RIGIDITY LECTURE NOTES III

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This lecture is entirely based on a joint work with F. Rodriguez Hertz, under preparation.

1. **Otal-Croke marked length spectrum rigidity.** *Let (S, g_1) and (S, g_2) be negatively curved surfaces. Denote by $[\gamma]$ the free homotopy class loops $S^1 \to S$. For any non-trivial $[\gamma]$ consider the unique geodesic representatives $\gamma_1 \in [\gamma]$ and $\gamma_2 \in [\gamma]$ for g_1 and g_2, correspondingly and assume that*

$$\ell_{g_1}(\gamma_1) = \ell_{g_2}(\gamma_2) \tag{$\blacklozenge$}$$

Then there exists an isometry $\sigma \colon (S, g_1) \to (S, g_2)$.

2. **Khalil-Lafont question.** *Consider additional data: two positive smooth functions $\varphi_1, \varphi_2 \colon S \to \mathbb{R}$. Instead of ($\blacklozenge$) assume that*

$$\forall[\gamma] \quad \int_{\gamma_1} \varphi_1(\gamma(t))dt = \int_{\gamma_2} \varphi_2(\gamma(t))dt \tag{$\heartsuit$}$$

Does it follow that (S, g_1) and (S, g_2) are homothetic? That is, does there exists a constant $c > 0$ and an isometry $\sigma \colon (S, g_1) \to (S, c^2 g_2)$.
Note that if $\varphi_1 = \varphi_2 = 1$ then this is precisely MLS rigidity. Also note that if $g_1 = c^2 g_2$ then $(\varphi_1, \varphi_2) = (\varphi, c\varphi)$ verify ($\heartsuit$).

3. **Sharpened MLS rigidity.** *Let (S, g_1) and (S, g_2) be negatively curved surfaces. Let $\varphi_i : T^1 S \to \mathbb{R}$ be smooth functions such that φ_1 is not an abelian coboundary. Assume that for every homologically trivial homotopy class of loops $[\gamma] \in \pi_1(S)$, $[\gamma] \neq 0$, we have*

$$\int_{\gamma_1} \varphi_1 = \int_{\gamma_2} \varphi_2$$

Then there exists a constant $c > 0$ and an isometry $\sigma \colon (S, g_1) \to (S, c^2 g_2)$.

4. **Examples of non abelian coboundaries.** Let X be an Anosov vector field on a closed manifold M. A Hölder continuous function $\varphi \colon M \to \mathbb{R}$ is called an *abelian coboundary* if there exists a closed 1-form ω and a Hölder function (differentiable along X) such that

$$\varphi = Xu + \omega(X)$$

Notice that the decomposition $\varphi = Xu + \omega(X)$ is highly non-unique because we can change ω by any an exact 1-form. Indeed given any smooth function $v \colon M \to \mathbb{R}$ we can write a different decomposition

$$\varphi = (\omega + dv)(X) + X(u - v)$$

If X is the geodesic flow on (S, g_1) then there two classes of non abelian coboundaries (to which Sharpened MLS rigidity would apply).

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1. Function $\varphi\colon T^1S\to\mathbb{R}$ is not an abelian coboundary if it is non-negative and takes at least one positive value.
2. Function $\varphi\colon T^1S\to\mathbb{R}$ is not an abelian coboundary if it is a pullback of a non-zero function on the surface, in other words, $\varphi(v,x)=\varphi(x)$.

5. **An example without rigidity.** Let S be a surface equipped with a cohomologically non-trivial closed $1-$form $\omega\colon TS\to\mathbb{R}$. Consider two non-isometric Riemannian metrics g_1 and g_2 on S. Then the corresponding unit tangent bundles are naturally embedded in the full tangent bundle $T^1_{g_i}S\subset TS$ and we can define $\varphi_i: T^1_{g_i}S\to\mathbb{R}$ by $\varphi_i(x,v)=\omega_x(v)$. Let $\gamma_i\subset T^1_{g_i}S$ be homotopic unit-speed closed g_i-geodesics. Then

$$\int_{\gamma_1}\varphi_1(\gamma_1(s))ds=\int_{\gamma_1}\omega(\gamma_1(s))ds=\langle[\omega],[\gamma]\rangle=\int_{\gamma_2}\omega(\gamma_2(s))ds=\int_{\gamma_2}\varphi_2(\gamma_2(s))ds$$

where $[\omega]$ is the cohomology class of ω and $[\gamma]$ is the homology class of γ_i, $i=1,2$.

6. **Abelian Livshits theorem.** We follow Sharp and say that a transitive Anosov flow $X^t\colon M\to M$ is *homologically full* if every integral homology class contains a closed orbit of X^t.

Theorem. *Assume that $X^t\colon M\to M$ is a homologically full transitive Anosov flow and let $\varphi\in C^r(M)$, $r>0$, $\varphi: M\to\mathbb{R}$ such that*

$$\int_\gamma\varphi=0$$

for all homologically trivial closed orbits γ. Then there is a C^∞ smooth closed 1-form ω on M and a function $u\in C^{r-\epsilon}(M)$, where $\epsilon>0$ is arbitrarily small, such that

$$\varphi=Xu+\omega(X)$$

Proof. By work of Sharp homologically trivial orbits equidistribute according to a certain equilibrium state. In particular, it follows that the homologically trivial orbits are dense. However, one can avoid using Sharp's machinery and give a simpler proof by using shadowing.

Let $\hat M$ be *the universal abelian cover* of M, that is, the cover which corresponds to the commutator subgroup $[\pi_1M,\pi_1M]$. Note that homologically trivial periodic orbits in M lift to periodic orbits in $\hat M$. Hence periodic orbits of the lifted flow are dense in $\hat M$ and, by applying the standard Smale argument we conclude that $X^t\colon\hat M\to\hat M$ is a transitive flow. Hence we can carry out the standard proof of Livshits theorem on $\hat M$. The conclusion is that the lift $\hat\varphi\colon\hat M\to\mathbb{R}$ is a coboundary (in the usual sense), which translates into φ being an abelian coboundary. □

7. **Matching rigidity for Anosov flows.**

Theorem. *Let $X^t_i\colon M\to M$, $i=1,2$ be $C^{1+\alpha}$ 3-dimensional transitive Anosov flows. Assume they are orbit equivalent via $H\colon M\to M$. Let $\varphi_i: M\to\mathbb{R}$ be C^1 functions. If $\int_{\gamma_1}\varphi_1=\int_{H_*\gamma_1}\varphi_2$ for every X_1-closed orbit γ_1, then one of the following holds:*

1. *φ_i are X_i abelian coboundary;*
2. *H is C^1 after adjusting it through a time change.*

8. **Another application to 3-dimensional Anosov flows.** Recall from the first lecture.

Theorem (de la Llave-Moriyon, Pollicott)**.** *Assume that X_1^t and X_2^t are orbit equivalent 3-dimensional transitive Anosov flows. Assume*

$$\forall p \in Per(X_1): \ \ T_p = \mathrm{per}_{X_1}(p) = \mathrm{per}_{X_2}(h(p)) = T_{h(p)} \tag{A1}$$

Also assume that the differentials of Poincaré return maps for all periodic points are conjugate:

$$\forall p \in Per(X_1) \ \exists C: \ \ DX_1^{T_p}(p) = C \circ DX_2^{T_{h(p)}}(h(p)) \circ C^{-1} \tag{A2}$$

Then X_1^t and X_2^t are smoothly conjugate.

In the first lecture we discussed what happens if (A2) is dropped.

If one drops the assumption (A1) and, keeps (A2) instead, then the matching theorem could be applied to infinitesimal stable and unstable jacobians yields the following.

Theorem. *Assume that X_1^t and X_2^t are orbit equivalent 3-dimensional transitive Anosov flows. Assume that the differentials of Poincaré return maps for all periodic points are conjugate (A2). Then X_1^t and X_2^t are smoothly orbit equivalent.*

9. **Outline of the proof of Sharpened MLS rigidity.**

9.1. *Reduction to a reparametrization.* Applying the Matching Theorem to the geodesic flows X_1 and X_2 we obtain a C^{1+} orbit equivalence H. Let

$$\tilde{X}_1 = DH(X_1)$$

Then obviously,

$$\tilde{X}_1 = \rho X_2, \rho > 0$$

that is $\tilde{X}_1$ is a C^1 flow which is reparametrization of X_2. Our goal is to prove that $\tilde{X}_1^t$ and X_2/c are conjugate.

9.2. *Matching of homologically trivial spectra.* Both flows $\tilde{X}_1$ and X_2 are contact. Denote by α and β the contact $1-$forms for X_2 and $\tilde{X}_1$, respectively. Then we have that $d\alpha$ is an exact X_2-invariant $2-$form. On the other hand, by using Cartan's formula, $d\beta$ is also X_2-invariant. Hamenstädt proved that such form is unique, that is $d\beta = c d\alpha$, $c > 0$. Hence the $1-$form

$$\mu = \beta - c\alpha$$

is closed. Plugging X_2 yields a formula for ρ in terms of μ

$$\rho = \frac{1}{c + \mu(X_2)}$$

9.3. *Sharpening the reparametrization.* Our goal now is to show that μ is exact. Then the periods of $Y_1 = c\tilde{X}_1$ and $Y_2 = X_2$ match and we would conclude that Y_1 and Y_2 are conjugate. To do this we rely on work of R. Sharp, which is based on earlier work of Katsuda-Sunada.

If $Y^t\colon M \to M$ is a homologically full Anosov flow then the functional $\beta\colon H^1(M, \mathbb{R}) \to \mathbb{R}$ given by

$$\beta([\theta]) = P_Y(\theta(Y)) = \sup_{\mu}\{h_\mu + \int_M \theta(Y) d\mu\}$$

attains a unique minimum at $\xi_Y \in H^1(M,\mathbb{R})$. Geodesic flows are special among homologically full flows:

Fact 1. If Y is a geodesic flow then $\xi_Y = 0$;

Using the minimizer property it is not hard to study the behavior of the minimizer under the reparametrizations.

Fact 2. If $Y_1 = Y_2/(1+\omega(Y_2))$ then

$$\xi_1 = \xi_2 + \beta(\xi_2)[\omega]$$

Since both Y_1 and Y_2 come from geodesic flows this boils down to $\beta(\xi_2)[\omega] = 0$. But one can also check that $\beta(\xi_2) = h_{\mu_{\xi_2}}(Y_2) > 0$. Hence $\omega = dv$.

We conclude that Y_1 and Y_2 are conjugate and, hence, we can apply Otal-Croke theorem to obtain the posited isometry $\sigma\colon (S,g_1) \to (S,c^2g_2)$.