BEYOND2019 - SPECIFICATION - NOTES FOR LECTURE 2

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Disclaimer. These notes are my preliminary attempt to provide a little more detail for my second lecture during the 2019 *Dynamics Beyond Uniform Hyperbolicity* conference at CIRM, and are still quite informal and incomplete. The goal of this second lecture is to describe "obstructions to expansivity" for partially hyperbolic systems, and formulate a general uniqueness result. The third lecture will describe further applications, including billiards and geodesic flows. Once again there is more material here than I really expect to get through in 50 minutes, so there will be some editing after the fact...

Correction: *During the live lecture I made a remark that the role of expansivity was to guarantee that for a fixed* $\epsilon > 0$*, the set of Bowen balls* $\{B_n(x,\epsilon) : x \in X, n \in \mathbb{N}\}$ *generates the Borel* σ*-algebra. In fact this is not correct: for example, if* f *is a circle rotation then* $B_n(x,\epsilon) = B(x,\epsilon)$ *for all* n*, and these sets generate the Borel* σ*-algebra because we can take intersections of them to obtain arbitrarily small balls. What I should have said is that expansivity guarantees that every partition of sufficiently small diameter is a generating partition for every invariant measure. Then one observes that it suffices to be a generating partition for measures of large entropy, and this motivates the definition of* $h^\perp_{\text{exp}}$.

1. Uniformly hyperbolic systems

Now we describe how Bowen's approach works when X is a transitive locally maximal hyperbolic set for a diffeomorphism f. First we recall some basic definitions.

1.1. Topological entropy.

Definition 1.1. Given $n \in \mathbb{N}$, the nth *dynamical metric* on X is

$$d_n(x,y) := \max\{d(f^k x, f^k y) : 0 \le k < n\}. \tag{1.1}$$

The *Bowen ball of order* n *and radius* $\epsilon > 0$ *centered at* $x \in X$ is

$$B_n(x,\epsilon) := \{y \in X : d_n(x,y) < \epsilon\}. \tag{1.2}$$

A set $E \subset X$ is called (n,ϵ)*-separated* if $d_n(x,y) > \epsilon$ for all $x,y \in E$ with $x \neq y$; equivalently, if $y \notin B_n(x,\epsilon)$ for all such x,y.

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We define entropy in a more general way than is standard, reflecting our focus on the "space of finite-length orbit segments" $X \times \mathbb{N}$ as the relevant object of study; this replaces the language $\mathcal{L}$ that we used in the symbolic setting. The analogy is that a cylinder $[w]$ for a word in the language corresponds to a Bowen ball $B_n(x, \epsilon)$ associated to an orbit segment $(x, n) \in X \times \mathbb{N}$, where we reiterate that $(x, n) \in X \times \mathbb{N}$ should be thought of as representing the orbit segment $(x, fx, f^2x, \dots, f^{n-1}x)$. Given a collection of orbit segments $\mathcal{D} \subset X \times \mathbb{N}$, for each $n \in \mathbb{N}$ we write

$$\mathcal{D}_n := \{x \in X : (x, n) \in \mathcal{D}\} \tag{1.3}$$

for the collection of points that begin a length-n orbit segment in $\mathcal{D}$.

Definition 1.2 (Topological entropy)**.** Given a collection of orbit segments $\mathcal{D} \subset X \times \mathbb{N}$, the *entropy* of $\mathcal{D}$ at scale $\epsilon > 0$ is

$$h(\mathcal{D}, \epsilon) := \overline{\lim_{n\to\infty}} \frac{1}{n} \log \max\{\#E : E \subset \mathcal{D}_n \text{ is } (n, \epsilon)\text{-separated}\}, \tag{1.4}$$

and the entropy of $\mathcal{D}$ is

$$h(\mathcal{D}) := \lim_{\epsilon\to 0} h(\mathcal{D}, \epsilon). \tag{1.5}$$

When $\mathcal{D} = Y \times \mathbb{N}$ for some $Y \subset X$, we write $h_{\text{top}}(Y, \epsilon) = h(Y \times \mathbb{N}, \epsilon)$ and $h_{\text{top}}(Y) = \lim_{\epsilon\to 0} h_{\text{top}}(Y, \epsilon)$. In particular, when $\mathcal{D} = X \times \mathbb{N}$ we write $h_{\text{top}}(X, f) = h_{\text{top}}(X) = h(X \times \mathbb{N})$ for the *topological entropy* of $f : X \to X$.

When different orbit segments in $\mathcal{D}$ are given weights according to their ergodic sum w.r.t. a given potential φ, we obtain a notion of *topological pressure*, which we will discuss in a later lecture.

Theorem 1.3 (Variational principle)**.** *Let X be a compact metric space and $f : X \to X$ a continuous map. Then*

$$h_{\text{top}}(X, f) = \sup_{\mu\in\mathcal{M}_f(X)} h_\mu(f). \tag{1.6}$$

The following construction forms one half of the proof of the variational principle.

Proposition 1.4 (Building a measure of almost maximal entropy)**.** *With X, f as above, fix $\epsilon > 0$, and for each $n \in \mathbb{N}$, let $E_n \subset X$ be an (n, ϵ)-separated set. Consider the Borel probability measures*

$$\nu_n := \frac{1}{\#E_n} \sum_{x\in E_n} \delta_x, \quad \mu_n := \frac{1}{n} \sum_{k=0}^{n-1} f_*^k \nu_n = \frac{1}{n} \sum_{k=0}^{n-1} \nu_n \circ f^{-k}. \tag{1.7}$$

Let μ_{n_j} be any subsequence that converges in the weak-topology to a limiting measure μ. Then $\mu \in \mathcal{M}_f(X)$ and*

$$h_\mu(f) \geq \overline{\lim_{j\to\infty}} \frac{1}{n_j} \log \#E_{n_j}. \tag{1.8}$$

In particular, for every $\delta > 0$ there exists $\mu \in \mathcal{M}_f(X)$ such that $h_\mu(f) \geq h_{\text{top}}(X, f, \delta)$.

Proof. See [Wal82, Theorem 8.6]. □

Corollary 1.5. *Let X, f be as above, and suppose that there is $\delta > 0$ such that $h_{\mathrm{top}}(X, f, \delta) = h_{\mathrm{top}}(X, f)$. Then there exists a measure of maximal entropy for (X, f). Indeed, given any sequence $\{E_n \subset X\}_{n=1}^{\infty}$ of maximal (n, δ)-separated sets, every weak*-limit point of the sequence μ_n from (1.7) is an MME.*

In our applications, it will often be relatively easy to verify that $h_{\mathrm{top}}(X, f, \delta) = h_{\mathrm{top}}(X, f)$ for some $\delta > 0$, and so Corollary 1.5 establishes existence of a measure of maximal entropy. Thus the real challenge is to prove uniqueness, and this will be our focus.

1.2. **Expansivity and specification.** Now we formulate two crucial properties that are satisfied by uniformly hyperbolic systems.

Definition 1.6 (Expansivity)**.** Given $x \in X$ and $\epsilon > 0$, let

$$\Gamma_\epsilon^+(x) := \{y \in X : d(f^n y, f^n x) < \epsilon \text{ for all } n \geq 0\} = \bigcap_{n \in \mathbb{N}} B_n(x, \epsilon) \tag{1.9}$$

be the forward infinite Bowen ball. If f is invertible, let

$$\Gamma_\epsilon^-(x) := \{y \in X : d(f^n y, f^n x) < \epsilon \text{ for all } n \geq 0\} \tag{1.10}$$

be the backward infinite Bowen ball, and let

$$\Gamma_\epsilon(x) := \Gamma_\epsilon^+(x) \cap \Gamma_\epsilon^-(x) = \{y \in X : d(f^n y, f^n x) < \epsilon \text{ for all } n \in \mathbb{Z}\} \tag{1.11}$$

be the bi-infinite Bowen ball. The system (X, f) is *positively expansive at scale* $\epsilon > 0$ if $\Gamma_\epsilon^+(x) = \{x\}$ for all $x \in X$, and *(two-sided) expansive at scale* $\epsilon > 0$ if $\Gamma_\epsilon(x) = \{x\}$. The system is *(positively) expansive* if there exists $\epsilon > 0$ such that it is (positively) expansive at scale ϵ.

Proposition 1.7. *If X is a locally maximal hyperbolic set for a diffeomorphism f, then (X, f) is expansive.*

Sketch of proof. Choose $\epsilon > 0$ small enough that given any $x, y \in X$ with $d(x, y) < \epsilon$, the local leaves $W^s(x)$ and $W^u(y)$ intersect in a unique point $[x, y] \in X$. Write

$$d^u(x, y) = d(x, [x, y]) \quad \text{and} \quad d^s(x, y) = d(y, [x, y]).$$

Hyperbolicity[1] gives $\lambda > 0$ such that

$$d^u(f^n x, f^n y) \geq e^{\lambda n} d^u(x, y) \text{ if } d(f^k x, f^k y) < \epsilon \text{ for all } 0 \leq k \leq n, \tag{1.12}$$

$$d^s(f^{-n} x, f^{-n} y) \geq e^{\lambda n} d^s(x, y) \text{ if } d(f^{-k} x, f^{-k} y) < \epsilon \text{ for all } 0 \leq k \leq n. \tag{1.13}$$

In particular, if $y \in \Gamma_\epsilon(x)$ then $d^u(f^n x, f^n y)$ is uniformly bounded for all n, so $d^u(x, y) = 0$, and similarly for d^s, which implies that $x = [x, y] = y$. □

One important consequence of expansivity is the following.

Proposition 1.8. *If (X, f) is expansive at scale ϵ, then $h_{\mathrm{top}}(X, f, \epsilon) = h_{\mathrm{top}}(X, f)$.*

[1] In an adapted metric, if necessary.

As with entropy, the following formulation of the specification property is given for a collection of orbit segments $\mathcal{D} \subset X \times \mathbb{N}$, and thus is not quite the classical one, but reduces to (a version of) the classical definition when we take $\mathcal{D} = X \times \mathbb{N}$. Observe that when X is a shift space and we associate to each (x, n) the word $x_{[1,n]} \in \mathcal{L}(X)$, the following agrees with the definition from last time.

Definition 1.9 (Specification)**.** A collection of orbit segments $\mathcal{D} \subset X \times \mathbb{N}$ has the *specification property at scale* $\delta > 0$ if there exists $\tau \in \mathbb{N}$ such that for every $(x_1, n_1), \dots, (x_k, n_k) \in \mathcal{D}$, there exist $0 = T_1 < T_2 < \cdots < T_k \in \mathbb{N}$ and $y \in X$ such that

$$f^{T_i}(y) \in B_{n_i}(x_i, \delta) \text{ for all } 1 \le i \le k,$$

so that starting from time T_i the orbit of y shadows the orbit of x_i, and writing $s_i = T_i + n_i$ for the time at which this shadowing ends, we have

$$s_i \le T_{i+1} \le s_i + \tau \text{ for all } 1 \le i < k;$$

see Figure 1. We say that $\mathcal{D}$ has the *specification property* if the above holds for every $\delta > 0$. We say that (X, f) has the specification property if $X \times \mathbb{N}$ does.

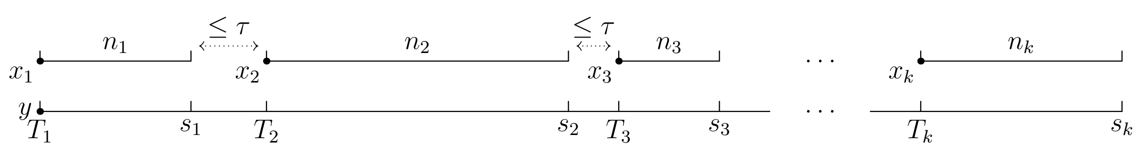

FIGURE 1. Bookkeeping in the specification property.

Proposition 1.10. *If X is a topologically transitive locally maximal hyperbolic set for a diffeomorphism f, then (X, f) has the specification property.*

Sketch of proof. Given $\delta > 0$, let $W^s_\delta(x)$ and $W^u_\delta(x)$ denote the stable and unstable leaves through x of size δ. Topological transitivity and compactness give $\tau \in \mathbb{N}$ such that for every $x, y \in X$ there is $t \in \{0, 1, \dots, \tau\}$ with $f^t(W^u_\delta(x)) \cap W^s_\delta(y) \neq \emptyset$. Given $(x_1, n_1), \dots, (x_k, n_k) \in X \times \mathbb{N}$, we construct iteratively $y_j \in X$ and $T_j \in \mathbb{N}$ such that y_j has the desired shadowing property for all $1 \le i \le j$: that is, $f^{T_i}(y_j) \in B_{n_i}(x_i, \delta)$ for all $1 \le i \le j$. Then taking $y = y_k$ suffices. See Figure 2 for a visual guide to the following estimates.

Start by putting $y_1 = x$ and $T_1 = 0$. Given y_j and T_j write $s_j := T_j + n_j$ and choose $t_j \in \{0, 1, \dots, \tau\}$ such that $f^{t_j}(W^u_\delta(f^{s_j}(y_j)) \cap W^s_\delta(x_{j+1}) \neq \emptyset$. Write $T_{j+1} = s_j + t_j$ and let y_{j+1} be such that $f^{T_{j+1}}(y_{j+1})$ lies in this intersection; that is,

$$f^{s_j}(y_{j+1}) \in W^u_\delta(f^{s_j}(y_j)) \quad \text{and} \quad f^{T_{j+1}}(y_{j+1}) \in W^s_\delta(x_{j+1}).$$

By hyperbolicity we have

$$W^s_\delta(x_{j+1}) \subset B_{n_{j+1}}(x_{j+1}, \delta) \quad \Rightarrow \quad d_{n_{j+1}}(x_{j+1}, f^{T_{j+1}} y_{j+1}) < \delta, \tag{1.14}$$

which we use in the form

$$d_{n_i}(x_i, f^{T_i} y_i) < \delta \text{ for all } i. \tag{1.15}$$

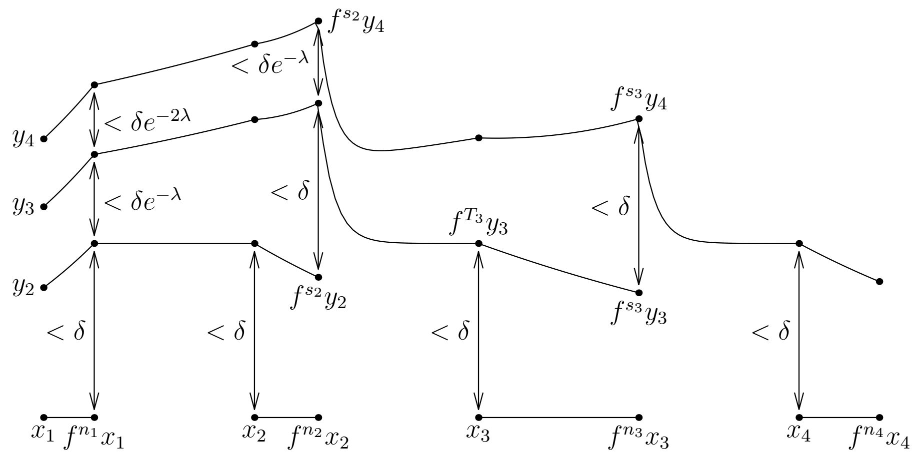

FIGURE 2. Specification for a uniformly hyperbolic set.

Moreover, for $i \leq j$ we have

$$d_{n_i}(f^{T_i}(y_j), f^{T_i}(y_i)) \leq d^u(f^{s_i}(y_j), f^{s_i}(y_i)) \leq \sum_{\ell=i}^{j-1} d^u(f^{s_i}(y_\ell), f^{s_i}(y_{\ell+1}))$$
$$\leq \sum_{\ell=i}^{j-1} e^{-\lambda(s_\ell - s_i)} d^u(f^{s_\ell}y_\ell, f^{s_\ell}y_{\ell+1}) \leq \sum_{\ell=i}^{j-1} e^{-\lambda(\ell-i)}\delta < \frac{\delta}{1-e^{-\lambda}}.$$

Together with (1.15) this gives

$$d_{n_i}(x_i, f^{T_i}y_j) \leq d_{n_i}(x_i, f^{T_i}y_i) + d_{n_i}(f^{T_i}y_i, f^{T_i}y_j) \leq \delta + \frac{\delta}{1-e^{-\lambda}} =: \delta',$$

which proves specification at scale δ'. □

Remark 1.11. The fact that f^{-1} contracts uniformly along W^u was used in an essential way in the proof, in order to guarantee convergence of the geometric series. However, contraction of f along W^s was only used to establish the inclusion in (1.14), and in particular it would suffice to know that $\|Df|_{E^s}\| \leq 1$.

Bowen's original uniqueness result [Bow75], which we outlined last time, was actually given not for shift spaces, but for more general expansive systems.

Theorem 1.12: Expansivity and specification (Bowen)

Let X be a compact metric space and $f\colon X \to X$ a continuous map. Suppose that $\epsilon > 28\delta > 0$ are such that f has expansivity at scale ϵ and the specification property at scale δ. Then (X, f) has a unique measure of maximal entropy.

Remark 1.13. If f has both specification and expansivity at scale δ, then it has specification at all scales, and this is the hypothesis in Bowen's original paper. The proof only uses specification at a fixed scale, and we formulate the result this way in preparation for our generalization, which uses a weaker expansivity property that does *not* let the specification property be transmitted to smaller scales.

2. Role of expansivity

The proof of Theorem 1.12 mirrors the strategy in the symbolic case:

(1) show that the usual construction of an MME gives an ergodic Gibbs measure;
(2) prove that an ergodic Gibbs measure must be the unique MME.

Now we examine the role played by expansivity.

2.1. **Construction of a Gibbs measure.** In the symbolic setting, the first step to building a Gibbs measure was to prove the following *counting bounds*:

$$e^{nh_{\mathrm{top}}(X)} \leq \#\mathcal{L}_n \leq Qe^{nh_{\mathrm{top}}(X)}. \tag{2.1}$$

Then one considered measures ν_n giving equal weight to every n-cylinder, and proved a Gibbs property for any limit point of the measures $\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} \sigma_*^k \nu_n$. For non-symbolic systems, the role of cylinders is played by Bowen balls, and one proves the following.

Proposition 2.1. *Let X be a compact metric space and $f\colon X \to X$ a continuous map with the specification property at scale $\delta > 0$. Let $E_n \subset X$ be a maximal (n, δ)-separated set for each n, and consider the measures*

$$\mu_n := \frac{1}{\#E_n} \sum_{x\in E_n} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k x}. \tag{2.2}$$

Then there is $K \geq 1$ such that every weak limit point μ of the sequence μ_n is f-invariant and satisfies*

$$K^{-1}e^{-nh_{\mathrm{top}}(X,f,7\delta)} \leq \mu(B_n(x, 7\delta)) \leq Ke^{-nh_{\mathrm{top}}(X,f,7\delta)} \quad \textit{for all } x \in X, n \in \mathbb{N}. \tag{2.3}$$

Remark 2.2. Our formulation here differs from that of Bowen in that we do not assume specification at all small scales, merely at scale δ. Note that this means we get a Gibbs property at scale 7δ, but not necessarily at smaller scales. The multiple scales appearing in Proposition 2.1 arise because Bowen balls can overlap without being nested. In the symbolic setting, if $\delta = \frac{1}{4}$ and $y \in B_n(x, \delta)$, then $B_n(y, \delta) = B_n(x, \delta) = [w]$ where $w := y_{[1,n]} = x_{[1,n]}$.[2] In the non-symbolic setting, if $y \in B_n(x, \delta)$ then the most we can say is that $B_n(y, \delta) \subset B_n(x, 2\delta)$, and vice versa. Thus the various counting arguments that mimic (2.1), and other arguments using the specification property, require us to change scales several times. We will skip the details of this here, though, and merely state the conclusions.

Corollary 2.3. *Let X, f be as above and suppose that $h_{\mathrm{top}}(X, f, 7\delta) = h_{\mathrm{top}}(X, f)$. Then every μ as above satisfies the Gibbs property*

$$K^{-1}e^{-nh_{\mathrm{top}}(X,f)} \leq \mu(B_n(x, 7\delta)) \leq Ke^{-nh_{\mathrm{top}}(X,f)}. \tag{2.4}$$

[2] In other words, in a shift space, each d_n is an *ultrametric*, for which the triangle inequality is strengthened to $d_n(x, z) \leq \max\{d_n(x, y), d_n(y, z)\}$.

In particular, if f has specification at scale δ and is expansive at scale 7δ, then Corollary 2.3 applies, thanks to the following.

Proposition 2.4. *If (X, f) is expansive at scale ϵ, then $h_{\text{top}}(X, f, \epsilon) = h_{\text{top}}(X, f)$.*

Two proof ideas. We outline two proofs in the positively expansive case.

One argument uses a compactness argument to show that for every $0 < \delta < \epsilon$, there is $N \in \mathbb{N}$ such that $B_N(x, \epsilon) \subset B(x, \delta)$ for all $x \in X$. This implies that $B_{n+N}(x, \epsilon) \subset B_n(x, \delta)$ for all x, and then one can show that the definition of topological entropy via (n, ϵ)-separated sets gives the same value at δ as at ϵ.

Another method, which is better for our purposes, is to observe that since ϵ-expansivity gives $\bigcap_n B_n(x, \epsilon) = \{x\}$ for all x, one can easily show that for every $\nu \in \mathcal{M}_f(X)$, we have:

(2.5) if β is a partition with d_n-diameter $< \epsilon$, then β is generating for (f^n, ν).

Given a maximal (n, ϵ)-separated set E_n, we can choose a partition β_n such that each element of β_n is contained in $B_n(x, \epsilon)$ for some $x \in E_n$, so β_n has exactly $\#E_n$ elements. Then we have

$$h_\mu(f) = \frac{1}{n} h_\mu(f^n) = \frac{1}{n} h_\mu(f^n, \beta_n) \leq \frac{1}{n} H_\mu(\beta_n) \leq \frac{1}{n} \log \#E_n. \tag{2.6}$$

Sending $n \to \infty$ gives $h_\mu(f) \leq h_{\text{top}}(X, f, \epsilon)$, and taking a supremum over all $\mu \in \mathcal{M}_f(X)$ proves that $h_{\text{top}}(X, f, \epsilon) = h_{\text{top}}(X, f)$. □

2.2. **Ergodicity.** Observe that we have not yet claimed anything about ergodicity of the Gibbs measure μ. In the symbolic case, the argument for the Gibbs property can be used to deduce that there is $c > 0$ and $k \in \mathbb{N}$ such that for every $v, w \in \mathcal{L}$ and $j \geq |v| + k$, we have

Not quite this... only for a sequence of j with bounded gaps.

$$\mu([v] \cap \sigma^{-j}[w]) \geq c\mu[v]\mu[w].$$

Since any Borel set can be approximated (w.r.t. μ) by unions of cylinders, this in turn implies that

$$\overline{\lim_{j\to\infty}} \mu(V \cap \sigma^{-j} W) \geq c\mu(V)\mu(W)$$

for all $V, W \subset X$, which gives ergodicity. In the non-symbolic setting, one can still mimic the Gibbs argument to prove that for every $(x, n), (y, m) \in X \times \mathbb{N}$ and any $j \geq n + k$, we have

$$\mu(B_n(x, 7\delta) \cap f^{-j} B_m(y, 7\delta)) \geq c\mu(B_n(x, 7\delta))\mu(B_m(y, 7\delta)). \tag{2.7}$$

To establish ergodicity from this one needs to approximate arbitrary Borel sets by sets whose μ-measure we control; this can be done by using a sequence of *adapted partitions* β_n, for which each element of β_n contains a Bowen ball $B_n(x, 7\delta)$ and is contained inside a Bowen ball $B_n(x, 14\delta)$. Expansivity implies that this sequence of partitions is generating w.r.t. μ, so the rest of the argument goes through as before, and establishes ergodicity. As we saw in the proof of Proposition 2.4, this is also enough to guarantee that $h_{\text{top}}(X, f, 7\delta) = h_{\text{top}}(X, f)$. We summarize our conclusions as follows.

Proposition 2.5. *Let X, f, δ, μ be as in Proposition 2.1. Suppose that f is expansive at scale 28δ. Then μ is ergodic and satisfies the (true) Gibbs property* (2.4).

2.3. **Adapted partitions and uniqueness.** The proof that an ergodic Gibbs measure is the unique MME has the following generalization to the non-symbolic setting.

Proposition 2.6. *Let X be a compact metric space, $f\colon X \to X$ a continuous map, and μ an ergodic f-invariant measure on X. Suppose that there is $\rho > 0$ such that*

- *f is expansive (or positively expansive) at scale 4ρ;*
- *there is $K \geq 1$ such that μ satisfies the lower Gibbs bound $\mu(B_n(x,\rho)) \geq K^{-1}e^{-nh_{\mathrm{top}}(X,f)}$ for every $x \in X$, $n \in \mathbb{N}$.*

Then μ is the unique MME for (X, f).

Outline of proof. As before, the first step is to reduce to proving that an invariant measure $\nu \perp \mu$ must have $h_\nu(f) < h_\mu(f)$; this is unchanged from the symbolic case. The next step there was to choose $D \subset X$ with $\mu(D) = 1$ and $\nu(D) = 0$, and approximate D by a union of cylinders; then writing[3]

$$nh_\nu(f) = h_\nu(f^n) = h_\nu(f^n, \alpha_0^{n-1}) \leq H_\nu(\alpha_0^{n-1}) = \sum_{w\in\mathcal{L}_n} -\nu[w]\log\nu[w], \tag{2.8}$$

and splitting the sum between cylinders in D_n and those in D_n^c, one eventually proves that $h_\nu(f) < h_\mu(f)$ by using the Gibbs bound $\mu[w] \geq e^{-|w|h_{\mathrm{top}}(X)}$.

In the non-symbolic setting, the approximation of D follows just as in the paragraph after (2.7). Moreover, we can obtain an analogue of (2.8) by replacing α_0^{n-1} with a partition β_n such that every element of β_n is contained in $B_n(x, 2\rho)$ for some point x in a maximal $(n, 2\rho)$-separated set E_n. Finally, as long as we also arrange that each element of β_n contain $B_n(x,\rho)$,[4] we can use the lower Gibbs bound to complete the proof just as in the symbolic case. □

Remark 2.7. In the two-sided expansive case, the same argument works, replacing d_n and B_n with their two-sided versions.

3. Obstructions to expansivity

As explained in the previous section, the role of expansivity in the proof of Theorem 1.12 is to guarantee that certain sequences of partitions are generating with respect to every invariant ν. In fact, in every place where this property is used, it is enough to know that this holds for all ν with sufficiently large entropy.

More precisely, at the end of the proof, in (the analogue of) (2.8), it suffices to know that α_0^{n-1} is generating for (f^n, ν) when ν is an arbitrary MME, because if ν is not an MME then we already have $h_\nu < h_\mu$, which was the goal. This is also sufficient for the approximation of D by elements of the partitions β_n, and thus Proposition 2.6 remains true if we replace expansivity with the assumption that for every MME ν, we have $\Gamma_\epsilon(x) = \{x\}$ for ν-a.e. x.

[3]Observe the similarity between (2.6) and (2.8).

[4]Such a β_n is called an *adapted partition.*

In Proposition 2.1, the argument for ergodicity required a similar generating property. Finally, in Proposition 2.4, it suffices to have this generating property w.r.t. a family of measures ν over which $\sup_\nu h_\nu(f) = h_{\text{top}}(X, f)$.

With these observations in mind, we make the following definitions.

Definition 3.1 ([BF13])**.** An f-invariant measure μ is *almost expansive at scale* ϵ if $\Gamma_\epsilon(x) = \{x\}$ for μ-a.e. x; equivalently, if the *non-expansive set* $\mathrm{NE}(\epsilon) = \{x \in X : \Gamma_\epsilon(x) \neq \{x\}\}$ has $\mu(\mathrm{NE}(\epsilon)) = 0$. Replacing Γ_ϵ by Γ_ϵ^+ gives NE^+ and a notion of *almost positively expansive*.

Definition 3.2 ([CT14])**.** The *entropy of obstructions to expansivity at scale* ϵ is

$$\begin{aligned} h^\perp_{\exp}(X, f, \epsilon) &:= \sup\{h_\mu(f) : \mu \in \mathcal{M}^e_f(X) \text{ is not almost expansive at scale } \epsilon\} \\ &= \sup\{h_\mu(f) : \mu \in \mathcal{M}^e_f(X) \text{ and } \mu(\mathrm{NE}(\epsilon)) > 0\}. \end{aligned}$$

We write $h^\perp_{\exp}(X, f) = \lim_{\epsilon \to 0} h^\perp_{\exp}(X, f, \epsilon)$ for the *entropy of obstructions to expansivity*, without reference to scale. The entropy of obstructions to positive expansivity $h^\perp_{\exp^+}$ is defined analogously.

From the discussion above, we see that in Proposition 2.5 we can replace the assumption of expansivity with the assumption that $h^\perp_{\exp}(X, f, 7\delta) < h_{\text{top}}(X, f)$, since then every ergodic ν with $h_\nu(f) > h^\perp_{\exp}(X, f, 7\delta)$ is almost expansive, so the Proposition goes through.[5] Similarly in Proposition 2.6, it suffices to assume that $h^\perp_{\exp}(X, f, 4\rho) < h_{\text{top}}(X, f)$.

Now we have all the pieces for a uniqueness result using non-uniform expansivity.

Theorem 3.3: Small obstructions to expansivity [CT14]

Let X be a compact metric space and $f\colon X \to X$ a continuous map. Suppose that $\epsilon > 28\delta > 0$ are such that $h^\perp_{\exp}(X, f, \epsilon) < h_{\text{top}}(X, f)$, and that f has the specification property at scale δ. Then (X, f) has a unique measure of maximal entropy.

Remark 3.4. As in the symbolic case, if specification is upgraded to periodic specification, then provided one excludes "non-expansive" periodic orbits, we get uniform counting bounds on the number of periodic orbits, and the unique MME is their limiting distribution.

4. Derived-from-Anosov systems

We describe a class of smooth systems for which expansivity fails but the entropy of obstructions to expansivity is small.[6] The following example is due to Mañé [Mañ78]; we primarily follow the discussion in [CFT19], and refer to that paper for further details and references.

[5] See [CT14, Proposition 2.7] for a detailed proof that $h_{\text{top}}(X, f, 7\delta) = h_{\text{top}}(X, f)$ in this case.

[6] This section was omitted from the live lecture due to time constraints.

4.1. **Construction of the Mañé example.** Fix a matrix $A \in SL(3,\mathbb{Z})$ with simple real eigenvalues $\lambda_u > 1 > \lambda_s > \lambda_{ss} > 0$, and corresponding eigenspaces $F^{u,s,ss} \subset \mathbb{R}^3$. Let $f_0 \colon \mathbb{T}^3 \to \mathbb{T}^3$ be the hyperbolic toral automorphism defined by A, and let $\mathcal{F}^{u,s,ss}$ be the corresponding foliations of $\mathbb{T}^3$. Define a perturbation f of f_0 as follows.

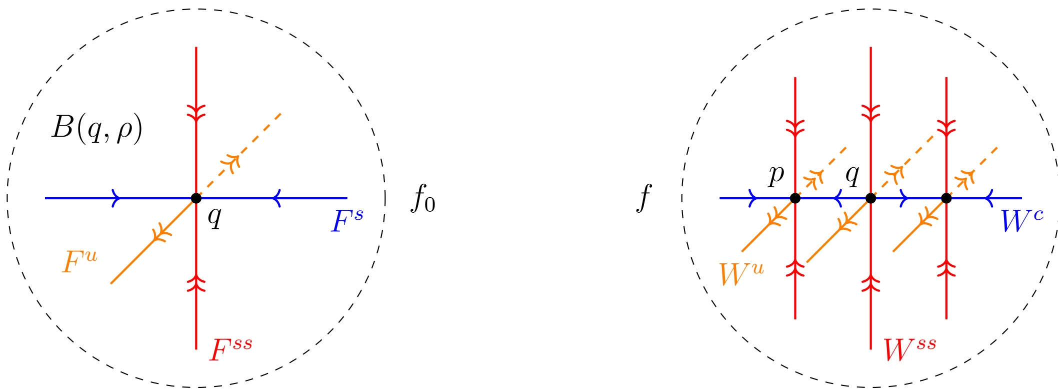

Figure 3. Mañé's construction.

Fix $\rho > \rho' > 0$ such that f_0 is expansive at scale ρ. Let $q \in \mathbb{T}^3$ be a fixed point of f, and set $f = f_0$ outside of $B(q,\rho)$. Inside $B(q,\rho)$, perform a pitchfork bifurcation in the center direction as shown in Figure 3, in such a way that

- the foliation $W^c := \mathcal{F}^s$ remains f-invariant, and we write $E^c = TW^c$;
- the cones around F^u and F^{ss} remain invariant and uniformly expanding for Df and Df^{-1}, respectively, so they contain Df-invariant distributions $E^{u,ss}$ that integrate to f-invariant foliations $W^{u,ss}$;
- $E^{cs} = E^c \oplus E^{ss}$ integrates to a foliation W^{cs};
- outside of $B(q,\rho')$, we have $\|Df|_{E^{cs}}\| \leq \lambda_s < 1$.

Thus f is partially hyperbolic with $T\mathbb{T}^3 = E^u \oplus E^c \oplus E^{ss} = E^u \oplus E^{cs}$. Observe that

$$\lambda_c(f) := \sup\{\|Df|_{E^{cs}(x)}\| : x \in \mathbb{T}^3\} > 1 \tag{4.1}$$

because the center direction is expanding at q.

Now consider a diffeomorphism $g \colon \mathbb{T}^3 \to \mathbb{T}^3$ that is C^1-close to f. Such a g remains partially hyperbolic, with

$$\lambda_c(g) > 1 > \lambda_s(g) := \sup\{\|Df|_{E^{cs}(x)}\| : x \in \mathbb{T}^3 \setminus B(q,\rho')\}. \tag{4.2}$$

Existence of a unique MME was proved for such g by Ures [Ure12] and by Buzzi, Fisher, Sambarino, and Vásquez [BFSV12], using the fact that there is a semiconjugacy from g back to the hyperbolic toral automorphism f_0. We outline an alternate proof using Theorem 3.3, which has the benefit of extending to class of nonzero potential functions as well [CFT19].

4.2. **Estimating the entropy of obstructions.** At first glance, the approach via expansivity and specification looks problematic: although the map g behaves as if it is uniformly hyperbolic outside of $B(q, \rho)$, the presence of fixed points with different indices inside this ball causes expansivity to fail. Indeed, let p denote one of the two fixed points created via the pitchfork bifurcation, and let x be any point on the leaf of W^c that connects p to q. Then for every $\epsilon > 0$, the bi-infinite Bowen ball $\Gamma_\epsilon(x)$ is a non-trivial curve in W^c, rather than a single point.

However, with a little more care we see that this failure of expansivity can only happen for points whose orbits satisfy rather severe restrictions.

Lemma 4.1. *Let g be a partially hyperbolic diffeomorphism with a splitting $E^u \oplus E^c \oplus E^s$ such that E^c is 1-dimensional and integrable. Then there is $\epsilon_0 > 0$ such that $\Gamma_{\epsilon_0}(x) \subset W^c(x)$ for every x. Moreover, for every $\lambda > 0$ there is $\epsilon > 0$ such that*

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log \|Dg^{-n}|_{E^c(x)}\| > \lambda \quad \Rightarrow \quad \Gamma_\epsilon(x) = \{x\}. \tag{4.3}$$

Sketch of proof. Following the argument for expansivity in the uniformly hyperbolic setting, we choose ϵ_0 such that whenever $d(x, y) < \epsilon_0$, we can get from x to y by moving a distance d^s along a leaf of W^s, then a distance d^c along a leaf of W^c, then a distance d^u along a leaf of W^u. The argument given there shows that if $y \in \Gamma_{\epsilon_0}(x)$ then we must have $d^s(x, y) = d^u(x, y) = 0$, which implies that $y \in W^c(x)$. For (4.3), we observe that if the condition on Dg^{-n} is satisfied, then there are arbitrarily large n such that

$$\|Dg^{-n}|_{E^c(x)}\| > ce^{\lambda n}. \tag{4.4}$$

Choosing $\epsilon > 0$ sufficiently small that $|\log \|Dg|_{E^c(z)}\| - \log \|Dg|_{E^c(z')}\|| < \lambda/2$ whenever $d(z, z') < \epsilon$, we see that any $y \in \Gamma_\epsilon(x)$ satisfies

$$d(g^{-n}x, g^{-n}y) \geq ce^{\lambda n/2} d(x, y) \tag{4.5}$$

for all n satisfying (4.4). Since n can become arbitrarily large, this implies that $d(x, y) = 0$. □

Remark 4.2. Replacing backwards time with forwards time, the analogous result for positive Lyapunov exponents is also true: $\overline{\lim} \frac{1}{n} \log \|Dg^n|_{E^c(x)}\| > \lambda$ implies that $\Gamma_\epsilon(x) = \{x\}$.

For the Mañé examples, we can use (4.2) to control $\|Dg^{-n}|_{E^c(x)}\|$ in terms of how much time the orbit of x spends outside $B(q, \rho)$; together with Lemma 4.1, this allows us to estimate the entropy of NE(ϵ). To formalize this, we write $\chi = \mathbf{1}_{\mathbb{T}^3 \setminus B(q,\rho)}$ and observe that by the definition of $\lambda_c(g)$ and $\lambda_s(g)$ in (4.1) and (4.2), we have

$$\|Dg^{-n}|_{E^c(x)}\| \geq \lambda_s(g)^{-s_n(x)} \lambda_c(g)^{-(n - s_n(x))} \quad \text{where } s_n(x) := \sum_{k=0}^{n-1} \chi(g^{-k}x).$$

It follows that

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log \|Dg^{-n}|_{E^c(x)}\| \geq -(\overline{r}(x) \log \lambda_s(g) + (1 - \overline{r}(x)) \log \lambda_c(g)) \tag{4.6}$$

where we write

$$\overline{r}(x) = \varlimsup_{n\to\infty} \frac{1}{n} s_n(x) = \varlimsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi(g^{-k}x).$$

Fix $\lambda \in (0, -\log \lambda_s(g))$ and let $r > 0$ satisfy $-(r \log \lambda_s(g) + (1-r)\log \lambda_c(g)) > \lambda$. Then Lemma 4.1 and (4.6) show that for a sufficiently small $\epsilon > 0$, we have

$$\text{NE}(\epsilon) \subset \{x : \overline{r}(x) < r\}. \tag{4.7}$$

Recalling that the uniform counting bounds in (2.1) (or rather, their analogue for non-symbolic systems) give $\Lambda_n^{\text{sep}}(X, f_0) \leq Qe^{nh_{\text{top}}(X,f_0)}$ for some constant L that is independent of n, one can use this characterization of non-expansive points to prove the following.

Lemma 4.3 ([CFT18, §3.4])**.** *Writing $H(t) = -t\log t - (1-t)\log(1-t)$ for the usual bipartite entropy function, the Mañé examples satisfy*

$$h_{\text{exp}}^{\perp}(g, \epsilon) < r(h_{\text{top}}(X, f_0) + \log Q) + H(2r).$$

Idea of proof. Given an ergodic measure μ that satisfies $\mu(\text{NE}(\epsilon))$ and thus satisfies $\varlimsup \frac{1}{n} S_n \chi(g^{-n}x) \leq r$ for μ-a.e. x, the Katok entropy formula [Kat80] can be used to show that $h_\mu(f) \leq h(\mathcal{C})$, where

$$\mathcal{C} := \{(x, n) \in \mathbb{T}^3 \times \mathbb{N} : S_n\chi(x) \leq rn\}. \tag{4.8}$$

To estimate $h(\mathcal{C})$, the idea is to partition an orbit segment $(x, n) \in \mathcal{C}$ into pieces lying entirely inside or outside of $B(q, \rho)$. There can be at most rn pieces lying outside, so the number of transition times between inside and outside is at most $2rn$. The number of ways of choosing these transition times is thus at most

$$\binom{n}{2rn} = \frac{n!}{(2rn)!((1-2r)n)!} \approx e^{H(2r)n},$$

where the approximation can be made more precise using Stirling's formula or a rougher elementary integral estimate. This contribues the $H(2r)$ term to the estimate; the remaining terms are roughly due to the observation that given a pattern of transition times for which the segments lying outside $B(q, \rho)$ have lengths $k_1, \dots, k_m$, the number of ϵ-separated orbit segments in $\mathcal{C}$ associated to this pattern is at most

$$\prod_{j=1}^{m} \Lambda_{k_i}^{\text{sep}}(X, f_0, \epsilon) \leq \prod_{j=1}^{m} Qe^{k_i h_{\text{top}}(X,f_0)} \leq Q^m e^{rnh_{\text{top}}(X,f_0)} \leq (Qe^{h_{\text{top}}(X,f_0)})^{rn},$$

since no entropy is produced by the sojourns inside $B(q, \rho)$. □

Since there is a semi-conjugacy from g to f_0, we have $h_{\text{top}}(X, g) \geq h_{\text{top}}(X, f_0)$. Thus we have $h_{\text{exp}}^{\perp}(g) < h_{\text{top}}(g)$ whenever r satisfies

$$r(h_{\text{top}}(X, f_0) + \log Q) + H(2r) < h_{\text{top}}(X, f_0). \tag{4.9}$$

Recall that r must be chosen large enough such that $\lambda_s(g)^r \lambda_c(g)^{1-r} < 1$. Equivalently, for a given value of r, the perturbation must be chosen small enough for this to hold (that is, λ_c must be close enough to 1). Thus given f_0, we can find r small enough such that (4.9) holds, and then for any sufficiently small perturbation the above argument guarantees that $h^{\perp}_{\mathrm{exp}}(X,g) < h_{\mathrm{top}}(X,g)$.

Remark 4.4. Since $\Gamma_\epsilon(x) \subset W^c(x)$, which is one-dimensional, it is not hard to show that $h_{\mathrm{top}}(W^c(x)) = 0$, and thus $h_{\mathrm{top}}(\Gamma_\epsilon(x)) = 0$ [CY05, CFT19]; in other words, f is *entropy expansive*. Entropy expansivity implies that $h_{\mathrm{top}}(X,f,\epsilon) = h_{\mathrm{top}}(X,f)$ [Bow72], which is sufficient for the construction of a Gibbs measure in Proposition 2.1.[7] However, there does not seem to be any way to use entropy expansivity to carry out the arguments for ergodicity and uniqueness, since we have no way to conclude that the Bowen balls generate the Borels on any sufficiently large set. In fact, for the Bonatti–Viana examples introduced in [BV00],[8] entropy expansivity can fail [BF13] even while the condition $h^{\perp}_{\mathrm{exp}} < h_{\mathrm{top}}$ is satisfied [CFT18].

4.3. **Coarse specification.** In order to apply Theorem 3.3 to the Mañé examples, one must investigate the specification property. In fact, recalling the uniformly hyperbolic proof, we see that the only thing we are missing is uniform contraction along W^{cs}, which is replacing W^s from that proof. But this contraction would enter the proof of specification only by guaranteeing that

$$W^{cs}_\delta(x) \subset B_n(x,\delta) \text{ for all } x. \tag{4.10}$$

Since contraction in W^{cs} can fail for the Mañé example only in $B(q,\rho')$, one can easily show that (4.10) continues to hold as long as $\delta > 2\rho'$, and thus g has specification at these scales. Choosing ρ' to be small enough relative to ρ, Theorem 3.3 applies and establishes existence of a unique MME.

Remark 4.5. It is essential here that we only require specification at a fixed finite scale, because it fails at sufficiently small scales. Indeed, observe that for sufficiently small $\delta > 0$, the forward infinite Bowen ball $\Gamma^+_\delta(q)$ is the 1-dimensional local stable leaf $W^{ss}_\delta(q)$. Suppose that g has specification at scale δ with gap size τ, and let x be any point whose orbit never enters $B(q,\rho)$. Specification gives $y \in W^u_\delta(x)$ and $0 \le k \le \tau$ such that $f^k(y) \in W^{ss}_\delta(q)$;[9] In other words, $f^{-\tau}(W^{ss}_\delta(q))$ intersects *every* local unstable leaf associated to an orbit that avoids $B(q,\rho)$. But this is impossible because the dimensions are wrong.[10]

5. The general result

Even though the Mañé example has specification at a suitable scale, it is still useful to formulate a general result that combines the symbolic result using decompositions

[7] This comes with the caveat that Proposition 2.1 also requires the specification property, which we have not discussed yet for these examples.

[8] These are 4-dimensional analogues of the Mañé examples that involve two separate perturbations and have a dominated splitting $T\mathbb{T}^4 = E^{cu} \oplus E^{cs}$ but are not partially hyperbolic.

[9] Use specification to get $y_n \in f^n(B_n(x,\delta)) \cap f^{-k_n}(B_n(q,\delta))$ for $0 \le k_n \le \tau$, choose k such that $k_n = k$ for infinitely many values of n, and let y be a limit point of the corresponding y_n.

[10] Note that $f^{-\tau}(W^{ss}_\delta(q))$ intersects a local leaf of W^{cu} in at most finitely many points, and thus thus intersects at most finitely many of the corresponding local leaves of W^u; however, there are uncountably many of these corresponding to points that never enter $B(q,\rho)$.

with Theorem 3.3 by allowing both expansivity and specification to fail, provided the obstructions have small entropy. This allows us to cover some new classes of examples, as we will see later, and is also important in dealing with nonzero potential functions, an issue which will be explored further in Dan Thompson's lectures next week.

Recall that a decomposition of the language $\mathcal{L}$ of a shift space consists of $\mathcal{C}^p, \mathcal{G}, \mathcal{C}^s \subset \mathcal{L}$ such that every $w \in \mathcal{L}$ can be written as $w = u^p v u^s$ where $u^p \in \mathcal{C}^p$, $v \in \mathcal{G}$, and $u^s \in \mathcal{C}^s$. For non-symbolic systems, we replace $\mathcal{L}$ with the space of orbit segments $X \times \mathbb{N}$ and make the following definition.

Definition 5.1. A *decomposition* for $X \times \mathbb{N}$ consists of three collections $\mathcal{C}^p, \mathcal{G}, \mathcal{C}^s \subset X \times \mathbb{N}_0$ for which there exist three functions $p, g, s \colon X \times \mathbb{N} \to \mathbb{N}_0$ such that for every $(x, n) \in X \times \mathbb{N}$, the values $p = p(x, n)$, $g = g(x, n)$, and $s = s(x, n)$ satisfy $p + g + s = n$, and

$$(x, p) \in \mathcal{C}^p, \quad (f^p x, g) \in \mathcal{G}, \quad (f^{p+s} x, s) \in \mathcal{C}^s.$$

Given a decomposition, for each $M \in \mathbb{N}$ we write

$$\mathcal{G}^M := \{(x, n) \in X \times \mathbb{N} : p(x, n) \leq M \text{ and } s(x, n) \leq M\}.$$

Theorem 5.2: Small obstructions to expansivity [CT16]

Let X be a compact metric space and $f \colon X \to X$ a continuous map. Suppose that $\epsilon > 28\delta > 0$ are such that $h^\perp_{\text{exp}}(X, f, \epsilon) < h_{\text{top}}(X, f)$, and that the space of orbit segments $X \times \mathbb{N}$ admits a decomposition $\mathcal{C}^p\mathcal{G}\mathcal{C}^s$ such that

(I) every collection $\mathcal{G}^M$ has specification at scale δ, and
(II) $h(\mathcal{C}^p \cup \mathcal{C}^s, \delta) < h_{\text{top}}(X, f)$.

Then (X, f) has a unique measure of maximal entropy.

Remark 5.3. If $\mathcal{G}$ has specification at all scales, then a short continuity argument proves that every $\mathcal{G}^M$ does as well, which establishes (I).

References

[BF13] J. Buzzi and T. Fisher, *Entropic stability beyond partial hyperbolicity*, J. Mod. Dyn. **7** (2013), no. 4, 527–552. MR 3177771

[BFSV12] J. Buzzi, T. Fisher, M. Sambarino, and C. Vásquez, *Maximal entropy measures for certain partially hyperbolic, derived from Anosov systems*, Ergodic Theory Dynam. Systems **32** (2012), no. 1, 63–79. MR 2873158 (2012k:37067)

[Bow72] Rufus Bowen, *Entropy-expansive maps*, Trans. Amer. Math. Soc. **164** (1972), 323–331. MR 0285689

[Bow75] ______, *Some systems with unique equilibrium states*, Math. Systems Theory **8** (1974/75), no. 3, 193–202. MR 0399413

[BV00] Christian Bonatti and Marcelo Viana, *SRB measures for partially hyperbolic systems whose central direction is mostly contracting*, Israel J. Math. **115** (2000), 157–193. MR 1749677

[CFT18] Vaughn Climenhaga, Todd Fisher, and Daniel J. Thompson, *Unique equilibrium states for Bonatti-Viana diffeomorphisms*, Nonlinearity **31** (2018), no. 6, 2532–2570. MR 3816730

[CFT19] Vaughn Climenhaga, Todd Fisher, and Daniel J. Thompson, *Equilibrium states for Mañé diffeomorphisms*, Ergodic Theory Dynam. Systems (to appear (2019)), 28 pages, arXiv:1703.05722.

[CT14] Vaughn Climenhaga and Daniel J. Thompson, *Intrinsic ergodicity via obstruction entropies*, Ergodic Theory Dynam. Systems **34** (2014), no. 6, 1816–1831. MR 3272773

[CT16] ———, *Unique equilibrium states for flows and homeomorphisms with non-uniform structure*, Adv. Math. **303** (2016), 745–799. MR 3552538

[CY05] William Cowieson and Lai-Sang Young, *SRB measures as zero-noise limits*, Ergodic Theory Dynam. Systems **25** (2005), no. 4, 1115–1138. MR 2158399

[Kat80] A. Katok, *Lyapunov exponents, entropy and periodic orbits for diffeomorphisms*, Inst. Hautes Études Sci. Publ. Math. (1980), no. 51, 137–173. MR 573822

[Mañ78] Ricardo Mañé, *Contributions to the stability conjecture*, Topology **17** (1978), no. 4, 383–396. MR 516217

[Ure12] R. Ures, *Intrinsic ergodicity of partially hyperbolic diffeomorphisms with a hyperbolic linear part*, Proc. Amer. Math. Soc. **140** (2012), no. 6, 1973–1985. MR 2888185

[Wal82] Peter Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics, vol. 79, Springer-Verlag, New York-Berlin, 1982. MR 648108