A note prepared for lectures :

Title : Transfer operator for
Anosov ~~flows~~ diffeomorphism.

Abstract We explain basic results about 'resonances' for Anosov diffeomorphism ([BT1], [BT2]). We use the method of wave packet transform, which is developed in [FT1] for Anosov flows.

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0. Introduction

$F : M \circlearrowleft$ C^∞ Anosov diffeomorphism

$\exists$ decomposition $TM = E^u \oplus E^s$

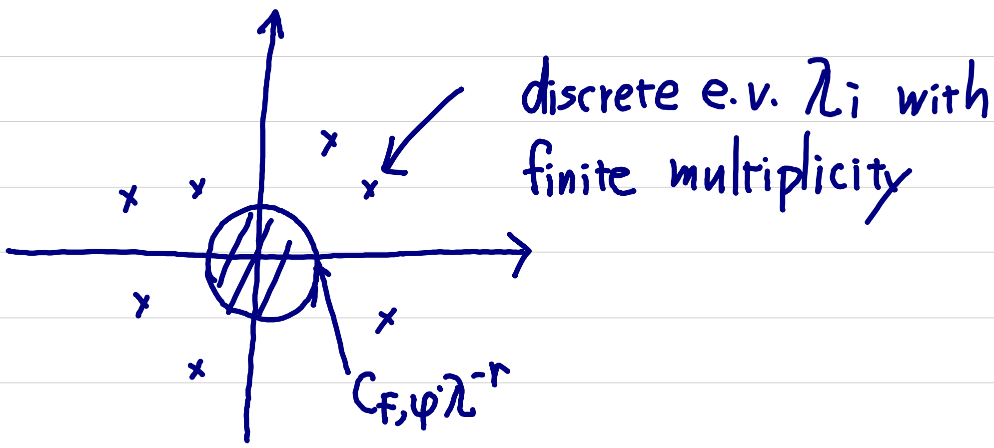
$\exists \lambda > 1$ s.t. $|DF^{-1}|_{E^u}| < \lambda^{-1}$, $|DF|_{E^s}| < \lambda^{-1}$.

<u>Transfer operator</u> $\mathcal{L} = \mathcal{L}_{F,G} : C^\infty(M) \circlearrowleft$

$$\mathcal{L}u(x) = (G \cdot u)(F^{-1}(x)) \qquad G \in C^\infty(M).$$

<u>Thm A</u> ([BKL], [GL], [BT1])

For any $r > 0$, there exists a Hilbert space $C^\infty(M) \subset \mathcal{H}^r(M) \subset (C^\infty(M))'$ s.t. $\mathcal{L} : \mathcal{H}^r(M) \circlearrowleft$ has discrete spectrum on $|z| > C_{F,G}\lambda^{-r}$.



<u>Thm. B</u> ([BT2]) As $n \to \infty$, we have

$$\mathrm{Tr}^b \mathcal{L}^n := \sum_{x : F^n_x = x} \frac{G^{(n)}(x)}{|\det(I - DF^{-n}_{(x)})|} = \sum_{i=1}^{m} \lambda_i^n + O\left(\left(C_{F,G}\lambda^{-r}\right)^n\right)$$

where $G^{(n)}(x) = \sum_{i=0}^{n-1} G(F^i(x))$.

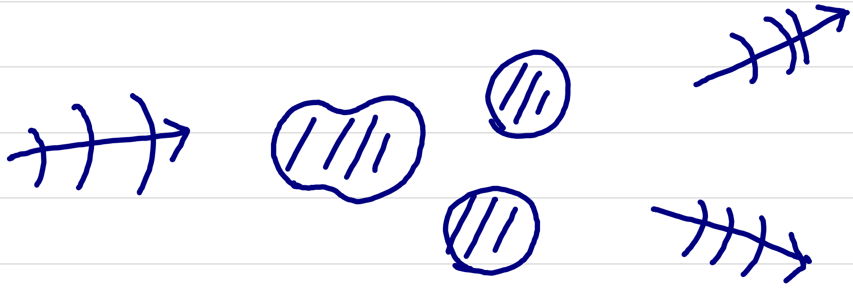
<u>Remark 1</u> $\{\lambda_i\}$ is intrinsic to $\mathcal{L}$ (or (F,G)), and called Ruelle-Pollicott resonances.

<u>Remark 2</u> For a concrete (non-trivial) example of R-P resonance, See [B].

1 What is 'resonances'?

1.1 Scattering problem ([LP])

Wave equation on the outside of obstacles.



Propagation of waves is described by a one-param. gr $U^t : X \circlearrowleft$ of unitary operators.

$\exists D_+, D_- \subset X$ closed subspaces (outgoing, incoming)

s.t. 0) $D_+ \cap D_- = \{0\}$

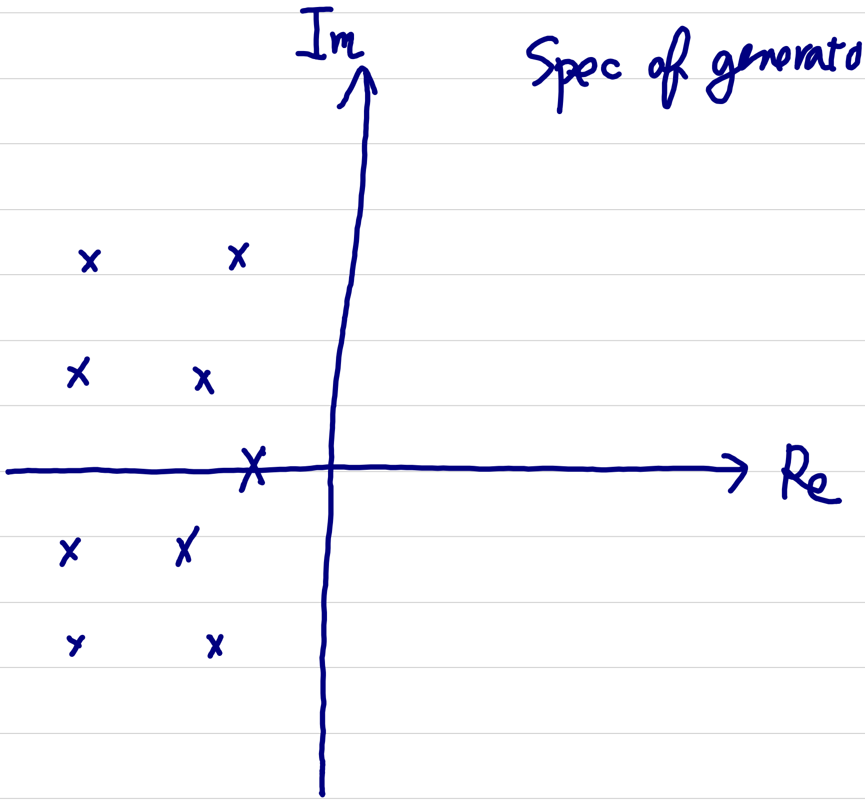
1) $U^t D_+ \subset D_+ \quad (t > 0) \qquad \bigcap_{t \geq 0} U^t D_+ = \{0\}$

$U^t D_- \subset D_- \quad (t < 0) \qquad \bigcap_{t \leq 0} U^t D_- = \{0\}$

2) $\overline{\bigcup_{t \in \mathbb{R}} U^t D_+} = \overline{\bigcup_{t \in \mathbb{R}} U^t D_-} = X.$

U^t induces $Z^t : (D_-)^{\perp}/D_+ \circlearrowleft$.

We have $\|Z^t[u]\| \to 0$ $(t \to \infty)$ and the generater has discrete spectrum (!)



1.2 A matrix model for resonances (by Faure).

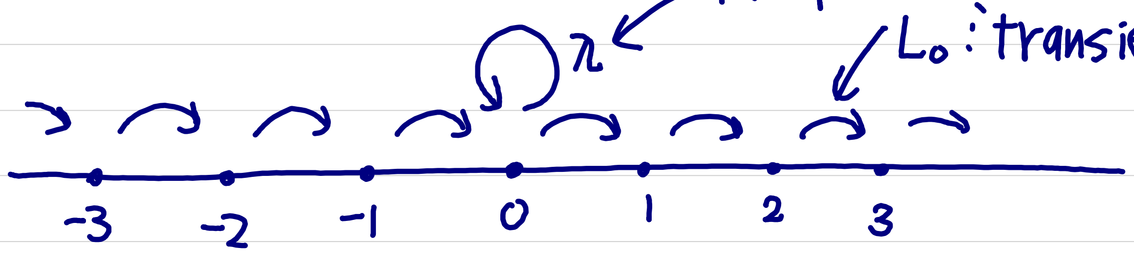
Consider (infinite) matrix

$$L_0 = [a_{ij}]_{i,j\in\mathbb{Z}}, \qquad K = [k_{ij}]_{i,j\in\mathbb{Z}}$$

$$a_{ij} = \begin{cases} 1 & \text{if } i=j+1 \\ 0 & \text{otherwise,} \end{cases} \qquad k_{ij} = \begin{cases} \mu & \text{if } (i,j)=(0,0) \\ 0 & \text{otherwise} \end{cases}$$

and set $L = L_0 + K$ $\quad K$: 'perturbation'

L_0: 'transient'



Let $D_+ = \ell^2(\mathbb{Z}_{>0})$, $D_- = \ell^2(\mathbb{Z}_{<0})$. Then

$\mathbb{Z} : (D_-)^{\perp}/D_+ \simeq \mathbb{C} \circlearrowleft$ is one-dimensional

and has 'resonance' $\{\lambda\}$.

<u>Remark</u> We do NOT find λ as a resonance for L^{-1}.

For $r>0$, let

$$\mathcal{H}^r = \{u:\mathbb{Z}\to\mathbb{C} \mid \|u\|_r := \textstyle\sum_{k\in\mathbb{Z}} e^{-2rk}|u(k)|^2 < +\infty\}$$

Then we see

$$\|L_0 : \mathcal{H}^r \circlearrowleft\| = e^{-r}, \quad K:\mathcal{H}^r\circlearrowleft \text{ finite rank.}$$

We can observe λ as a (legal) e.v of $L:\mathcal{H}^r\circlearrowleft$.

<u>Lemma C</u> $\mathcal{H}$: Hilbert sp.

$A:\mathcal{H}\circlearrowleft$ bounded op $\quad B:\mathcal{H}\circlearrowleft$ finite rank.

Then $(A+B)$ has discrete spectrum on $|z|>\|A\|$.

<u>Proof</u>

$$zI-(A+B) = \underbrace{(zI-A)}_{\text{invertible}}\ \underbrace{(I-(zI-A)^{-1}B)}_{\text{analytic family of finite rank op}}$$

<u>Remark</u> We may replace 'finite rank' by 'compact', and $\|A\|$ by $\rho(A)$.

Remark We have ([FT1])

$$\#\{\text{e.v of } (A+B) \text{ on } |z|>\rho_0+\varepsilon\} \leq C \, \|B\|_{Tr}$$

where C depends on ρ_0, $\|B\|$, ε.

Remark For $q>r$, set

$$\tilde{\mathcal{H}}^q = \{u : \mathbb{Z} \to \mathbb{C} \mid \|u\|_q^{\sim} = \sum e^{-2qk} |u(k)|^2 < +\infty\}.$$

Then $i : \mathcal{H}^r \to \tilde{\mathcal{H}}^q$ is compact and

$$\begin{aligned} \|Lu\|_r &\leq \|L_0 u\|_r + \|Ku\|_r \\ &\leq e^{-r} \|u\|_r + {}^{\exists}C \|u\|_q^{\sim} \end{aligned}$$

This is a Lasota-Yorke type inequality.

1.3 Resonances in simple hyp. dynamics.

① Let $T : \mathbb{R} \circlearrowleft \quad T(x) = \lambda x \qquad (\lambda > 1)$.

$L : L^2(\mathbb{R}) \circlearrowleft \quad Lu(x) = u(T^{-1}x)$.

If we set

$$D_+ = \{ u \in L^2(\mathbb{R}) \mid \operatorname{supp} u \cap (-1,1) = \emptyset \}$$

$$D_- = \{ u \in L^2(\mathbb{R}) \mid \operatorname{supp} \mathcal{F}u \cap (-1,1) = \emptyset \}$$

then

$$Z : (D_-)^{\perp} / D_+ \circlearrowleft$$

presents resonances $\{1, \lambda^{-1}, \lambda^{-2}, \cdots \}$.

eigenfunction $[1], [x], [x^2], \cdots$

② Let $F: S^1 = \mathbb{R}/2\pi\mathbb{Z} \circlearrowleft$ $F(\theta) = m\theta$ $(m \geq 2)$

and set

$$L: L^2(S^1) \circlearrowleft \quad Lu(\theta) = \sum_{F(\theta') = \theta} u(\theta').$$

If we set $D_+ = \emptyset$, $D_- = \{ u = \sum a_k e^{ik\theta} \mid a_0 = 0 \}$

then $\mathbb{Z} \cdot L^2(S^1)/D_+ \circlearrowleft$ presents resonance

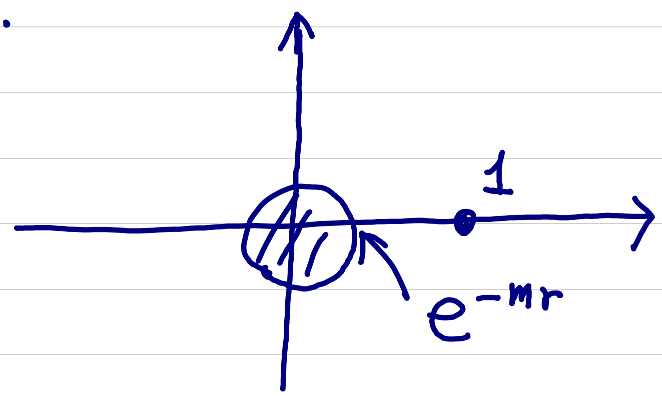
$\{1\}$.

<u>Remark</u> To observe this resonance as an eigenvalue

we consider the Sobolev space.

$$\mathcal{H}^r = W^{r,2}(S^1) = \{ u = \sum a_k e^{ik\theta} \mid \sum (k^{2r}+1)|a_k|^2 < +\infty \}$$

for $r > 0$.



2. Anatomy of transfer operators via wave packet transform.

2.1 hyperbolic diffeomorphism

Fix a decomposition $\mathbb{R}^d = \mathbb{R}^{d_u} \oplus \mathbb{R}^{d_s}$

We consider transfer op.

$L = L_{T,g} : C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$

$Lu = (g \cdot u)(T^{-1}x)$

where $g \in C_0^\infty(\mathbb{R}^d)$ and $T : \mathbb{R}^d \to \mathbb{R}^d$ C^∞ s.t. $\|T - T_0\|_{C^1} < \varepsilon \ll 1$ where

$$T_0 = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right] \begin{array}{l} \}d_u \\ \}d_s \end{array}$$

with $\|A^{-1}\| < \lambda^{-1}$, $\|B\| < \lambda^{-1}$.

<u>Remark</u> We may decompose $\mathcal{L}_{F,G}$ for an Anosov diffeo into those as above on local charts.

2.2 Wave packets

a) A metric on $T^*\mathbb{R}^d = \mathbb{R}^d_x \oplus \mathbb{R}^d_\xi$.

For $p = (x, \xi) \in T^*\mathbb{R}^d$, set

$$\|(v,\eta)\|_p^2 = (\langle\xi\rangle^{\alpha}|v|)^2 + (\langle\xi\rangle^{-\alpha}|\xi|)^2$$

where $\langle\xi\rangle = \sqrt{|\xi|^2+1}$. We assume

$$1/2 < \alpha < 1.$$

M1) $\|\cdot\|_p$ is compatible w.r.t the standard symplectic form $\Omega = dx \wedge d\xi$

M2) Coordinate changes in x induces a Lipschitz map w.r.t. $\|\cdot\|_p$. ($\Leftrightarrow \alpha > 1/2$)

M3) $\|\cdot\|_p$ is temperate in the sense that

$$\frac{\|w\|_{p'}}{\|w\|_p} \leq \exists C \langle\|p'-p\|_p\rangle^{\exists N} \qquad (\forall p, p', w)$$

b) Quadratic partition of unity on $(\mathbb{R}^q)^*$

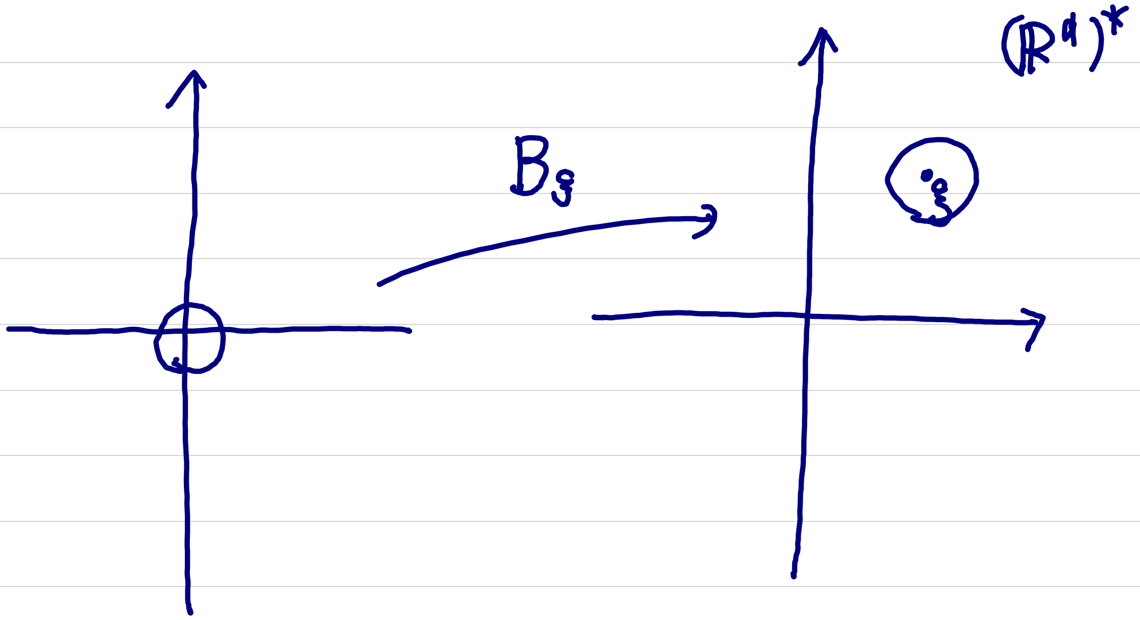
Let $\{\hat{\varphi}_\xi \in C^\infty((\mathbb{R}^q)^*)\}_{\xi\in(\mathbb{R}^q)^*}$ be a family of function s.t.

P1) $\int |\hat{\varphi}_\xi(\xi')|^2 d\xi \equiv 1$ for $\forall \xi'$

P2) For $\forall$ isometry $B_\xi : (\mathbb{R}^q, |\cdot|) \to (\mathbb{R}^q, \|\cdot\|_{(x,\xi)})$ with $B_\xi(0) = \xi$, $\hat{\varphi}_\xi \circ B_\xi$ is bounded in $\mathcal{S}$ uniformly in ξ. i.e

indep of ξ.

$$|\xi'|^\beta |D^\alpha(\hat{\varphi}_\xi \circ B_\xi)(\xi')| < C_{\alpha,\beta} \quad (\forall \alpha, \beta, \xi')$$



c) Wave packets

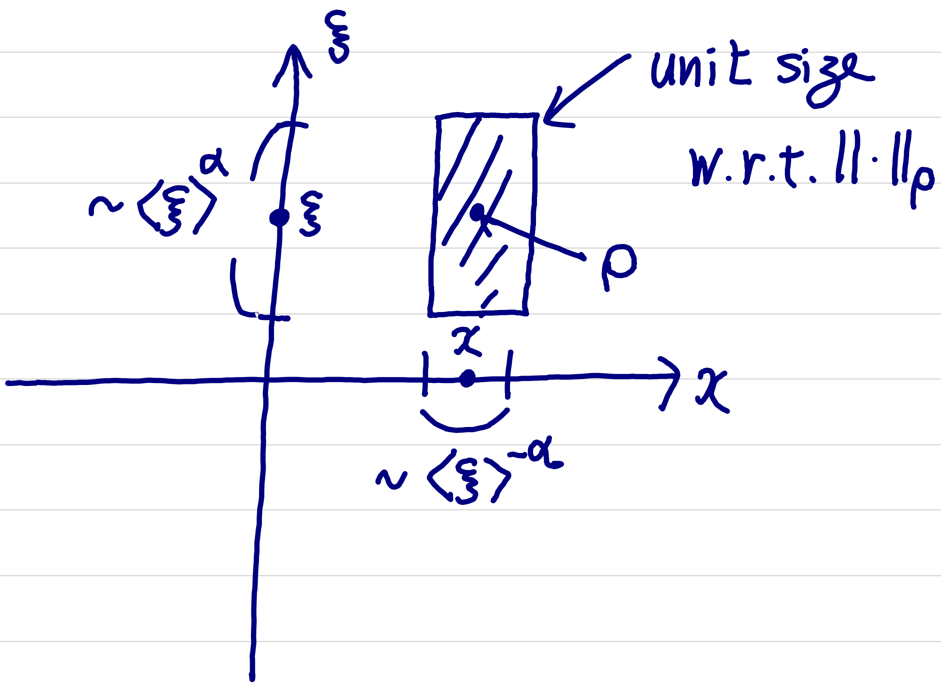
For $\rho = (x, \xi) \in T^*\mathbb{R}^d$, we set

$$\varphi_\rho(y) = \mathcal{F}^{-1}\hat{\psi}_\xi(y - x) \quad \in C^\infty(\mathbb{R}^d).$$

Then we have , for any $N > 0$,

$$|\varphi_\rho(y)| < C_N \langle \| (y-x, 0) \|_\rho \rangle^{-N}$$

$$|\mathcal{F}\varphi_\rho(\eta)| < C_N \langle \| (0, \eta - \xi) \|_\rho \rangle^{-N}$$



d) Wave packet transform

$$\Phi : L^2(\mathbb{R}^d) \longrightarrow L^2(T^*\mathbb{R}^d)$$
$$u \longmapsto \Phi u(\rho) = \int \overline{\varphi_\rho(y)}\, u(y)\, dy$$

$$\Phi^* . L^2(T^*\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$$
$$v \longmapsto \Phi^* v(y) = \int \varphi_\rho(y) v(\rho)\, dy$$

<u>Lemma</u> Φ is an L^2 isometric embedding

$\Phi^* \circ \Phi = \mathrm{id}$.

<u>Pf</u> $\Phi^* \circ \Phi\, u(y) = \int \varphi_\rho(y) \overline{\varphi_\rho(y')}\, u(y')\, d\rho\, dy'$

$$= \int \mathcal{F}_? |\varphi_\xi|^2 (y-y')\, u(y')\, d\rho\, dy'$$

$$= \int \delta(y-y')\, u(y')\, dy' = u(y).$$

$$\langle \Phi u, \Phi u' \rangle_{L^2} = \langle u, \Phi^* \circ \Phi u' \rangle_{L^2} = \langle u, u' \rangle_{L^2}. \quad //$$

Remark

$$u(y) = \Phi^*(\Phi u) = \underbrace{\int \Phi u(p) \cdot \varphi_p(y)\, dy}_{\text{decomposition into } \{\varphi_p\}.}$$

$$L^2(\mathbb{R}^d) \underset{\Phi^* \text{ reconstruction}}{\overset{\Phi \text{ decomp}}{\rightleftarrows}} L^2(T^*\mathbb{R}^d) = \mathrm{Im}\,\Phi \oplus \mathrm{Ker}\,\Phi^*.$$

2.3 Lifted transfer op.

$\hat{\mathcal{L}} = \Phi \circ \mathcal{L} \circ \Phi^*$ satisfies

$$\begin{array}{ccc} L^2(T^*\mathbb{R}^d) & \xrightarrow{\hat{\mathcal{L}}} & L^2(T^*\mathbb{R}^d) \\ \uparrow \Phi & & \uparrow \Phi \\ L^2(\mathbb{R}^d) & \xrightarrow{\mathcal{L}} & L^2(\mathbb{R}^d) \end{array}.$$

<u>Remark</u> $\hat{\mathcal{L}} = (\hat{\mathcal{L}}|_{\mathrm{Im}\,\Phi}) \oplus (\hat{\mathcal{L}}|_{\ker \Phi^*})$

$$\begin{array}{ccc} \hat{\mathcal{L}}|_{\mathrm{Im}\,\Phi} & & \hat{\mathcal{L}}|_{\ker \Phi^*} \\ \wr\wr & & \| \\ \mathcal{L} & & 0 \end{array}$$

<u>Remark</u> $\hat{\mathcal{L}}$ has smooth kernel (!)

$$\hat{\mathcal{L}} v(p') = \int \underbrace{\left(\int \overline{\varphi_{p'}} \cdot \mathcal{L}(\varphi_p)\, dy \right)}_{K_{\hat{\mathcal{L}}}(p', p)} v(p)\, dp$$

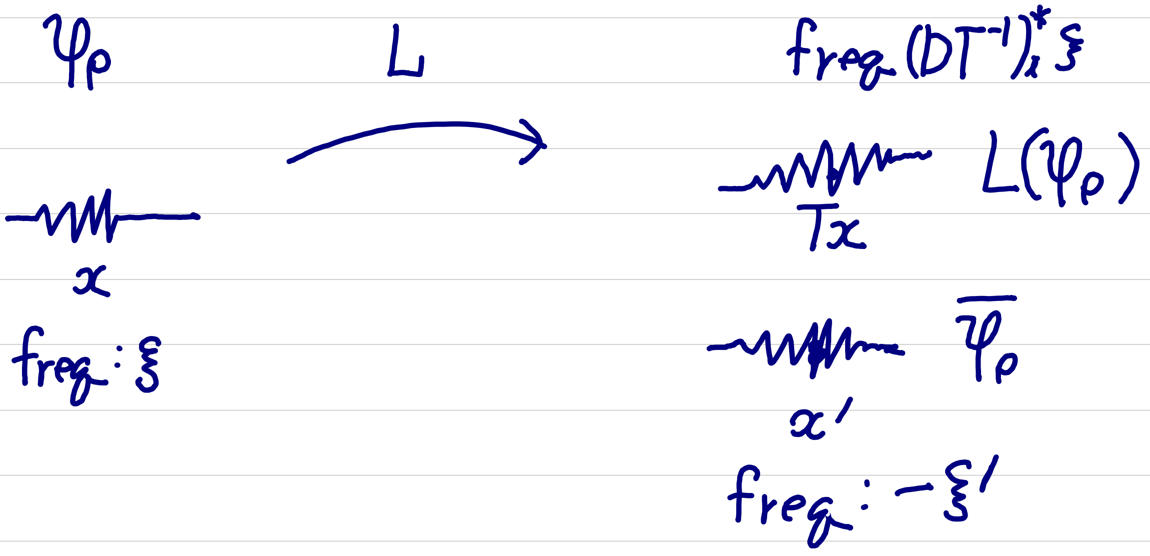
Thm (Kernel estimate)

$$|K_{\hat{L}}(\rho', \rho)| < C_{N,T,g}\langle \|\rho' - (DT^{-1})^{*}(\rho)\|_{\rho'} \rangle^{-N}$$

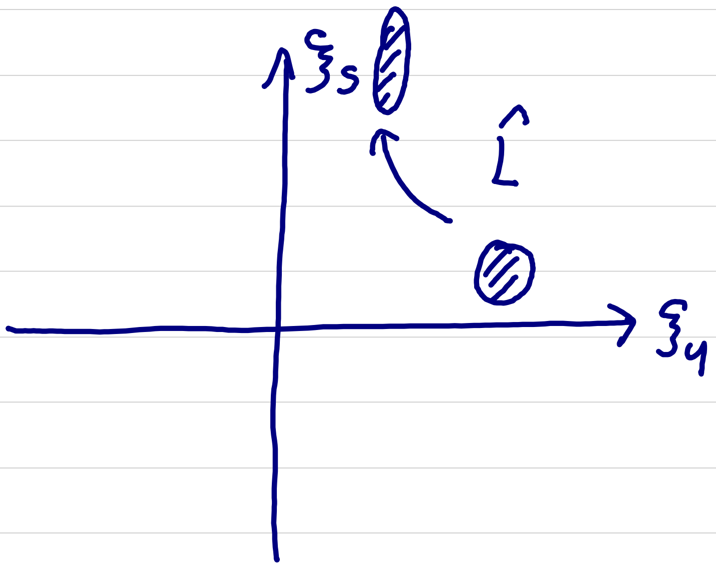
$$\left(\quad '' \quad < C_{N,T,g} \langle \| DT^{*}(\rho) - \rho \|_{\rho} \rangle^{-N} \right)$$

Proof By integration by parts

$$K_{\hat{L}}(\rho', \rho) = \int \overline{\varphi_{\rho'}(y)}\, \varphi_{\rho'}(T^{-1}(y))\, g(T^{-1}(y))\, dy \quad //$$



The action of $\hat{L}$.



Remark The kernel estimate above is coarse (but enough for our purpose).

3. Resonance for Anosov diffeomorphism

3.1 Weight function

Take a C^∞ function $W : T^*\mathbb{R}^d \to \mathbb{R}_+$ s.t

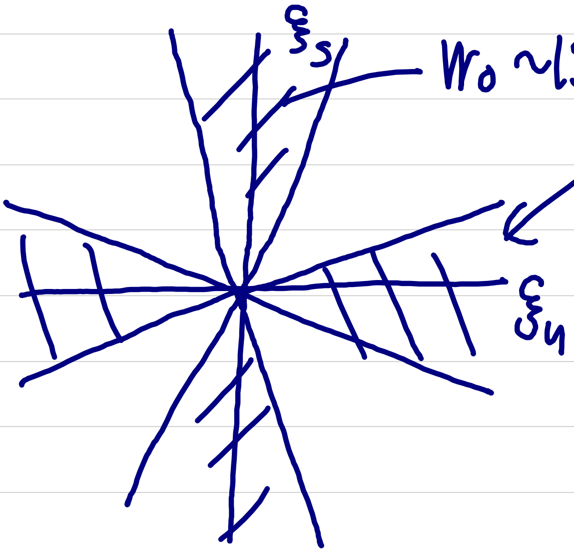
W1) $W((DF^{-1})^*(\rho)) \leq \lambda^{-1} \cdot W(\rho)$ if $|\xi| > {}^\exists C$.

W2) ${}^\exists N > 0$, ${}^\forall \varepsilon > 0$, ${}^\exists C_\varepsilon > 0$ s.t.

$$\frac{W(\rho')}{W(\rho)} \lesssim (1 + \varepsilon \|\rho' - \rho\|_\rho^N) \quad \text{if } |\xi| > C$$

where $\rho = (x, \xi)$.

<u>Construction</u> $W(x, \xi) = W_0(\xi)$



3.2 The anisotropic spaces.

<u>Define</u>

$$L^2(T^*\mathbb{R}^d, W^r) := \left\{ v : T^*\mathbb{R}^d \to \mathbb{C} \;\middle|\; \underset{\displaystyle \overset{\|\ \|_{L^2(W)}}{\shortparallel}}{} \|W^r v\|_{L^2} < \infty \right\}$$

$$H^r(\mathbb{R}^d) := \Phi^{-1}(L^2(T^*\mathbb{R}^d, W^r))$$

Then $\quad C_0^\infty(\mathbb{R}^d) \subset H^r(\mathbb{R}^d) \subset (C_0^\infty(\mathbb{R}^d))'$

and

$$\Phi : H^r(\mathbb{R}^d) \longrightarrow L^2(T^*\mathbb{R}^d, W^r)$$

is an isometric embedding.

$$\begin{tikzcd}
L^2(T^*\mathbb{R}^d, W^r) \arrow[r, "\hat{L}"] & L^2(T^*\mathbb{R}^d, W^r) \\
H^r(\mathbb{R}^d) \arrow[r, "L"] \arrow[u, "\Phi"'] & H^r(\mathbb{R}^d) \arrow[u, "\Phi"']
\end{tikzcd}$$

3.3 A decomposition of $\hat{L}$

Set

$$\Lambda = \left\{ (\rho', \rho) \in (T^*\mathbb{R}^d)^2 \;\middle|\; \begin{array}{l} W(\rho') < C_0 \lambda^{-1} W(\rho) \\ |\xi| > C \quad |\xi'| > C \end{array} \right\}$$

and decompose $\hat{\mathcal{L}}$ as

$$\hat{L} u = \int K_{\hat{L}}(\rho', \rho) u(\rho) d\rho$$

$$\begin{cases} \overset{\shortparallel}{L_0} := \int \mathbb{1}_\Lambda(\rho', \rho) K_{\hat{L}}(\rho', \rho) u(\rho) d\rho \\ + \\ K := \text{remainder} \end{cases}$$

<u>Remark</u> L_0 : 'transient'

K : decay fast as $|\xi|, |\xi'| \to \infty$.

<u>Thm</u> $\hat{L}$ extends to a bounded operator on $L^2(T^*\mathbb{R}^d, W^r)$ and

1) $\|L_0\|_{L^2(W)} \leq ((C_0\lambda^{-1})^r \cdot |\det DT|_\infty^{1/2}$

2) $K : L^2(W) \circlearrowleft$ compact (trace class)

<u>Proof</u> 1) follows from the kernel estimate and (W2) in the choice of W.

2) follows from the kernel estimate. //

Proof of Thm A

Take local charts $\{\kappa : U \to V \subset \mathbb{R}^q\}$ on M and a (quadratic) partition of unity $\{\psi_i\}$ appropriately (!). Then we have

$$\begin{tikzcd}
C^\infty(M) \arrow[r, "\mathcal{L}"] \arrow[d, "i"] & C^\infty(M) \arrow[d, "i"] & u \arrow[d] \\
\bigoplus_i \mathcal{H}^r \arrow[r, "(L_{ij})"] \arrow[d, "\oplus\Phi"] & \bigoplus \mathcal{H}^r \arrow[d, "\oplus\Phi"] & (\psi_i \cdot u)\circ\kappa^{-1} \\
\bigoplus_i L^2(W^r) \arrow[r, "(\widehat{L}_{ij})"] & \bigoplus_i L^2(W^r) &
\end{tikzcd}$$

If we set $\mathcal{H}^r(M) = ((\oplus\Phi)\circ i)^{-1}(\oplus L^2(W^r))$

we see $\mathcal{L} : \mathcal{H}^r(M) \circlearrowleft \simeq (\widehat{L}_{ij})|_{\mathrm{Im}(\oplus\Phi\circ i)}$.

Hence we get the conclusin by Lemma C and the last theorm.

3.3 Flat trace.

For $\hat{L}$, we define

$$\mathrm{Tr}^b \hat{L} := \int K_{\hat{L}}(\rho,\rho)\, d\rho .$$

<u>Lemma</u> $\mathrm{Tr}^b \hat{L} = \mathrm{Tr}^b L := \sum_{T(y)=y} \frac{g(y)}{|\det(I - DT_{(y)}^{-1})|}$

<u>Pr</u> $\mathrm{Tr}^b \hat{L} = \lim_{M\to\infty} \int_{|\xi|\le M} K_{\hat{L}}(\rho,\rho)\, d\rho$

$$= \lim_{M\to\infty} \int_{|\xi|\le M} \overline{\mathcal{F}\hat{\varphi}_{\xi}}(y-x)\, \mathcal{F}\hat{\varphi}_{\xi}(F_y^{-1}-x)\, g(F_y^{-1})\, dy\, dx\, d\xi$$

$$= \lim_{M\to\infty} \int_{|\xi|\le M} \mathcal{F}|\hat{\varphi}_{\xi}|^2(F_y^{-1}-y)\, g(F_y^{-1})\, d\xi\, dy$$

$$= \int \delta(F_{(y)}^{-1}-y)\, g(y)\, dy = \mathrm{Tr}^b L . \;//$$

Pf of Thm B We have

$$\begin{array}{ccc} \mathcal{H}^r(M) & \xrightarrow{\mathcal{L}} & \mathcal{H}^r(M) \\ \downarrow & & \downarrow \\ \bigoplus_i L^2(W) & \longrightarrow & \bigoplus L^2(W) \end{array}$$

$$\mathbb{L} = (\hat{L}_{ij}) = \mathbb{L}_0 + \mathbb{K}$$

Define $\mathrm{Tr}^b \mathbb{L}^n = \sum_i \mathrm{Tr}^b L_{ii}^n$. Then by the lemma, $\mathrm{Tr}^b \mathbb{L}^n = \mathrm{Tr}^b \mathcal{L}^n$.

Let $\mathbb{L} = \mathbb{P} + \mathbb{Q}$ be the spec. decomp. where $\mathbb{P}$ corresponds to the discrete e.v's with $|z| > \rho_0 > (C_0 \lambda^{-1})^r$. We show

$$\mathrm{Tr}^b \mathbb{Q}^n = \mathrm{Tr}^b ((\mathbb{L} - \mathbb{P})^n) < C \rho_0^n .$$

$\because$) $\mathbb{L} - \mathbb{P} = \underbrace{\mathbb{L}_0}_{\text{transient}} + \underbrace{\mathbb{K} - \mathbb{P}}_{\text{trace class.}}$

So

$$\mathrm{Tr}^b\,\mathbb{Q}^n = \mathrm{Tr}^b\left((\mathbb{L}_0+(\mathbb{K}-\mathbb{P}))^n\right) - \underbrace{\mathrm{Tr}^b\,\mathbb{L}_0^n}_{0}$$

Replacing $\mathcal{L}$ by $\mathcal{L}^n$, we may suppose

$$\begin{aligned}\|\mathbb{K}-\mathbb{P}\|_{L^2(W)} &= \|\mathbb{Q}-\mathbb{L}_0\|_{L^2(W)} \\ &\leq \|\mathbb{Q}\|_{L^2(W)} + \|\mathbb{L}_0\|_{L^2(W)} \\ &\leq \rho_0 + \rho_0 = 2\rho_0.\end{aligned}$$

Therefore

$$\begin{aligned}|\mathrm{Tr}^b\,\mathbb{Q}^n| &\leq C\|\mathbb{K}-\mathbb{P}\|_{\mathrm{Tr}}\cdot(\rho_0+2\rho_0)^n \\ &\leq C\,(3\rho_0)^n. \qquad //\end{aligned}$$

Appendix 1 Geometric spaces. ([GL])

$$\|u\|_{p,q} = \sup_{|\alpha|\le p} \sup_{W\in\Sigma} \sup_{\substack{\varphi\in C_0^{q+|\alpha|} \\ |\varphi|_{C^{q+|\alpha|}}\le 1}} \int_W \partial^\alpha h\, \varphi$$

where Σ is the space of 'almost' stable curves.

We define Banach spaces $B^{p,q}$ as completion of $C^\infty(\mathbb{R}^d)$ w.r.t $\|\cdot\|_{p,q}$.

In spirit, these spaces are similar to H^r and we can prove the same results as Thm A, B for them.

<u>Appendix B</u> For Anosov flows, we can develop a parallel argument. For the details, see [FT1]. The main difference is that $F^t : M \to M$ is partially hyperbolic. The function

generating v.f.

$$\omega : T^*M \to \mathbb{R}, \quad \omega(x, \xi) = \langle \xi, X(x) \rangle,$$

is invariant. But if we look at the level sets of ω, we see the same dynamics as in the case of Anosov diffeomorphism.