BEYOND2019 - SPECIFICATION - NOTES FOR LECTURE 3

VAUGHN CLIMENHAGA

Disclaimer. These notes are my preliminary attempt to provide a little more detail for my third lecture during the 2019 *Dynamics Beyond Uniform Hyperbolicity* conference at CIRM, and are still quite informal and incomplete. The goal of this third lecture is to describe some applications of the general uniqueness result that was introduced over the course of the first two lectures.

1. Recollection of general result

Let us recall the general definitions and abstract result from the previous lecture. Here X is a compact metric space and $f\colon X \to X$ is a continuous map.

Definition 1.1 ([BF13])**.** An f-invariant measure μ is *almost expansive at scale* ϵ if $\Gamma_\epsilon(x) = \{x\}$ for μ-a.e. x; equivalently, if the *non-expansive set* $\mathrm{NE}(\epsilon) = \{x \in X : \Gamma_\epsilon(x) \neq \{x\}\}$ has $\mu(\mathrm{NE}(\epsilon)) = 0$. Replacing Γ_ϵ by Γ_ϵ^+ gives NE^+ and a notion of *almost positively expansive*.

Definition 1.2 ([CT14])**.** The *entropy of obstructions to expansivity at scale* ϵ is

$$\begin{aligned} h^\perp_{\exp}(X, f, \epsilon) &:= \sup\{h_\mu(f) : \mu \in \mathcal{M}^e_f(X) \text{ is not almost expansive at scale } \epsilon\} \\ &= \sup\{h_\mu(f) : \mu \in \mathcal{M}^e_f(X) \text{ and } \mu(\mathrm{NE}(\epsilon)) > 0\}. \end{aligned}$$

We write $h^\perp_{\exp}(X, f) = \lim_{\epsilon\to 0} h^\perp_{\exp}(X, f, \epsilon)$ for the *entropy of obstructions to expansivity*, without reference to scale. The entropy of obstructions to positive expansivity $h^\perp_{\exp^+}$ is defined analogously.

Definition 1.3. A *decomposition* for $X \times \mathbb{N}$ consists of three collections $\mathcal{C}^p, \mathcal{G}, \mathcal{C}^s \subset X \times \mathbb{N}_0$ for which there exist three functions $p, g, s\colon X \times \mathbb{N} \to \mathbb{N}_0$ such that for every $(x, n) \in X \times \mathbb{N}$, the values $p = p(x, n)$, $g = g(x, n)$, and $s = s(x, n)$ satisfy $p + g + s = n$, and

$$(x, p) \in \mathcal{C}^p, \quad (f^p x, g) \in \mathcal{G}, \quad (f^{p+s} x, s) \in \mathcal{C}^s.$$

Given a decomposition, for each $M \in \mathbb{N}$ we write

$$\mathcal{G}^M := \{(x, n) \in X \times \mathbb{N} : p(x, n) \le M \text{ and } s(x, n) \le M\}.$$

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Theorem 1.4: Uniqueness with small obstructions [CT16]

Let X be a compact metric space and $f\colon X \to X$ a continuous map. Suppose that $\epsilon > 28\delta > 0$ are such that $h^{\perp}_{\text{exp}}(X, f, \epsilon) < h_{\text{top}}(X, f)$, and that the space of orbit segments $X \times \mathbb{N}$ admits a decomposition $\mathcal{C}^p\mathcal{G}\mathcal{C}^s$ such that

(I) every collection $\mathcal{G}^M$ has specification at scale δ, and
(II) $h(\mathcal{C}^p \cup \mathcal{C}^s, \delta) < h_{\text{top}}(X, f)$.

Then (X, f) has a unique measure of maximal entropy.

Remark 1.5. If $\mathcal{G}$ has specification at all scales, then a short continuity argument proves that every $\mathcal{G}^M$ does as well, which establishes (I).

2. Partially hyperbolic systems

Theorem 1.4 can be applied to a broad class of partially hyperbolic systems, which includes the Mañé examples.[1]

Theorem 2.1 (Not-yet-published result)**.** *Let $f\colon M \to M$ be a partially hyperbolic diffeomorphism with $TM = E^u \oplus E^c \oplus E^s$. Assume that $\dim E^c = 1$ and that every leaf of the foliations W^s and W^u is dense in M.*

Let $\varphi^c(x) = \log \|Df|_{E^c(x)}\|$, and given $\mu \in \mathcal{M}^e_f(X)$, let $\lambda^c(\mu) = \int \varphi^c \, d\mu$ be the center Lyapunov exponent *of μ. Consider the quantities*

$$\begin{aligned} h^+ &:= \sup\{h_\mu(f) : \mu \in \mathcal{M}^e_f(X), \lambda^c(\mu) \geq 0\}, \\ h^- &:= \sup\{h_\mu(f) : \mu \in \mathcal{M}^e_f(X), \lambda^c(\mu) \leq 0\}. \end{aligned} \tag{2.1}$$

Suppose that $h^+ \neq h^-$. Then f has a unique MME.

Remark 2.2. One can obtain an analogous result for equilibrium states for general Hölder potentials φ by replacing $h_\mu(f)$ in (2.1) by $h_\mu(f) + \int \varphi \, d\mu$ and using a generalization of Theorem 1.4 that will be discussed in Dan Thompson's lectures next week.

Remark 2.3. Since $h_{\text{top}}(X, f) = \max(h^+, h^-)$, the condition $h^+ = h^-$ is equivalent to the condition that either $h^+ < h_{\text{top}}(X, f)$ or $h^- < h_{\text{top}}(X, f)$. The only way for this condition to fail is if there is an ergodic MME with $\lambda^c = 0$, or if there are (at least) two ergodic MMEs for which λ^c takes both signs.

An elementary argument using properties of topological pressure shows that $h^+ = h^-$ if and only if the function $t \mapsto P(t\varphi^c)$ has a global minimum at $t = 0$. Thus one can restate the last line of Theorem 2.1 as the conclusion that f has a unique MME if there is $t \neq 0$ such that $P(t\varphi^c) < P(0) = h_{\text{top}}(f)$.

[1]The general result described here is still preliminary in the sense that while nearly all the details have been written down in one form or another, there is no preprint on arXiv yet.

First observe that arguments similar to those given for the Mañé example in the previous lecture's notes show that $h^\perp_{\rm exp}(f) \leq \min(h^+, h^-)$, so the condition $h^\perp_{\rm exp}(f) < h_{\rm top}(f)$ is satisfied whenever $h^+ \neq h^-$.

Remark 2.4. The upper bound on $h^\perp_{\rm exp}$ for the Mañé examples in the last set of notes is actually an upper bound on h^+ in that setting, verifying that $h^+(g) < h_{\rm top}(g)$ whenever the perturbation is small enough. Moreover, the leaves of W^u are all dense for these examples [PS06], so Theorem 2.1 applies to the Mañé examples.

The rest of the proof of Theorem 2.1 consists of finding a decomposition $\mathcal{C}^p, \mathcal{G}, \mathcal{C}^s$ for $X \times \mathbb{N}$ such that $\mathcal{G}$ has specification at all scales and $h(\mathcal{C}^p \cup \mathcal{C}^s) < h_{\rm top}(X, f)$. We describe the general argument in the case when $h^+ < h_{\rm top}(f)$, so intuitively, all of the large entropy parts of the system have negative central Lyapunov exponents.

2.1. **A small collection of obstructions.** We take $\mathcal{C}^s = \emptyset$. To describe $\mathcal{C}^p$, we first observe that the condition $h^+ < h_{\rm top}(f)$ implies that

$$\sup\{h_\mu(f) : \mu \in \mathcal{M}_f, \lambda^c(\mu) \geq 0\} < h_{\rm top}(f),$$

where the difference is that now the supremum allows non-ergodic measures as well, and then a weak*-continuity argument gives $r > 0$ such that

$$\sup\{h_\mu(f) : \mu \in \mathcal{M}_f, \lambda^c(\mu) \geq -r\} < h_{\rm top}(f). \tag{2.2}$$

We can relate the left-hand side of (2.2) to $h(\mathcal{C}^p)$, where

$$\mathcal{C}^p := \{(x, n) \in M \times \mathbb{N} : S_n\varphi^c(x) \geq -rn\}.$$

One relationship between these was mentioned when we bounded $h^\perp_{\rm exp}$ for the Mañé example (though the function being summed there was different). Here we want to go the other way and obtain an upper bound on $h(\mathcal{C}^p)$. For this we observe that if we let $E_n \subset \mathcal{C}^p_n$ be any (n, ϵ)-separated set, ν_n the equidistributed atomic measure on E_n, and $\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} f^k_*\nu_n$, then half of the proof of the variational principle [Wal82, Theorem 8.6] shows that any limit point of μ_n is f-invariant and has

$$h_\mu(f) \geq h(\mathcal{C}^p, \epsilon).$$

Moreover, $\lambda^c(\mu) = \int \varphi^c \, d\mu(x) \geq -r$ by weak*-convergence and the definition of $\mathcal{C}^p$. Together with (2.2), we conclude that $h(\mathcal{C}^p) < h_{\rm top}(f)$.

2.2. **A good collection with specification.** Of course we have not yet defined a decomposition because we did not say what $\mathcal{G}$ is, let alone prove that it has specification!

To this end, take an arbitrary orbit segment $(x, n) \in M \times \mathbb{N}$, and remove the longest possible element of $\mathcal{C}^p$ from its beginning. That is, let $p = p(x, n)$ be maximal with the property that $(x, p) \in \mathcal{C}^p$. Then we have

$$S_p\varphi^c(x) \geq -rp \text{ and } S_k\varphi^c(x) < -rk \text{ for all } p < k \leq n.$$

Subtracting the first from the second gives

$$S_{k-p}\varphi^c(f^p x) = S_k\varphi^c(x) - S_p\varphi^c(x) < -r(k-p),$$

which we can rewrite as

$$S_j\varphi^c(f^j x) < -rj \text{ for all } 0 \le j \le n-p.$$

In other words, as shown in Figure 1, we have[2]

$$(f^p x, n-p) \in \mathcal{G} := \{(y,m) : S_j\varphi^c(y) < -rj \text{ for all } 0 \le j \le m\}.$$

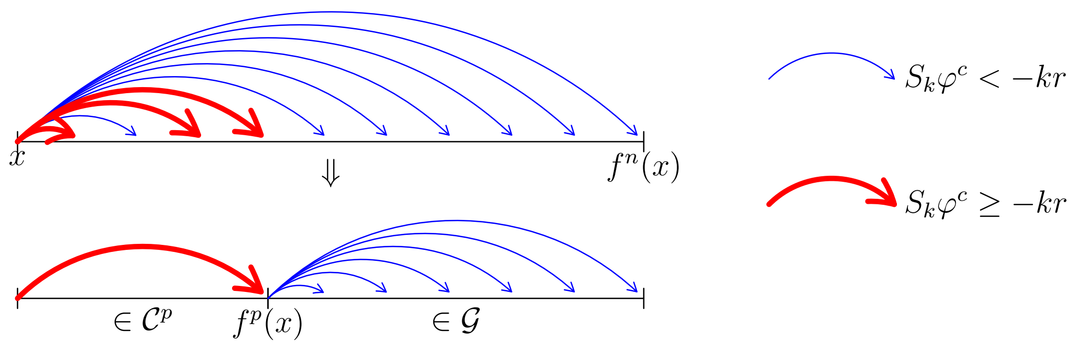

FIGURE 1. A decomposition $\mathcal{C}^p\mathcal{G}$ of the space of orbit segments.

Moreover, by choosing $\delta > 0$ sufficiently small that $|\varphi^c(y) - \varphi^c(z)| < r/2$ whenever $d(y,z) < \delta$, we see that if $(y,m) \in \mathcal{G}$ and $z \in B_m(y,\delta)$, then

$$\|Df^j|_{E^{cs}(z)}\| \le e^{-rj/2} \text{ for all } 0 \le j \le m. \tag{2.3}$$

This is enough to prove the specification property for $\mathcal{G}$. If E^{cs} is integrable, then one can simply use the proof from the uniformly hyperbolic case verbatim, using (2.3) to guarantee that

$$W^{cs}_\delta(x) \subset B_n(x,\delta) \text{ whenever } (x,n) \in \mathcal{G}. \tag{2.4}$$

But even if E^{cs} is not integrable, it is sufficient to take an "admissible manifold" through $f^n(x)$ that is tangent to the center-stable cone, then pull this manifold back by f^{-n} to obtain some $W^{cs}_\delta(x)$ that still satisfies (2.4) and that intersects the image under f^t of the unstable leaf from the end of the previous orbit to which (x,n) is being joined for specification.

3. Application to billiards: the Bunimovich stadium

Let Ω be a Bunimovich stadium with boundary

$$\partial\Omega = \Gamma_1 \cup Y_1 \cup \Gamma_2 \cup Y_2,$$

where Γ_1, Γ_2 are semicircles, and Y_1, Y_2 are parallel line segments; see Figure 2.

We consider the billiard map F associated to this table, which takes a unit vector pointing inwards from the boundary $\partial\Omega$ and moves along the associated direction until the next encounter with the boundary, at which point it changes direction

[2] There is a clear analogy between what we are doing here and the notion of *hyperbolic time* introduced by Alves [Alv00, ABV00].

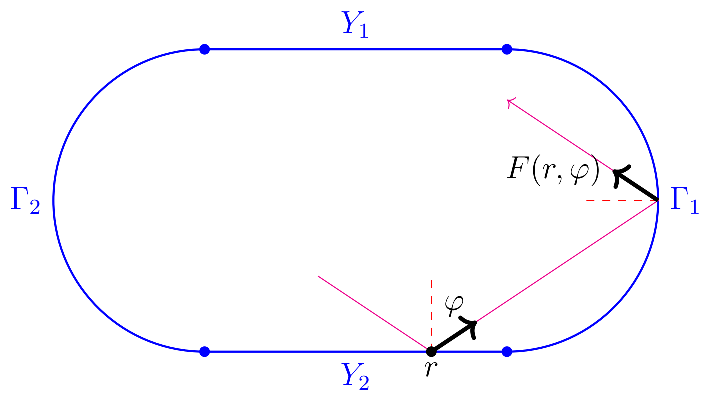

FIGURE 2. The Bunimovich stadium.

according to the law of reflection: outgoing angle is equal to incoming angle. Thus the phase space for F is given by

$$X := \{x = (r,\varphi) : r \in \partial\Omega, \varphi \in (-\pi/2, \pi/2)\}.$$

As shown in Figure 2, (r,φ) represents the unit vector based at point $r \in \partial\Omega$ and making an angle φ with the inward-pointing normal to $\partial\Omega$. It is often convenient to picture X as a rectangle by identifying $\partial\Omega$ with the interval $[0, |\partial\Omega|]$, where the endpoints are identified; see Figure 3.

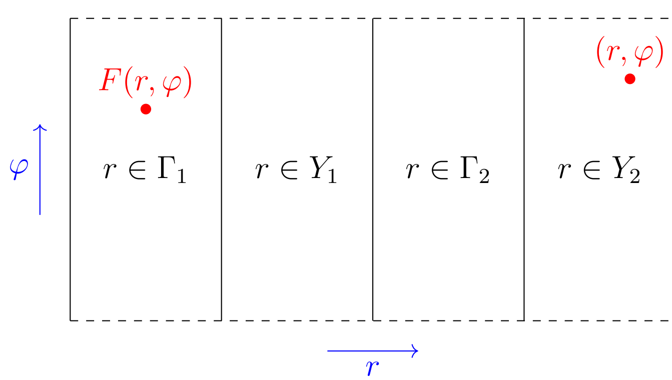

FIGURE 3. Phase space for the Bunimovich stadium map.

Theorem 3.1 (Jianyu Chen, V.C., Hong-Kun Zhang[3]). *The billiard map for the Bunimovich stadium has a unique measure of maximal entropy.*

Remark 3.2. As with the partially hyperbolic examples above, one can prove a corresponding result for equilibrium states, but there are some limitations on the class of potentials that can be considered: one needs φ to extend continuously to the

[3]This theorem is still preliminary in the sense that while nearly all the details have been written down in one form or another, there is no preprint on arXiv yet.

closure of the above rectangle in such a way that φ is constant on each component of $Y_i \times \{\pm\pi/2\}$,[4] and one also needs to impose a certain "pressure gap" condition.

To deduce Theorem 3.1 from the abstract result Theorem 1.4, we first need to compactify X so that F can be treated as a continuous map on a compact metric space. The obvious compactification has four discontinuity points where r lies on the intersection of Γ_i and Y_j and $\varphi = \pi/2$ or $-\pi/2$, whichever points along Y_j. To work around this problem we instead consider the two-point compactification $\overline{X}$ in which $\partial\Omega \times \{-\pi/2\}$ and $\partial\Omega \times \{\pi/2\}$ represent the two additional points (so $\overline{X}$ is topologically a sphere), which are fixed by F. Equip $\overline{X}$ with a metric ρ such that

$$\rho(x,y) \leq d(x,y) \text{ for all } x,y \in X, \tag{3.1}$$

and such that in addition, for every $\eta > 0$ there is $C > 0$ such that

$$d(x,y) \leq C\rho(x,y) \text{ for all } x,y \in X(\eta) := \{(r,\varphi) \in X : |\varphi| \leq \pi/2 - \eta\}. \tag{3.2}$$

Using the bound in (3.1), specification w.r.t. d (which we will prove for an appropriate $\mathcal{G}$) implies specification w.r.t ρ (which we need to apply Theorem 1.4). Similarly, (3.2) will allow us to obtain expansivity w.r.t. ρ.

The representation of X as a rectangle allows us to identify each tangent space T_xX with $\mathbb{R}^2$ in a canonical way, with basis given by partial derivatives w.r.t. r, φ. Given $v \in T_xX$, we write $v = (dr, d\varphi)$ for the representation of v w.r.t. this basis; then we fix $c > 0$ and consider the following cones in $\mathbb{R}^2 = T_xX$:

$$K^u := \{v : -dr \leq d\varphi \leq -c\,dr\} \quad \text{and} \quad K^s := \{c\,dr \leq d\varphi \leq dr\}.$$

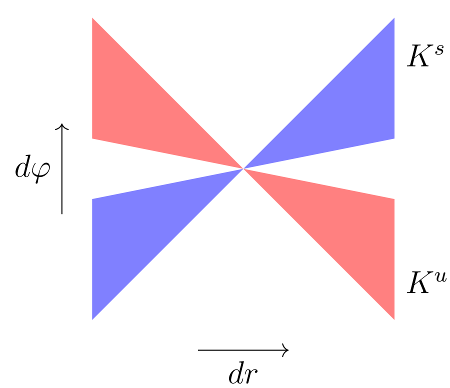

FIGURE 4. Invariant cones for the stadium map.

Now we consider the *regular set*

$$R_0 = (\Gamma_1 \cup \Gamma_2) \times (-\pi/2, \pi/2)$$

consisting of all collisions with one of the semi-circles. Say that an orbit segment (x,n) with endpoints in R_0 *crosses the stadium* if it hits both semi-circles

[4]This also requires us to use a different compactification than the one described here.

at least once; that is, if there are $k_1, k_2 \in \{0, 1, \ldots, n-1\}$ such that $F^{k_i}(x_i) \in \Gamma_i \times (-\pi/2, \pi/2)$. Let

$$\mathcal{G}^0 = \{(x, n) \in X \times \mathbb{N} : x, F^{n-1}(x) \in R_0 \text{ and } (x, n) \text{ crosses the stadium}\}.$$

The following uses properties of the Bunimovich stadium, and we omit the proof.

Lemma 3.3. *There are $\Lambda > 1$ and $c > 0$ such that if $(x, n) \in \mathcal{G}^0$, then*

$$DF_x^n(K^u) \subset K^u \quad \text{and} \quad (DF_x^n)^{-1}(K^s) \subset K^s,$$

and for any $v^u \in K^u$, $v^s \in K^s$, we have

$$\|DF_x^n v^u\| \geq \Lambda \|v^u\|, \quad \text{and} \quad \|(DF_x^n)^{-1} v^s\| \geq \Lambda \|v^s\|.$$

Now fix $\eta > 0$ and let

$$R_\eta = \{(r, \varphi) \in X : d(r, Y) \geq \eta \text{ and } |\varphi| \leq \pi/2 - \eta\}.$$

Then consider the collection of orbit segments

$$\mathcal{G}^\eta := \{(x, n) \in X \times \mathbb{N} : x, F^{n-1}(x) \in R_\eta \text{ and } (x, n) \text{ crosses the stadium}\}.$$

Using Lemma 3.3 and replacing W^s, W^u with curves tangent to the cones K^s, K^u, one can run the specification argument just as in the uniformly hyperbolic case and use topological transitivity to deduce that

Lemma 3.4. *For every $\eta > 0$, $\mathcal{G}^\eta$ has specification at all scales.*

Moreover, writing

$$\mathcal{B}^\eta = \{(x, n) : F^k x \in R_\eta^c = \overline{X} \setminus R_\eta \text{ for all } 0 \leq k < n\},$$

it is easy to see that $\mathcal{B}^\eta, \mathcal{G}^\eta, \mathcal{B}^\eta$ gives a decomposition for the space of orbit segments $\overline{X} \times \mathbb{N}$: indeed, given (x, n) it suffices to take $p, s \in \{0, 1, \ldots, n-1\}$ minimal such that $f^p(x) \in R_\eta$ and $f^{n-p}(x) \in R_\eta$; then clearly $(x, p) \in \mathcal{B}^\eta$, $(f^p x, n-p-s) \in \mathcal{G}^\eta$, and $(f^{n-p}x, s) \in \mathcal{B}^\eta$.

Using arguments as in §2.1, one can make the following observations.

- If μ_η is a limit point of the usual maximizing measure construction for $\mathcal{B}^\eta$, then $h_{\mu_\eta}(F) \geq h(\mathcal{B}^\eta)$ and $\mu_\eta(R_\eta^c) = 1$.
- If μ is a limit point of μ_η as $\eta \to 0$, then $\mu(R_0^c) = 1$ and thus $h_\mu(f) = 0$. In particular, by upper semi-continuity of entropy (which takes a proof!), $h_{\mu_\eta}(f) \to 0$ as $\eta \to 0$.

Combining these gives $\lim_{\eta \to 0} h(\mathcal{B}^\eta) = 0$, and in particular there is $\eta > 0$ such that $h(\mathcal{B}^\eta) < h_{\text{top}}(\overline{X}, F)$.

Finally, one can argue that any measure giving positive weight to some R_η is expansive (use Poincaré recurrence to get infinitely many returns to R_η, each of which gives a uniform amount of hyperbolicity), and thus $h_{\text{exp}}^\perp(F) = 0$. This verifies the hypotheses of Theorem 1.4 for $(\overline{X}, F)$, and thus there is a unique MME.

4. General result for flows

In order to state a version of Theorem 1.4 for flows, we need to give an appropriate definition of expansivity. Given a continuous flow $\{f_t\colon X \to X\}$, for each $x \in X$ and $\epsilon > 0$ we consider the set[5]

$$\Gamma_\epsilon(x) = \{y \in X : d(f_t x, f_t y) < \epsilon \text{ for all } t \in \mathbb{R}\}.$$

Definition 4.1. A flow-invariant measure μ is *almost expansive at scale* $\epsilon > 0$ if for μ-a.e. $x \in X$ there is $s > 0$ such that $\Gamma_\epsilon(x) \subset \{f_t x : t \in [-s, s]\}$.

Then we define the entropy of obstructions to expansivity in the same way as before:

$$h^\perp_{\exp}(\{f_t\}, \epsilon) := \sup\{h_\mu(f_1) : \mu \text{ is } \textit{not} \text{ almost expansive at scale } \epsilon\}.$$

Specification, decompositions, and entropy of a collection of orbit segments are defined for flows in the obvious way, by adapting the discrete-time definitions.

Theorem 4.2: A general uniqueness result for flows [CT16]

Let X be a compact metric space and $\{f_t\}\colon X \to X$ a continuous flow. Suppose that $\epsilon > 40\delta > 0$ are such that $h^\perp_{\exp}(X, \{f_t\}, \epsilon) < h_{\text{top}}(X, \{f_t\})$, and that the space of orbit segments $X \times \mathbb{N}$ admits a decomposition $\mathcal{C}^p\mathcal{G}\mathcal{C}^s$ such that

(I) every collection $\mathcal{G}^M$ has specification at scale δ, and
(II) $h(\mathcal{C}^p \cup \mathcal{C}^s, \delta) < h_{\text{top}}(X, \{f_t\})$.

Then $(X, \{f_t\})$ has a unique measure of maximal entropy.

5. Geodesic flows

Let M be a closed connected Riemannian manifold and $F = \{f_t\colon T^1M \to T^1M\}_{t\in\mathbb{R}}$ its geodesic flow; that is, $f_t(v) = \dot c_v(t)$, where $c_v\colon \mathbb{R} \to M$ is the unique unit speed geodesic with $\dot c_v(0) = v$.

If all sectional curvatures of M are negative at every point, then F is a transitive Anosov flow; in particular, it is expansive and has specification, and there is a unique measure of maximal entropy.

To go beyond negative curvature, one generally needs the tools of non-uniform hyperbolicity.[6] There are three further classes of manifolds that generally exhibit some kind of non-uniformly hyperbolic behaviour: nonpositive curvature; no focal points; and no conjugate points. The relationships are as follows:

$$\text{negative curv.} \Rightarrow \text{nonpositive curv.} \Rightarrow \text{no focal points} \Rightarrow \text{no conjugate points.}$$

[5] There is a difference between this and the original formulation of expansivity for flows by Bowen and Walters: they allow reparametrizations and thus obtain a potentially larger set Γ_ϵ. Our definition of expansivity (even without the "almost everywhere" part) is weaker than theirs, but is still sufficient for the uniqueness result.

[6] There are some examples of manifolds with some positive curvature where the geodesic flow is still Anosov, but one should not expect this in general.

The reverse implications all fail in general.

The definition of nonpositive curvature is easy: all sectional curvatures are ≤ 0 at every point. No focal points and no conjugate points are defined in terms of Jacobi fields. If we work in the universal cover $\widetilde{M}$, then no focal points is equivalent to the condition that $t \mapsto d(c_1(t), c_2(t))$ be nondecreasing whenever c_1, c_2 are geodesics with $c_1(0) = c_2(0)$, while no conjugate points is equivalent to the condition that this function never vanish for $t > 0$; in other words, there is at most one geodesic connecting any two points in $\widetilde{M}$.

5.1. **Nonpositive curvature and no focal points.** To prove uniqueness of the MME in the setting of nonpositive curvature, one must first rule out counterexamples such as direct products by imposing a "rank 1" condition; in dimension 2 this turns out to be equivalent to asking that the genus be ≥ 2. Under this condition, Knieper proved uniqueness of the MME in any dimension using Patterson–Sullivan measures on the ideal boundary [Kni98]. His result has recently been extended to manifolds with no focal points by Fei Liu, Fang Wang, and Weisheng Wu [LWW18].

An alternate proof of uniqueness of the MME for rank 1 manifolds with nonpositive curvature was recently given using specification-based techniques [BCFT18]; this also established existence and uniqueness of a broad class of equilibrium states, and will be the main focus of Dan Thompson's talks next week. This approach has been extended to surfaces without focal points by Dong Chen, Nyima Kao, and Kiho Park [CKP18]. Another specification-based proof of uniqueness of the MME for surfaces without focal points was given by Katrin Gelfert and Rafael Ruggiero [GR19].

5.2. **No conjugate points.** When M is merely assumed to have no conjugate points, life is substantially harder because many of the geometric tools used in the previous settings are no long available, such as convexity of horospheres, monotonicity of the distance function, and continuity of the stable and unstable foliations of T^1M (cf. the "dinosaur" example of Ballmann, Brin, and Burns [BBB87]).

Under the additional (strong) assumption that the flow is expansive, uniqueness of the MME was proved by Aurélien Bosché, a student of Knieper, in his Ph.D. thesis [Bos18]. The following result says that at least in dimension 2, we can remove the assumption of expansivity.

Theorem 5.1 (V.C., Gerhard Knieper, Khadim War [CKW19])**.** *Let M be a closed manifold of dimension 2, with genus ≥ 2, equipped with a smooth Riemannian metric without conjugate points. Then the geodesic flow on T^1M has a unique measure of maximal entropy.*

Remark 5.2. A higher-dimensional version of Theorem 5.1 is available [CKW19], but requires additional assumptions on M: existence of a 'background' metric with negative curvature; the divergence property; residually finite fundamental group; and a certain 'entropy gap' condition. All of these can be verified for every metric without conjugate points on a surface of genus 2.

Remark 5.3. It is a somewhat surprising consequence of the proof technique (to be described momentarily) that we have no idea how to extend this result to any nonzero potentials!

To prove Theorem 5.1, we will actually establish specification for the entire system, without the need for passing to a subcollection of orbit segments. The key tool is the *Morse Lemma*, which states that if g, g_0 are two metrics on M such that g has no conjugate points and g_0 has negative curvature, then there is a constant $R > 0$ such that if c, α are geodesic segments w.r.t. g, g_0, respectively, in the universal cover $\widetilde{M}$ that agree at their endpoints, then they remain within a distance R for along their entire length.

Since M is a surface of genus ≥ 2, it admits a metric of negative curvature. Given an orbit segment $(v,t) \in T^1M \times (0,\infty)$ for the g-geodesic flow, let p, q be the start and end points of some lift of the corresponding g-geodesic segment to the universal cover. Let $(w,s) \in T^1M \times (0,\infty)$ be the unique unit tangent vector that begins a g_0-geodesic segment starting at p and ending at q. Then $E\colon (v,t) \mapsto (w,s)$ defines a map from the space of g-orbit segments to the space of g_0-orbit segments with the property that (v,t) and $E(v,t)$ remain within R for their entire lengths.

Using this correspondence, one can take a finite sequence of g-orbit segments $(x_1,t_1),\dots,(x_k,t_k)$, find g_0-orbit segments $E(x_i,t_i)$ that remain within R, and shadow these (say, to within R) using the specification property for the (Anosov) g_0-geodesic flow by a single orbit segment (y,T). Then $E^{-1}(y,T)$ is a $3R$-shadowing orbit (w.r.t. g) for the original segments (x_i,t_i), for which the transition times are uniformly bounded. This argument, when properly fleshed out, proves that the geodesic flow for g has specification at scale $3R$.[7]

Now all we need is to prove that obstructions to expansivity at scale $120R$ have small entropy. The problem with this is that R itself, and especially $120R$, is likely much larger than the diameter of M. So at this point, it looks like the previous paragraph is completely vacuous – *any* orbit segment of the appropriate length shadows the (x_i,t_i) segments to within $3R$.

The way out is to pass to a finite cover. By gluing together enough copies of a fundamental domain for M,[8] one can find a finite covering manifold N whose injectivity radius is $> 360R$. Observe that

- the geodesic flow on T^1M is a finite-to-1 factor of the geodesic flow on T^1N, so there is an entropy-preserving bijection between their spaces of invariant measures, and in particular there is a unique MME for the geodesic flow over M if and only if there is a unique MME over N;
- the argument for specification that we gave above still works for the geodesic flow on N, with the same scale, because this scale comes from the Morse Lemma and is given at the level of the universal cover.

[7]In fact the scale ends up being a more complicated function of R, but the idea is the same.

[8]Formally, one needs to take a finite index subgroup of $\pi_1(M)$ that avoids all non-identity elements corresponding to a large ball in $\widetilde{M}$; this is possible because $\pi_1(M)$ is *residually finite*.

So it only remains to argue that $h^\perp_{\mathrm{exp}}(120R) < h_{\mathrm{top}}$ for the geodesic flow on N. This is done by observing that if $d(f_t v, f_t w) \le 120R$ for all $t \in \mathbb{R}$ but w does not lie on the orbit of v, then lifting to geodesics on $\widetilde{M}$ and using the fact that we are below the injectivity radius of N allows us to conclude that the lifts of v, w are tangent to distinct geodesics between the same pair of points on the ideal boundary $\partial\widetilde{M}$. Thus if μ is any ergodic invariant measure that is *not* almost expansive at scale $120R$, then μ gives full weight to the set of vectors tangent to such "non-unique geodesics".

On the other hand, if $h_\mu > 0$, then μ is a hyperbolic measure by the Margulis–Ruelle inequality, and thus by Pesin theory, μ-a.e. v has transverse stable and unstable leaves. These leaves are the normal vector fields to the stable and unstable horospheres, and thus these horospheres meet at a single point, meaning that the geodesic through v is the *unique* geodesic between its endpoints on the ideal boundary. By the previous paragraph, this means that μ is almost expansive. It follows that $h^\perp_{\mathrm{exp}}(120R) = 0 < h_{\mathrm{top}}$, and Theorem 4.2 establishes existence and uniqueness of the MME.

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