RIGIDITY LECTURE NOTES I

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What is rigidity? Weak form of equivalence or coincidence of some simple invariants implies a strong form of equivalence. Today we will discuss several examples of rigidity of rank one, *i.e.,* for maps, diffeomorphisms and flows.[1]

1. **Shub-Sullivan.** *Let $f, g\colon S^1 \to S^1$ be C^r, $r \geq 2$, expanding maps. Assume that there exists a absolutely continuous conjugacy: $h \circ f \circ h^{-1} = g$. Then f and g are C^r conjugate.*

Proof. To make the proof very simple also *assume that h is a homeomorphism* and recall the following.

Theorem (Krzyżewski-Sacksteder)**.** *If $f\colon S^1 \to S^1$ is a C^r expanding map. Then there exists a C^{r-1} smooth f-invariant measure $\mu_f = \rho_f(x)dx$.*

By ergodicity of expanding maps we have $h^*\mu_g = \mu_f$. We can assume that $h(0) = 0$. Then

$$I_f(x) = \int_0^x \rho_f(x)dx = \int_0^{h(x)} \rho_g(x)dx = I_g(h(x))$$

Functions I_f and I_g are C^r. Hence, by the implicit function theorem, h is also C^r. □

2. **Expanding maps rigidity.** *Let $E_d : x \mapsto dx$. Let $f\colon S^1 \to S^1$ be a degree d expanding map, $h \circ f = E_d \circ h$. Let λ_f the be Lyapunov exponent of f with respect to μ_f. Assume $\lambda_f = \log d$ then f is smoothly conjugate to E_d.*

Proof.

$$\lambda_f = h_{\mu_f}(f) = h_{h_*\mu_f}(L)$$

But $h_{Leb}(E_d) = \log d$. Hence by uniqueness of the measure of maximal entropy $h_*\mu_f = Leb$. Hence, by Shub-Sullivan, h is smooth. □

3. **Katok entropy rigidity.** *Let (S, g) be a negatively curved surface and let $X^t\colon T^1S \to T^1S$ be its geodesic flow. Denote by λ the Liouville measure. Then*

$$h_\lambda(X^t) = h_{top}(X^t)$$

if and only of g is a hyperbolic metric (constant curvature).

4. **Avez rigidity.** *If $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ is a C^r, $r \geq 2$, Anosov diffeomorphism which has C^2 stable and unstable foliations then f is C^r conjugate to a linear automorphism.*

Date: May 13, 2019.

[1]Disclaimer: The rich subject of rigidity of higher rank actions will be ignored.

5. **de la Llave-Marco-Moriyon rigidity.** *Let $f, g\colon \mathbb{T}^2 \to \mathbb{T}^2$ be conjugate C^r-smooth Anosov diffeomorphisms, $h \circ f = g \circ h$. Assume*

$$\forall p = f^k p \quad \exists C : \; Df^k(p) = C Dg^k(h(p)) C^{-1} \tag{$\star$}$$

Then h is $C^{r-\varepsilon}$ smooth.

Proof.

1. *Equilibrium states.* Given an Anosov diffeo f and a Hölder potential φ there exists a unique f-invariant measure $\mu_{f,\varphi}$, called *equilibrium state* which maximizes metric pressure $h_\mu(f) + \int \varphi d\mu$. Note that by uniqueness if ψ is *cohomologous* to φ, $\varphi = \psi + u \circ f - u$, then $\mu_{f,\varphi} = \mu_{f,\psi}$. The equilibrium state for $-\log Jac^u(f)$ is the SRB measure which has absolutely continuous conditional measure on unstable leaves of f. Denote this equilibrium state by m_f.
2. *Functoriality.* If $f = h^{-1} \circ g \circ h$ then $h^*\mu_{g,\varphi} = \mu_{f,\varphi\circ h}$. Follows directly from functoriality of metric entropy.
3. *Smoothness along foliations.*

   Theorem (Livshits). *If f is a transitive Anosov diffeomorphism and φ and ψ are Hölder potentials such that*

   $$\forall p = f^k p : \qquad \sum_{x \in \mathcal{O}(p)} \varphi(x) = \sum_{x \in \mathcal{O}(p)} \psi(x)$$

   then φ is cohomologous to ψ.

   Then ($\star$) verifies the assumption of Livshits and implies that $\varphi = -\log Jac^u f$ is cohomologous to $\psi \circ h = -\log Jac^u g \circ h$. And hence, by functoriality,

   $$m_f = \mu_{f,\varphi} = \mu_{f,\psi\circ h} = h^*\mu_{g,\psi} = h^* m_g$$
4. *Calculus.* If ξ is a measurable partition subordinate to W^u_f then for m_f almost every x we have that $h|_{\xi(x)}\colon \xi(x) \to h(\xi(x))$ is absolutely continuous and, hence, smooth. It follows that $h \in C^r_u(\mathbb{T}^2)$, that is, h is smooth along unstable leaves. Applying the same argument to f^{-1} and g^{-1} which are conjugate via the same h, $h \circ f^{-1} = g^{-1} \circ h$, we also obtain $h \in C^r_s(\mathbb{T}^2)$. Then it remains to prove that

   $$C^{r-\varepsilon}(\mathbb{T}^2) \subset C^r_s(\mathbb{T}^2) \cap C^r_u(\mathbb{T}^2)$$

   which is an easy exercise when $r = \infty$.

□

6. **Otal-Croke marked length spectrum rigidity.** *Let (S, g_1) and (S, g_2) be negatively curved surfaces. Denote by $[\gamma]$ the free homotopy class loops $S^1 \to S$. For any non-trivial $[\gamma]$ consider the unique geodesic representatives $\gamma_1 \in [\gamma]$ and $\gamma_2 \in [\gamma]$ for g_1 and g_2, correspondingly and assume that*

$$\ell_{g_1}(\gamma_1) = \ell_{g_2}(\gamma_2) \tag{$\blacklozenge$}$$

Then there exists an isometry $\sigma\colon (S, g_1) \to (S, g_2)$.

Recently local marked length spectrum rigidity was established by Guillarmou-Lafeuvre for higher dimensional negatively curved manifolds M. Namely, if g_2 is sufficiently close to g_1 in C^N topology, $N = \frac{3}{2}\dim M + 8$, then ($\blacklozenge$) implies isometry.

To make the proof very simple assume that $g_2 = \rho^2 g_1$ (which is, after moding out by Diff(S), a finite codimension assumption in the case of surfaces). This proof is due to Katok and precedes Otal-Croke.

Proof. Denote by λ_i the Riemannian volume and by A_i the total area of g_i, $i = 1, 2$. Using Birkhoff ergodic theorem and Anosov closing lemma we can approximate λ_1 by a measure supported on a single periodic geodesic γ: $\lambda_1 \approx \frac{A_1}{\ell_{g_1}(\gamma)}\delta_\gamma$, where δ_γ is the uniform measure on γ. Then

$$A_2 = \int_S \rho^2 d\lambda_1 = \frac{\int_S \rho^2 d\lambda_1 \int_S d\lambda_1}{\int_S d\lambda_1} \geq \frac{\left(\int_S \rho d\lambda_1\right)^2}{A_1} \approx \left(\frac{A_1}{\ell_{g_1}(\gamma)}\right)^2 \frac{\left(\int_\gamma \rho d\delta_\gamma\right)^2}{A_1}$$
$$= A_1 \frac{\ell_{g_2}(\gamma)^2}{\ell_{g_1}(\gamma)^2} \geq A_1 \frac{\ell_{g_2}(\bar\gamma)^2}{\ell_{g_1}(\gamma)^2} = A_1,$$

where $\bar\gamma$ is the g_2-geodesic homotopic to γ. Hence we obtain that $A_2 \geq A_1$ and, using the symmetric argument we also have $A_1 \geq A_2$. It follows that the equality must be achieved in the Cauchy-Shwartz inequality above. Hence ρ is constant, and, hence, $\rho = 1$. □

7. **Rigidity of Anosov flows in dimension 3.** Anosov flows $X_1^t, X_2^t\colon M \to M$ are called *conjugate* via $h\colon M \to M$ if

$$\forall t \qquad h \circ X_1^t = X_2^t \circ h$$

If X_1^t and X_2^t are C^1 close transitive Anosov flows then Anosov structural stability yields an *orbit equivalence* $h\colon M \to M$ which send orbits of X_1 to orbits of X_2 preserving the time direction. However, typically a true conjugacy does not exists. Indeed, similarly to (♦), the periods of periodic orbits provide an obstruction to existence of the conjugacy. Applying the flow version of Livshits theorem to $D_{X_1}h - 1$ yields the following characterization.

Let X_1^t and X_2^t be orbit equivalent transitive Anosov flows. Assume that

$$\forall p \in Per(X_1): \ \ \mathrm{per}_{X_1}(p) = \mathrm{per}_{X_2}(h(p))$$

Then X_1^t is conjugate to X_2^t. Further, following in the footsteps of the proof for 2-dimensional Anosov diffeos one can obtain the following.

Theorem (de la Llave-Moriyon, Pollicott)**.** *Assume that X_1^t and X_2^t are conjugate transitive Anosov flows. Also assume, analogously to ($\star$), that the differentials of Poincaré return maps for all periodic points are conjugate. Then X_1^t and X_2^t are smoothly conjugate.*

Note that in the setting of geodesic flows any conjugacy is automatically smooth (one has that the conjugacy is volume preserving as an intermediate step in Otal's MLS rigidity proof and, hence, is smooth by following the de la Llave-Moriyon argument). That is, the assumption on differentials of return maps at periodic orbits is redundant. More generally, Feldman-Ornstein showed that the same holds for transitive contact Anosov flows. Hamenstädt generalized this result to higher dimensions assuming additionally C^1 stable and unstable foliations. We offer the following generalization.

Theorem (A.G. – F. Rodriguez Hertz)**.** *Let X_1^t and X_2^t be conjugate 3-dimensional transitive Anosov flows. Then either the conjugacy is smooth or X_1^t is a constant roof suspension of an Anosov diffeomorphism of $\mathbb{T}^2$.*