$

$$h^{\perp}_{\exp}(f,\epsilon) \leq \sup\{h_\mu(f) : |\lambda^c(\mu)| \leq r,\ \mu \in \mathcal{M}_f^e\}$$

$$\Downarrow$$

$$h^{\perp}_{\exp}(f) = \lim_{\epsilon\to 0} h^{\perp}_{\exp}(f,\epsilon) \leq \sup\{h_\mu(f) : \lambda^c(\mu) = 0,\ \mu \in \mathcal{M}_f^e\}$$

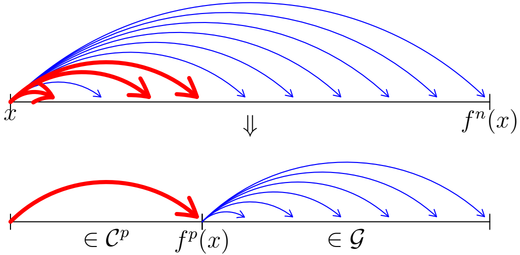
2nd step: Take μ_n with $\lambda^c(\mu_n) \to 0$ and $h^{\perp}_{\exp}(\frac{1}{n}) \leq h_{\mu_n}(f) + \frac{1}{n}$

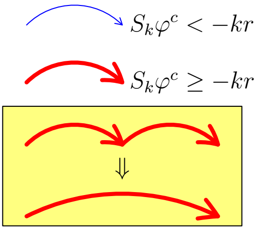
Mañé example: $\lambda^c(\mu) = 0 \Rightarrow$ most weight near q, so small entropy.

One-dimensional center: decomposition for specification

Suppose ‘most’ orbit segments have contracting E^c. Fix $r > 0$,

$$\mathcal{C}^p = \{(x,n) : S_n\varphi^c(x) \geq -nr\}, \quad p(x,n) = \max\{p : (x,p) \in \mathcal{C}^p\}$$





$$\mathcal{G} = \{(x,n) : S_k\varphi^c(x) < -kr \text{ for all } 0 \leq k \leq n\}, \qquad \mathcal{C}^s = \emptyset$$

Entropy of $\mathcal{C}^p$, specification for $\mathcal{G}$

Use $\mathcal{C}^p$ orbit segments $(S_n\varphi^c(x) \geq -nr)$ in "MME construction":

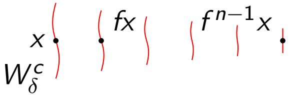
$$\exists \mu \in \mathcal{M}_f \text{ such that } \lambda^c(\mu) \geq -r \quad \text{and} \quad h_\mu(f) \geq h(\mathcal{C}^p).$$

This gives the following lemma:

If $h^+ := \sup\{h_\mu(f) : \lambda^c(\mu) \geq 0\} < h_{\text{top}}(f)$, then $\exists\ r > 0$ s.t.

$$h(\mathcal{C}^p) \leq \sup\{h_\mu(f) : \lambda^c(\mu) \geq -r\} < h_{\text{top}}(f).$$

W^u dense $\Rightarrow \mathcal{G}$ $(\|Df^k|_{E^c}\| \leq e^{-rk}\ \forall k \in [0,n])$ has specification:



$\Rightarrow$ $W^{cs}_\delta(x) \subset B_n(x,\delta)$ for all $(x,n) \in \mathcal{G}$

Conclusions for one-dimensional center

Theorem (Following C.–Fisher–Thompson; applies to Mañé)

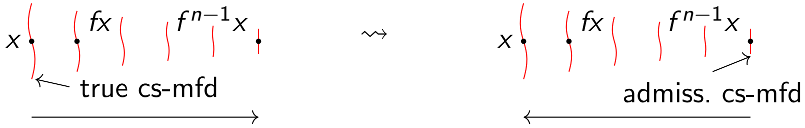
If f is partially hyperbolic with $\dim E^c = 1$, E^c *integrable, all leaves of* W^u *dense, and* $h^+ := \sup\{h_\mu(f) : \lambda^c(\mu) \geq 0\} < h_{\mathrm{top}}(f)$, *then f has a unique measure of maximal entropy.*

Same if W^s *dense and* $h^- := \sup\{h_\mu(f) : \lambda^c(\mu) \leq 0\} < h_{\mathrm{top}}(f)$.

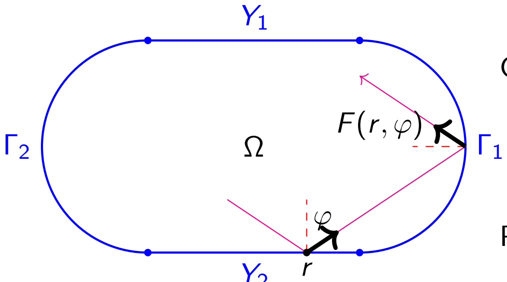
The following are equivalent to "$\min(h^+, h^-) < h_{\mathrm{top}}$":

- $h^+ \neq h^-$
- $P(t\varphi^c)$ does **not** have a minimum at $t = 0$

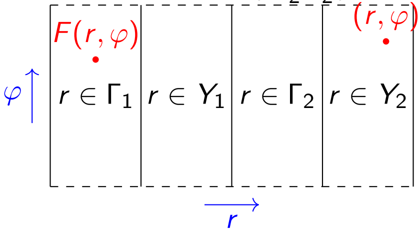
In fact, theorem is true without assuming E^c integrable:



Bunimovich stadium billiard map



$X = \partial\Omega \times (-\frac{\pi}{2}, \frac{\pi}{2})$



Collision info: (r, φ)

- Location: $r \in \partial\Omega$
- Angle with normal: $\varphi \in I = (-\frac{\pi}{2}, \frac{\pi}{2})$

Phase space $X = \partial\Omega \times I$

- Open annulus
- Must compactify
- Obvious choice has 4 discontinuities
- $\overline{X} = X \cup \{2 \text{ fixed points}\} = S^2$
- $F\colon S^2 \to S^2$ has same MMEs

A unique MME for the stadium

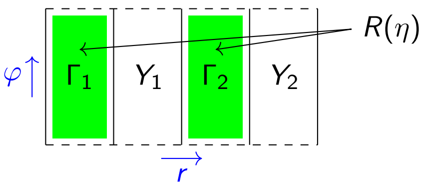
Theorem (Jianyu Chen, V.C., Hong-Kun Zhang – preliminary)

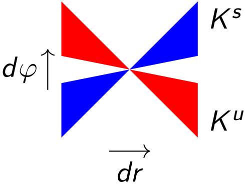
The billiard map for the Bunimovich stadium has a unique MME.

Fix $\eta > 0$, let $\mathcal{G}$ be the set of (x, n) that start and end in

$$R(\eta) := \{(r, \varphi) \in X : d(r, Y) > \eta \text{ and } |\varphi| < \pi/2 - \eta\}$$

and cross the stadium at least once (hit both components of $R(\eta)$)





Lemma: $(x, n) \in \mathcal{G} \Rightarrow DF_x^n(K^u) \subset K^u$ and $(DF_x^n)^{-1}(K^s) \subset K^s$, with uniform expansion estimates.

- This + transitivity is enough to give specification

A decomposition for the stadium

Let $\mathcal{C} = \{(x,n) : F^k(x) \in R(\eta)^c \text{ for all } 0 \le k < n\}$. Then $\mathcal{C}, \mathcal{G}, \mathcal{C}$ is a decomposition:

- $p = p(x,n) \in [0,n]$ minimal such that $F^p(x) \in R(\eta)$;
- $s = s(x,n) \in [0,n]$ minimal such that $F^{n-s}(x) \in R(\eta)$.

μ an "MME" for $\mathcal{C} \Rightarrow \mu(R(\eta)^c) = 1$, so

$$\begin{aligned}\lim_{\eta\to 0} h(\mathcal{C}) &\le \lim_{\eta\to 0} \sup\{h_\mu(F) : \mu(R(\eta)^c) = 1\} \\ &= \sup\{h_\mu(F) : \mu(R(0)^c) = 1\}.\end{aligned}$$

But $R(0)^c$ is just fixed points and period 2 orbits. So $h(\mathcal{C}) \to 0$.

The lemma on $\mathcal{G}$ also shows $h^\perp_{\exp}(F) = 0$.

General result for flows

Theorem (C.–Thompson 2016)

X compact metric space, $f_t\colon X \to X$ continuous flow, $\epsilon > 40\delta > 0$.

Assume: $h^{\perp}_{\exp}(\epsilon) < h_{\mathrm{top}}(f)$. $\sup\{h_\mu : \Gamma_\epsilon(x) \neq f_{[-t,t]}(x)\ \mu\text{-a.e.}\}$

Assume: Decomposition $\mathcal{C}^p, \mathcal{G}, \mathcal{C}^s$ of $X \times \mathbb{R}^+$ such that

1. $\mathcal{G}^M$ *has specification at scale δ for every $M > 0$;*
2. $h(\mathcal{C}^p \cup \mathcal{C}^s, \delta) < h_{\mathrm{top}}(f)$. $\overline{\lim}\, \frac{1}{n} \log \#([\mathcal{C}^p \cup \mathcal{C}^s]_n, \delta)$

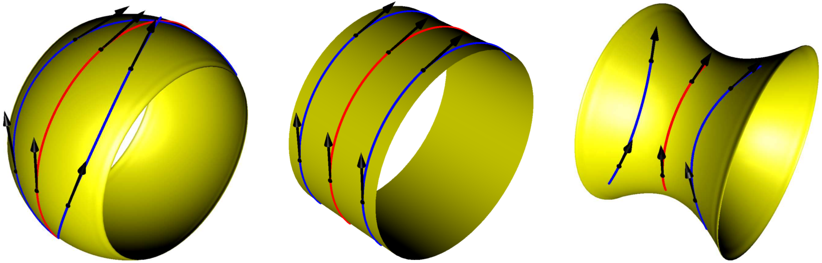
Then $(X, \{f_t\})$ has a unique measure of maximal entropy.

Geodesic flow and curvature

M a closed Riemannian manifold, $f_t \colon T^1M \to T^1M$ geodesic flow

$$v \in T^1M \;\rightsquigarrow\; c_v \text{ geodesic with } \dot{c}_v(0) = v \;\rightsquigarrow\; f_t(v) := \dot{c}_v(t)$$

Hyperbolicity associated to curvature: $K < 0 \Rightarrow$ Anosov

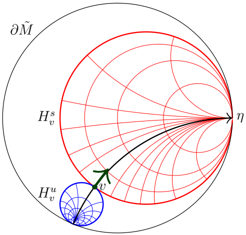


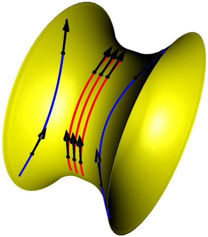
$K > 0$ $\quad$ $K = 0$ $\quad$ $K < 0$

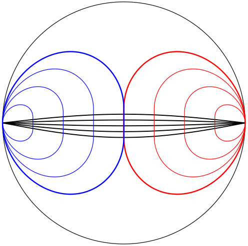
Horospheres as tool for studying hyperbolicity

Work in universal cover $\widetilde{M}$ and use ideal boundary $\partial\widetilde{M}$.

Negative curvature: $W^{s,u}(v) \leftrightarrow$ normal vector fields to horospheres

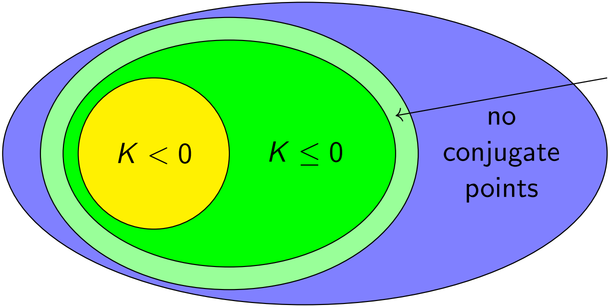






Works under weaker conditions, but horospheres may have higher-order tangencies, or even overlap nontrivially. Corresponds to zero angle (or nontrivial intersection) between W^s and W^u.

Hierarchy of hyperbolicity conditions for geodesic flows



all inclusions proper

No focal points (NFP): balls in $\widetilde{M}$ are convex

No conjugate points (NCP): $p \neq q \in \widetilde{M}$ determine unique geodesic

For NCP, bijection between $T^1M \times (0, \infty)$ and $(\widetilde{M}^2 - \mathrm{diag})/\pi_1 M$

Gallery of uniqueness results for geodesic flows

Negative curvature: Bowen–Margulis (1970s), and $\varphi \neq 0$

Nonpositive curvature: Knieper (1998), only MME

Nonpositive curvature again:
Burns–C.–Fisher–Thompson (2018)

Some $\varphi \neq 0$: Dan's talks next week will have more details

No focal points:

Katrin Gelfert, Rafael Ruggiero (published 2019): dim 2, MME
Fei Liu, Fang Wang, Weisheng Wu (arXiv 2018): any dim, MME
Dong Chen, Nyima Kao, Kiho Park (arXiv 2018): dim 2, some φ

Theorem (V.C., Gerhard Knieper, Khadim War, 2019-arXiv)

Let M be a surface of genus ≥ 2 without conjugate points. Then the geodesic flow on T^1M has a unique MME.

Comparison of hyperbolicity conditions

things in $\widetilde{M}$	$K < 0$	$K \leq 0$	NFP	NCP
$t \mapsto d(c_1(t), c_2(t))$ when $c_1(0) = c_2(0)$	strictly convex	convex	monotonic	positive
horospheres	str. cvx	convex		???
$v \mapsto E^{s,u}_v = T_v W^{s,u}_v$	Hölder	continuous		???
$c_1(\pm\infty) = c_2(\pm\infty)$	$c_1 = c_2$	flat strip		???

Comparison of hyperbolicity conditions

things in $\widetilde{M}$	$K < 0$	$K \leq 0$	NFP	NCP
$t \mapsto d(c_1(t), c_2(t))$ when $c_1(0) = c_2(0)$	strictly convex	convex	monotonic	positive
horospheres	str. cvx	convex		???
$v \mapsto E_v^{s,u} = T_v W_v^{s,u}$	Hölder	continuous		???
$c_1(\pm\infty) = c_2(\pm\infty)$	$c_1 = c_2$	flat strip		???

It looks like all of our tools have vanished! What are we to do? Is anything left? In dimension 2, genus ≥ 2 we have the following:

- $\widetilde{M}$ is a disc, every $p \neq q$ connected by a unique geodesic
- $\partial\widetilde{M}$ still makes sense, as do horospheres
- $h_\mu > 0 \Rightarrow W_v^s \cap W_v^u$ trivial μ-a.e. $\Rightarrow H_v^s \cap H_v^u$ trivial
- if $w \in \Gamma_{\frac{1}{3}\,\mathrm{inj}\,M}(v)$, then lifting gives same in $\widetilde{M}$, so either $w \in \{\dot{c}_v(t)\}_t$, or $H_v^s \cap H_v^u$ nontrivial: thus $h^{\perp}_{\exp}(\frac{1}{3}\,\mathrm{inj}\,M) = 0$
- there is a *different* metric g_0 with negative curvature...

No conjugate points: Morse lemma and specification

Morse lemma: Let g, g_0 be two metrics on M such that g_0 has negative curvature and g has no conjugate points. Then there is $R > 0$ such that all $p, q \in \widetilde{M}$, the g-geodesic and g_0-geodesic connecting p to q are within Hausdorff distance R of each other.

$$\begin{tikzcd}
& (\widetilde{M}^2 - \mathrm{diag})/\pi^1 M \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\begin{array}{c} T^1 M \times (0,\infty) \\ g\text{-orbit segments} \end{array} \arrow[rr, leftrightarrow, "R\text{-shadowing}"] & & \begin{array}{c} T^1 M \times (0,\infty) \\ g_0\text{-orbit segments} \end{array}
\end{tikzcd}$$

Now given orbit segments $(x_1, t_1), \ldots, (x_k, t_k)$ for g,

- R-shadow each one by an orbit segment for g_0;
- R-shadow this list by a single g_0 orbit segment (g_0-spec.);
- R-shadow this single orbit segment by a g-orbit segment.

Thus the g-geodesic flow has specification at scale $(\approx)$ $3R$

Salvation by residual finiteness

Surface M of genus ≥ 2 with no conjugate points:

- the geodesic flow has $h^{\perp}_{\mathrm{exp}}(\frac{1}{3}\,\mathrm{inj}\,M) = 0 < h_{\mathrm{top}}$;
- the flow has specification at scale $3R$. (R from Morse)

If $40 \cdot 3R < \frac{1}{3}\,\mathrm{inj}\,M$, then the general theorem gives a unique MME.

But we have no reason to expect this... probably R is very large.

Salvation by residual finiteness

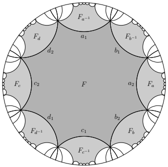
Surface M of genus ≥ 2 with no conjugate points:

- the geodesic flow has $h^{\perp}_{\exp}(\frac{1}{3}\operatorname{inj} M) = 0 < h_{\mathrm{top}}$;
- the flow has specification at scale $3R$. (R from Morse)

If $40 \cdot 3R < \frac{1}{3}\operatorname{inj} M$, then the general theorem gives a unique MME.

But we have no reason to expect this... probably R is very large.

Solution: Replace M with a finite cover N with $\operatorname{inj} N > 360R$.



- Entropy-preserving bijection between $\mathcal{M}_f(T^1 M)$ and $\mathcal{M}_f(T^1 N)$
- Theorem gives unique MME on $T^1 N$
- Thus there is a unique MME on $T^1 M$

Why possible? dim $M = 2$ implies $\pi_1(M)$ is *residually finite.*

Higher dimensions and open questions

Method works for higher-dim M with no conjugate points **if**

1. $\exists$ Riemannian metric g_0 on M with negative curvature;
2. divergence property: $c_1(0) = c_2(0) \Rightarrow d(c_1(t), c_2(t)) \to \infty$;
3. $\pi_1(M)$ is residually finite;
4. $\exists h^* < h_{\mathrm{top}}$ such that if μ-a.e. v has non-trivially overlapping horospheres, then $h_\mu \leq h^*$.

First is a real topological restriction: rules out Gromov example.

Second and third might be redundant? No example satisfying (1) where they are known to fail

Fourth is true if $\{v : H_v^s \cap H_v^u \text{ trivial}\}$ contains an open set. Unclear if this is always true.

Higher dimensions and open questions

Method works for higher-dim M with no conjugate points **if**

1. $\exists$ Riemannian metric g_0 on M with negative curvature;
2. divergence property: $c_1(0) = c_2(0) \Rightarrow d(c_1(t), c_2(t)) \to \infty$;
3. $\pi_1(M)$ is residually finite;
4. $\exists h^* < h_{\text{top}}$ such that if μ-a.e. v has non-trivially overlapping horospheres, then $h_\mu \leq h^*$.

First is a real topological restriction: rules out Gromov example.

Second and third might be redundant? No example satisfying (1) where they are known to fail

Fourth is true if $\{v : H^s_v \cap H^u_v \text{ trivial}\}$ contains an open set. Unclear if this is always true.

What about $\varphi \neq 0$? Not clear how to extend these techniques.