RIGIDITY LECTURE NOTES II

ANDREY GOGOLEV

Today we will discuss rigidity of toral automorphisms similarly to rigidity of expanding maps and Anosov diffeos discussed last time. Any matrix $L \in SL(d, \mathbb{Z})$ induces a torus automorphism $L\colon \mathbb{T}^d \to \mathbb{T}^d$. If L is hyperbolic and $f\colon \mathbb{T}^d \to \mathbb{T}^d$ is an Anosov diffeo which is homotopic to L then, by work of Franks and Manning, f is conjugate to L, $h \circ f = L \circ h$. We are interested in higher regularity of h. Recall that the obstructions are carried by periodic orbits

$$\forall p = f^k p \quad \exists C : \quad Df^k(p) = CL^kC^{-1} \tag{$\star$}$$

Recall that if $d = 2$ then vanishing of obstructions implies that h is as regular as f by work of de la Llave-Marco-Moriyon.

1. **Periodic data rigidity in dimension 3.** There are two cases to consider for 3-dimensional automorphisms with 2-dimensional unstable subbundle: the comformal case of a pair complex conjugate eigenvalues and the case when unstable subbundle admits a dominated splitting.

Theorem (Kalinin-Sadovskaya)**.** *Assume that $L\colon \mathbb{T}^3 \to \mathbb{T}^3$ has a pair of complex eigenvalue with $0 < \lambda_1 < 1 < |\lambda_2| = |\lambda_3|$ and that C^r Anosov diffeomorphism f is conjugate to L, $h \circ f = L \circ h$. If ($\star$) then h is $C^{r-\varepsilon}$.*

Theorem (A.G. – Guysinsky)**.** *Assume that $L\colon \mathbb{T}^3 \to \mathbb{T}^3$ has real spectrum $0 < \lambda_1 < 1 < \lambda_2 < \lambda_3$. Consider C^r smooth Anosov diffeomorphism f, which is conjugate to L; $h \circ f = L \circ h$. If ($\star$) then h is $C^{r-\varepsilon}$.*[1]

The bootsrap of regularity of h is carried out in several steps: to Lipschitz, to C^1 and then to $C^{r-\varepsilon}$. In the conformal case Kalinin and Sadovskaya built an invariant conformal structure on the unstable subbundle of f. Then presence of the conformal structure allows to employ apply one-dimensional techniques to the 2-dimensional problem and obtain regularity along the unstable foliation.

2. **De la Llave counterexample for dimensions ≥ 4.** In general, rigidity of automorphism in dimensions ≥ 4 is false. The following example is due to de la Llave.

Let A and B be hyperbolic automorphisms of $\mathbb{T}^2$ with $Av = \lambda v$ and $Bu = \mu u$ where $\mu > \lambda > 1$. Consider the product automorphism $L(x, y) = (Ax, By)$ and its perturbation

$$f(x, y) = (Ax + \varphi(y)v, By),$$

where $\varphi(y) = \sin(2\pi y_1)$. Then the conjugacy h has the form

$$h(x, y) = (x + \psi(y)v, y)$$

Date: May 14, 2019.

[1]To obtain the global version one needs to use, additionally, more recent results of Wang-Sun on approximation of Lyapunov exponents, Velozo on $SL(2, \mathbb{R})$ cocycles and also thesis of Potrie.

and can be calculated explicitly in the form of series. Let $r_0 = \log\lambda/\log\mu$. Note that $r_0 < 1$. One can then check that ψ is C^{r_0}, but no more regular.

3. **Lyapunov spectrum rigidity.** Let $L\colon \mathbb{T}^d \to \mathbb{T}^d$ be an automorphism and let $f\colon \mathbb{T}^d \to \mathbb{T}^d$ be a volume preserving C^1 small perturbation of L, $h \circ f = L \circ h$. Instead of assuming that the Lyapunov exponents at periodic match we can make the same assumption for volume Lyapunov exponents, that is,

$$\chi_i^f = \chi_i^L,\ i = 1, \dots d \tag{$\star$}$$

If f is not ergodic we can take average Lyapunov exponent.

3.1. *Dimension 2.* If $d = 2$ then

$$h_{vol}(f) = \chi_f = \chi_L = h_{vol}(L) = h_{h^*vol}(f)$$

Hence, by uniqueness of the measure of maximal entropy we have $h^*vol = vol$. Then one concludes that h is smooth following the proof in Lecture 1.

3.2. *Dimension 3.*

Theorem (Saghin-Yang)**.** *Assume that $L\colon \mathbb{T}^3 \to \mathbb{T}^3$ has real spectrum $0 < \lambda_1 < 1 < \lambda_2 < \lambda_3$. Assume that f is a C^1 small perturbation of L which satisfies ($\star$). Then f is smoothly conjugate to L.*

Proof. Similarly to the 2-dimensional case one can use Pesin's formula to obtain $h^*vol = vol$. Then, just as in the 2-dimensional case, $h \in C_s^\infty(\mathbb{T}^3)$, that is, h is smooth along the stable foliation. By, comparing the rates at which points diverge in the universal cover, one has $h(W_f^{wu}) = W_L^{wu}$. However, one cannot conclude smoothness along W_f^{wu} because the conditional measures of volume typically fail to be absolutely continuous. This is the main issue and we need the following lemma to overcome it.

Lemma (Ledrappier)**.** *Let W be a uniformly expanding foliation for a preserving diffeomorphism $f\colon M \to M$. Let m be an ergodic invariant measure. Let ξ be a measurable, Markov partition subordinate to W. Denote by $m_{\xi(x)}$ the conditional measures on $\xi(x)$. Conditional entropy $H(f^{-1}\xi|\xi)$ is defined by*

$$H(f^{-1}\xi|\xi) = \int_M -\log m_{\xi(x)}(f^{-1}(\xi(fx))dm$$

Then the conditional measures m_ξ are absolutely continuous if and only if

$$H(f^{-1}\xi|\xi) = \int_M \log Jac(f|_W)dm$$

We can apply the lemma to W_f^{wu} and vol. Indeed,

$$H(f^{-1}\xi|\xi) = H(L^{-1}(h(\xi))|h(\xi)) = \log\lambda_2$$

and

$$\int_{\mathbb{T}^3} \log Jac^{wu}(f)dvol = \chi_f^{wu} = \log\lambda_2$$

hence W_f^{wu} is absolutely continuous and the rest of the proof proceeds in the same way as for periodic data rigidity. □

Theorem (AG-Kalinin-Sadovskaya)**.** *Assume that $L\colon \mathbb{T}^3 \to \mathbb{T}^3$ has a pair of complex eigenvalues $0 < \lambda_1 < 1 < |\lambda_2| = |\lambda_3|$. Assume that f is a C^1 small perturbation of L which satisfies ($\star$). Then f is smoothly conjugate to L.*

Proof. The challenge here is to establish regularity of h along the two dimensional unstable foliation. To do that we need to control two pieces of data for f: jacobian and quasi-conformal distortion.

- *The unstable jacobian $Jac^u(f)$ is continuously cohomologous to constant*
$$Jac^u(f) = \frac{\rho(fx)}{\rho(x)}|\lambda_2|^2$$
The proof uses absolute continuity of W^u and then the measurable Livshits theorem for scalar cocycles.
- *There exists a Hölder continuous Riemannian metric on E^u such that $Df|_{E^u}$ is conformal*
$$\|Dfv_x\| = a(x)\|v_x\|, \quad \forall v_x \in E^u$$
The tool for proving this is the *trichotomy of Kalinin-Sadovskaya.* Namely, if 2×2 matrix cocycle over f has only one volume exponent then:
  1. the cocycle is conformal;
  2. the cocycle admits a continuous invariant line bundle;
  3. the cocycle admits an invariant pair of transverse line bundles;

  Note that in our setting we have that at the fixed point p, $Df|_{E^u}(p)$ is close to an irrational rotation which eliminates possibilities 2 and 3.
- *The conjugacy h is Lipschitz along W^u.*
We approximate h by an h_0 which is C^1 along W^u. Then define
$$h_n = L^n \circ h \circ f^{-n}$$
By using conformality and the fact that $a(x)$ is cohomologous to a constant it is easy to check that h_n are uniformly bounded in C^1 topology and, in fact converge to h.
- It follows from the Rademacher theorem that h is differentiable almost everywhere along W^u, that is,
$$D^u h \circ D^u f = D^u L \circ D^u h$$
has a measurable solution $D^u h$. Then by a result of Sadovskaya it must have a continuous version and hence, $h \in C^{1+}(W^u)$. After that the classical de la Llave bootsrap argument kicks in.

□

Combining the above techniques yields a higher dimensional result.

Theorem (Saghin-Yang/ AG-Kalinin-Sadovskaya)**.** *Let f be a C^1 small perturbation of $L\colon \mathbb{T}^d \to \mathbb{T}^d$, where L^4 is a hyperbolic irreducible automorphism such that no three eigenvalues have the same absolute value. Then the conjugacy $h \in C^{1+Holder}$.*

Remark. In contrast to periodic data rigidity, the Lyapunov spectrum rigidity is an "extremal" property of the automorphism. It is easy to produce non-linear Anosov diffeomorphisms with the same Lyapunov spectrum which are not C^1 conjugate.

4. **Rigidity of partially hyperbolic automorphisms.**

4.1. *Dimension 3.*

Theorem (Saghin-Yang). *Let $L\colon \mathbb{T}^3 \to \mathbb{T}^3$ be the product partially hyperbolic automorphism*

$$L(x, y, z) = (A(x, y), z)$$

and let f be a volume preserving perturbation with the same (average) Lyapunov exponents ($\star$). Then f is smoothly conjugate to

$$L'(x, y, z) = (A(x, y), z + \alpha(x, y))$$

Proof. Consider the semi-conjugacy $h\colon \mathbb{T}^3 \to \mathbb{T}^2$, $h \circ f = A \circ h$. Let ξ be a measurable, Markov partition partition subordinate to W^u_f (by pulling-back the Markov partition $h(\xi)$ for A).
Invariance Principle. (Avila-Viana, Tahzibi-Yang)

$$H_{vol}(f^{-1}(\xi)|\xi) \leq H_{h_* vol}(A^{-1}(h(\xi))|h(\xi))$$

and the equality holds if and only if the conditional measures are invariant under the center holonomy. We have

$$\chi^{uu}_f = H_{vol}(f^{-1}\xi|\xi) \leq H_{h_* vol}(A^{-1}(h(\xi))|h(\xi)) \leq \chi^{uu}_L = \chi^{uu}_A$$

By the assumption on the Lyapunov spectrum ($\star$) we have that both inequalities above are, in fact, equalities. Hence, the center holonomy is absolutely continuous, hence, smooth, both within W^{cu} and W^{cs}. Therefore W^c is a smooth circle fibration. Straightening this fibration we can conjugate f to a diffeomorphism of the form

$$(x, y, z) \mapsto (g(x, y), \alpha_{(x,y)}(z))$$

Then applying the earlier 2-dimensional argument to A and g we obtain that g is smoothly conjugate to A. □

4.2. *Dimension 4.*

Theorem. *Let $L\colon \mathbb{T}^4 \to \mathbb{T}^4$ be an irreducible partially hyperbolic diffeomorphism, $\lambda_1 < |\lambda_2| = |\lambda_3| = 1 < \lambda_4$. And let f be a volume preserving C^{infty} small perturbation with the same Lyapunov exponents ($\star$). Then f is smoothly conjugate to L.*

This relies on two big reults:

F. Rodriguez Hertz dichotomy: Either

1. f is conjugate to L (and conjugacy is smooth along W^c via a KAM argument); or
2. f is accessible.

Avila-Viana: If f is accessible then f has at least one non-zero center exponent.

Combining this with ($\star$) we have that f is conjugate to L. We only need to check smoothness along stable and unstable foliations. Berg proved that volume is the unique measure of maximal entropy for L. As before we have:

$$h_{h_* vol}(L) = h_{vol}(f) = \chi^{uu}_f = \chi^{uu}_L = h_{vol}(L)$$

Hence, the same argument as in dimension 2, we obtain smoothness along W^{uu}, and, similarly, along W^{ss}.