Effective algorithms in ACSV

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$$\text{ACSV} =$$

Analytic Combinatorics in Several Variables

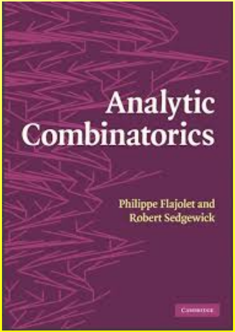
$$= \textbf{Anacomb}{++}$$

Analytic Combinatorics

Generating functions code arrays such as $\{a_n : n \in \mathbb{Z}^+\}$ or $\{\mathbf{a_r} : \mathbf{r} \in (\mathbb{Z}^+)^{\mathbf{d}}\}$ into power series $\mathbf{f(z)} = \sum_{\mathbf{n}} \mathbf{a_n z^n}$ or in the multivariate case,

$$\sum_{\mathbf{r}} \mathbf{a_r z^r} = \sum_{\mathbf{r}} \mathbf{a_r z_1^{r_1} \cdots z_d^{r_d}} \,.$$

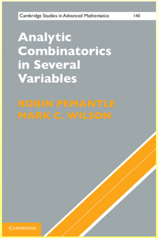
When the coefficients obey recursions, this coding often produces power series that represent nice objects in the analytic sense: functions that are rational, algebraic, or solve linear differential equations with polynomial coefficients.



Analytic Combinatorics is the study of how to get information back out of this encoding when the generating function is a nice function.

Plan

ACSV considers specifically coefficient arrays of more than one dimension.



PLAN:

I Remind why ACSV is interesting for AofA crowd

II Possible behaviors of coefficients of rational multivariate GF's

III How multivariate coefficients are extracted

IV Topology: a story in pictures

V New effective algorithms

Multivariate generating functions are used

- to count combinatorial classes
- to compute recursively defined probabilities
- to encode "integrable" ensembles

Applications of interest include:

Queuing theory, lattice point enumeration, enumeration and analysis of search trees, transfer matrices, lattice paths, quantum walks, sequence alignment and matching, special functions and random tilings.

Queuing theory

Determination of the stationary distribution of a system of d queues is useful for the analysis of algorithms. Encoding this as a d-variate probability generating function, the queuing recursion can lead to a number of types of function.

One possibility (see Bertozzi and McKenna) is a function of the form

$$\frac{e^{\ell_0(z)}}{\prod_{j=1}^{m} \ell_j(z)}$$

where $\ell_0, \ldots, \ell_m$ are affine functions of $z_1, \ldots, z_d$.

Many other outcomes are possible, even when $d = 2$. The ramifications of various boundary conditions are discussed in *Random walks in the quarter plane* (Fayolle et al.).

Enumeration of lattice points in polytopes

Efficiently counting integer points that satisfy a set of inequalities is a classic problem, related to the volume computation for the Birkhoff polytope.

This may be an end in itself, or a tool in the analysis of an algorithm that explores this set of points.

The generating function will be a sum of relatively simple rational functions of the form (see DeLoera et al.)

$$\frac{z^m}{\prod_j (1 - z^{b^{(j)}})} .$$

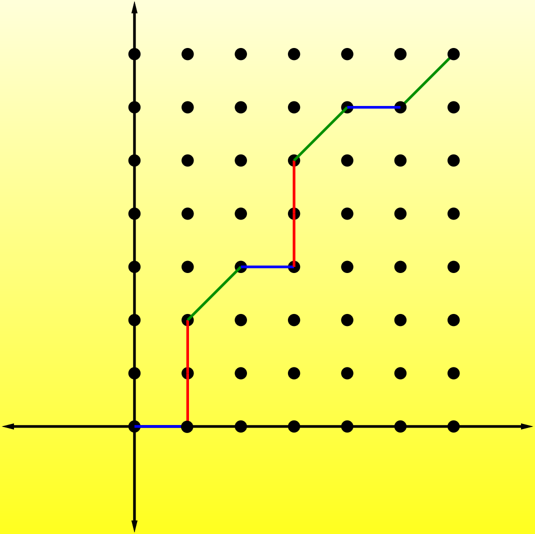
Trees

Many types of trees can be counted by generating functions. For example, bivariate generating functions counting by path length and height satisfy the implicit equations

$$\begin{aligned} Q(u,z) &= \frac{z}{1-Q(z,zu)} && \text{trees} \\ Q(u,z) &= zQ(u,zu)^2 && \text{binary trees} \\ \frac{\partial}{\partial z}Q(u,z) &= Q(u,zu)^2 && \text{binary search trees} \end{aligned}$$

Lattice paths

Let a_{rs} count the number of lattice paths from the origin to (r,s) with steps in a prescribed set, such as $\{$ N, E, NE $\}$.



One of 19,825 lattice paths from the origin to (6,7)

Transfer matrices

Lattice path enumeration is a special case of the so-called *transfer matrix* method.

Let M be a $k \times k$ matrix whose entries are monomials λz^r. Then $(I - tM)^{-1}$ is the sum over chains $i = x_0, x_1, \ldots, x_\ell = j$ of $\nu t^\ell z^m$ where the weight ν is the product of the constants λ for the matrix entries $M_{x_p, x_{p+1}}$ along the chain and the index m is the sum of the indices r along the chain.

Quantum computing

A proposed building block for quantum computation is the quantum random walk. Given a $k \times k$ unitary matrix U and k lattice steps $m^{(1)}, \ldots, m^{(k)}$ in $\mathbb{Z}^d$, there is a quantum walk with those steps, whose amplitudes are determined by repeated applications of the unitary operator.

The multivariate generating function for the amplitude $a_{n,i,j}$ of going from state i to state j in n steps with displacement r is given by

$$P(z)_{ij} = \sum_{n,r} a_{n,i,j} z^r t^n = (I - tDU)^{-1} .$$

Sequence alignment and matching

Given k sequences on a finite alphabet, what is the optimal alignment, and is this better than what one would get for k independent random sequences of the same lengths? The answer is not well understood when $k > 2$.

Distributions of statistics of the optimal alignment require one to count non-isomorphic alignments. The generating function for number of non-isomorphic alignments of sequences of lengths i, j and k with minimum block size b is given by

$$\frac{1 + \frac{(xyz)^n}{1-xyz}}{1 - ((1+x)(1+y)(1+z) - 1 - xyz)\left(1 + \frac{(xyz)^b}{-xyz+1}\right)}.$$

Special functions

Nonnegativity and asymptotics of coefficients of many classes of generating functions have been studied from the points of view of special functions and statistical physics.

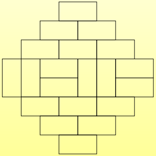
Szegö showed that the $-\beta$ power of the $(d-1)^{st}$ elementary symmetric function of $1-x_1,\ldots,1-x_d$ has nonnegative coefficients when $\beta \geq 1/2$.

Scott and Sokal vastly generalized this to negative powers of the Tutte polynomials on classes of graphs including series-parallel graphs.

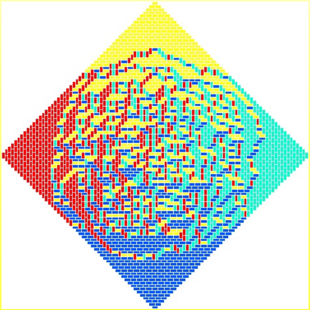
Other such families were studied by Lewy and Askey, then in greater generality by Kauers and Zeilberger. One interesting class is the Gillis-Reznick-Zeilberger class

$$F_{c,d}(z) = \frac{1}{1 - x_1 - x_2 - \cdots - x_d + c(x_1 x_2 \cdots x_d)}\,.$$

Random tilings



Let $a_{n,i,j}$ be the number of domino tilings of the order-n Aztec diamond, in which (i,j) is paired with its North neighbor.



The generating function is given by

$$F(x,y,z) = \sum a_{n,i,j} x^i y^j z^n = \frac{yz/2}{1-(x+x^{-1}+y+y^{-1})z/2+z^2}$$

II: Phenomena

Scope of ACSV

ACSV extends the ideas of univariate singularity analysis to multivariate generating functions.

Much of the ACSV literature concentrates on rational functions.

This loses less generality than you would think because all algebraic functions and many D-finite functions are representable as generalized diagonals of rational functions (see Wilson and Raichev 2007, 2012).

Also, there are direct results for algebraic functions (see Greenwood 2018).

Research area: How can one do singularity analysis on implicitly defined multivariate functions such as `search trees`?

Example of diagonal representation

Let

$$f(x,y) = x\sqrt{1-x-y}\,.$$

Then $[x^i y^j]f = [x^{i+j} y^j z^i]g$, where

$$g(x,y,z) = \frac{xz(2+x+xy+3z+z^3)}{2+x+xy+z}\,.$$

Pole variety

Restricting to the rational case, the most important aspect of a multivariate generating function is the *pole variety*, that is the complex algebraic variety $\mathcal{V}$ where the denominator vanishes.

Analytic methods will be discussed in detail later.

For now, keep in mind that the multivariate generalization of singularity analysis has to do with the geometry of $\mathcal{V}$.

By asymptotics "in the direction $\hat{r}$" we mean asymptotic behavior of r as $|r| \to \infty$ with the normalized vector $r/|r|$ converging to the unit vector $\hat{r}$.

Univariate methods for multivariate generating functions

Unlike the univariate case, where rational functions can only generate quasi-polynomials, phenomena for multivariate rational functions are quite varied.

Essentialy, only one multivariate behavior can be computed by univariate methods. GF-sequence methods (see Bender, Richmond, Gao, Hwang, etc.) represent

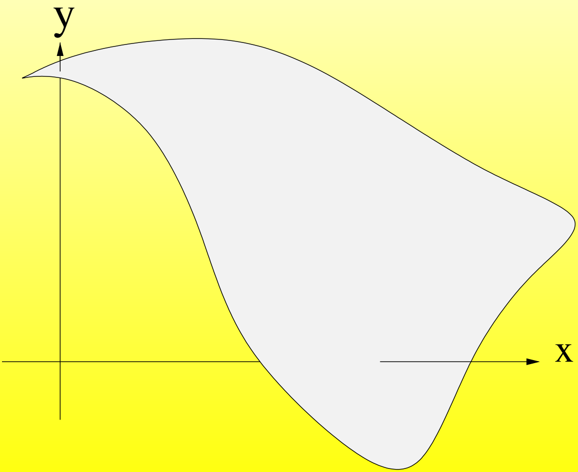
$$F(z) = \sum_{n=0}^{\infty} G_n(z_1, \ldots, z_{d-1}) z_d^n$$

and rely on establishing the asymptotic equivalence in a suitable region: $G_n \sim A(z_2, \ldots, z^d) g(z_1, \ldots, z_{d-1})^n$.

When this holds, the coefficients $\{a_r\}$ obey a Gaussian limit. Because this was the only known method, almost all the multivariate asymptotic results you will find in the literature between 1986 and 2006 are Gaussian limit theorems.

Smooth case

ACSV shows the Gaussian case to be a corollary of smoothness of $\mathcal{V}$; therefore we will refer to this as the *smooth case*.



Smooth case = Gaussian case = GF-sequence case

The behavior of $\{a_r\}$ is then given by a Central Limit and Local Large Deviation estimate.

Multivariate methods show the following asymptotic behavior when $\mathcal{V}$ is smooth (and under a few further conditions):

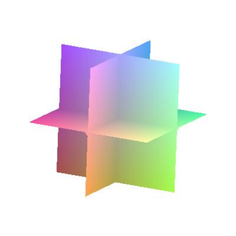
$$a_r \sim C_{\hat{r}} |r|^{(1-d)/2} z_*(\hat{r})^{-r} .$$

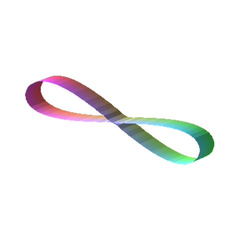
The formula holds piecewise over a finite collection of cones in r-space. This is simultaneously a central limit theorem and a local large deviation estimate.

This was known to physicists in some form thirty years ago. See, e.g., Chayes[2] 1986 on "Ornstein-Zernicke behavior".

Normal intersections

When $\mathcal{V}$ is the union of smooth sheets intersecting transversely (left), or is locally such a union (right), the coefficients have different formulas on different cones in r-space.





Counting integer solutions

For example, the generating function $1/\prod_{j=1}^{5}(1 - z^{b^{(j)}})$ counts solutions to $Ax = r$ when $b^{(j)}$ are the columns of

$$A := \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$ and r varies over $(\mathbb{Z}^+)^3$.

The counts grow like five different polynomials in 5 conical regions of $(\mathbb{Z}^+)^3$ (see De Loera and Sturmfels).

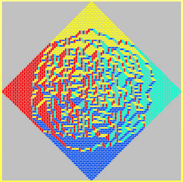
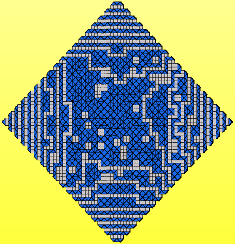
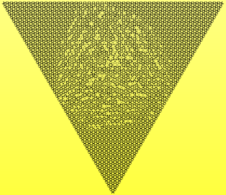
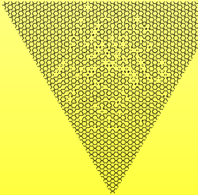
The stationary probabilities of points in the system of d queues introduced by Bertozzi and McKenna behave very similarly, due to the similar form of their generating functions.

Ensembles

In many applications, d-variate coefficients represent $d-1$ aspects of a large probability ensemble, with the last variable counting the system size.

As the size n goes to infinity, there is a feasible region where the probability of individual points is of order n^{-p} for some p, and an infeasible region where the probability of points is exponentially small.

For example, random tilings such as the Aztec Diamond pictured earlier, or diabolo tilings, or cube groves, or double dimer configurations, etc., look like these respective samples.

The feasible regions are already visible even when n is small, around 30 for most of these pictures.

Cone points

In the geometry of the generating function, in all of these cases, the asymptotics are driven by a cone point at $(1,1,1)$.

This is an isolated singularity where in local, rotated coordinates, the denominator looks like the quadratic

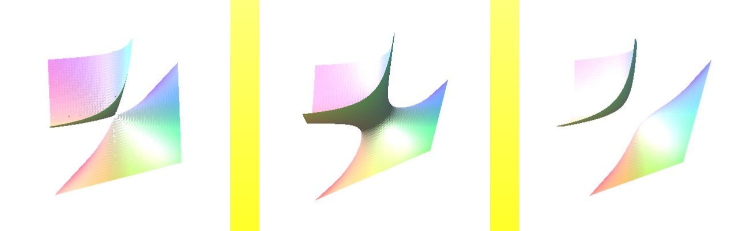
$$x_1^2 - \sum_{j-2}^{d} a_j^2 x_j^2 \, .$$

In one of the four cases, namely the second one, the cone is not quadratic but quartic.

Phase transitions

Cone points are not generic. In a family of generating functions with a free parameter, a finite number of parameter values will exhibit this behavior.

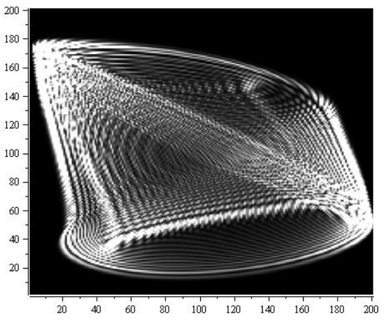
For example, in the GRZ family, this left-hand picture of $\mathcal{V}$ shows the geometry at criticality, transitioning between the two noncritical geometries in the middle and right figure.



Topology at criticality is particularly interesting!

Similar phenomenon, different class of generating function

When $\mathcal{V}$ is smooth but has high contact with the unit torus, coefficients can also behave this way. For example the quantum walk in $d+1$ spacetime variables always has $\mathcal{V}$ intersecting T^{d+1} in a manifold of dimension d rather than $d-1$. The feasible region is parametrized by this intersection, typically producing intensities like this.



III: Analytic machinery

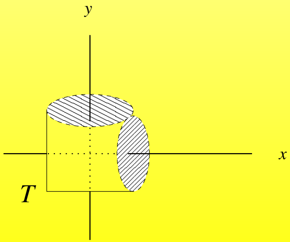
(How coefficients are extracted)

Getting from $F(X)$ to asymptotics for a_R

We begin with Cauchy's multivariate integral formula:

$$a_R = \left(\frac{1}{2\pi i}\right)^d \int_T Z^{-R} F(Z) \, \frac{dZ}{Z} \, .$$

Here T is a small torus, a product of circles winding once about the origin in each coordinate direction.

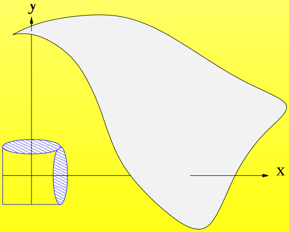


Domain of analyticity

If the generating function $F(Z) = P(Z)/Q(Z)$ is rational then F is analytic away from the singular variety

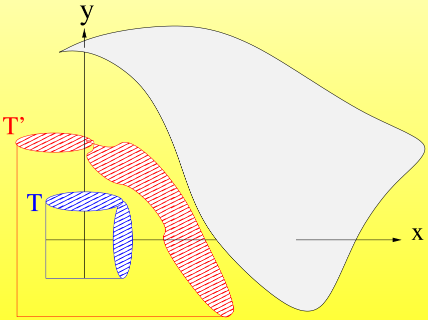
$$\mathcal{V} := \{Z : Q(Z) = 0\}.$$

More generally, one might have $F(Z) = G(Q(Z)^\alpha)$ or $F(Z) = G(\log Q(Z))$, where G is analytic but there is a branch singularity on $\mathcal{V}$.



Moving the chain of integration

We can move the contour of integration freely within the manifold $\mathcal{M} := (\mathcal{V} \cup \{\prod_{i=1}^{d} z_i = 0\})^c$.

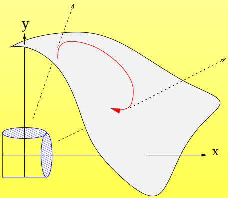


Once we cross $\mathcal{V}$, we pick up a residue.

Residue identity

The topological intersection of the d-cycle T with the variety $\mathcal{V}$ is a $(d-1)$-cycle $\mathcal{C}$, well defined at the level of homology.

The residue of the integrand $\omega := z^{-r}F(z)dz/z$ is a $(d-1)$-form $\mathrm{Res}(\omega)$.



The Cauchy integral reduces to an integral over $\mathcal{C}$:

$$(2\pi i)^d a_R = \int_T \omega = \int_{\mathcal{C}} \mathrm{Res}(\omega) \; .$$

IV: Topological machinery

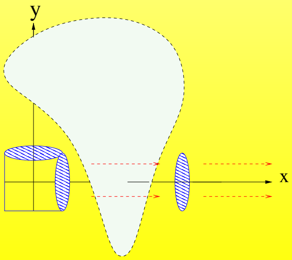
(How coefficients are extracted, part II)

We just saw that

$$(2\pi i)^d a_R = \int_T \omega = \int_{\mathcal{C}} \mathrm{Res}(\omega) \, .$$

The residue $\mathrm{Res}(\omega)$ is easily found via computer algebra.

The cycle $\mathcal{C} \subseteq \mathcal{V}$ is a topological invariant no matter how the original torus T is expanded across $\mathcal{V}$. We begin with a canonical expansion, say we expand the first coordinate to infinity while keeping the others fixed.



We now redraw this using a *height function*.

Height function

To evaluate an integral containing the monomial z^{-r} asymptotically as $r \to \infty$, we need to push the contour "down" in the sense of making the term z^{-r} as small as possible.

Equivalently, we want a contour minimizing the maximum value of

$$h(z) := -r \cdot (\log |z_1|, \ldots, \log |z_d|) \, .$$

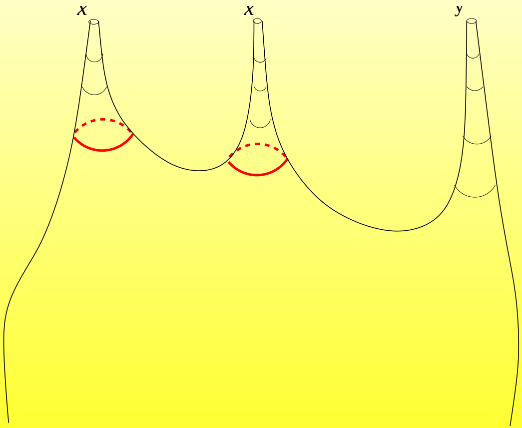
same as minimizing: $$h(z) := -\hat{r} \cdot (\log |z_1|, \ldots, \log |z_d|) \, .$$

In the next few pictures, $\mathcal{V}$ is drawn so that the height h decreases as you move down the screen.

$\mathcal{V}$ pictured by height

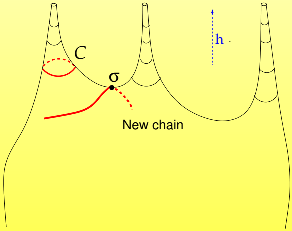
Where $\mathcal{V}$ intersects the x- or y-axis, height is infinite.

Expanding the 2-torus along the x-axis creates a ring (a 1-torus) around each such infinite height peak.

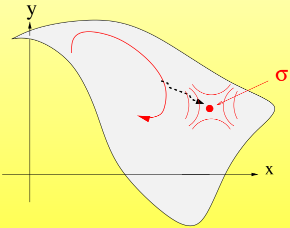


Pushing down = stationary phase

Pushing the contour down as far as it will go produces a stationary phase integral.



height view



regular view

The minimax is always located at critical point σ of h on $\mathcal{V}$.
The phase of the integrand is h, hence is stationary at σ.

Computer algebra

The critical points solve polynomial equations. Simplest case:

$$\begin{aligned} Q(z) &= 0 \\ r_d z_1 \frac{\partial Q}{\partial z_1}(z) &= r_1 z_d \frac{\partial Q}{\partial z_d}(z) \\ \vdots \quad & \quad \vdots \\ r_d z_{d-1} \frac{\partial Q}{\partial z_{d-1}}(z) &= r_{d-1} z_d \frac{\partial Q}{\partial z_d}(z)\,. \end{aligned}$$

Computer algebra can easily:

- Find the critical points.
- Compute the expansions of the residue form there.
- In most cases, read off asymptotic expressions such as

$$\int_{\text{near } \sigma} \sim \left(\frac{\sqrt{r^2+s^2}-s}{r} \right)^{-r} \cdot \left(\frac{\sqrt{r^2+s^2}-r}{s} \right)^{-s} \cdot \sqrt{\frac{1}{2\pi}} \sqrt{\frac{rs}{\sqrt{r^2+s^2}(r+s-\sqrt{r^2+s^2})^2}}$$

What's left to do

There may be many critical points.

1. How can we be certain the contour can be deformed to hang from a critical point?
2. To which such point(s) can the contour be deformed?
3. Once it gets into this position, what does the contour look like?

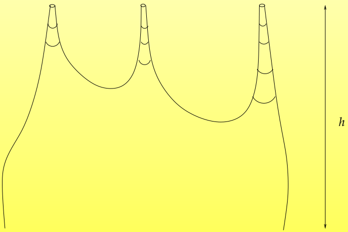
These questions are the province of Morse Theory.

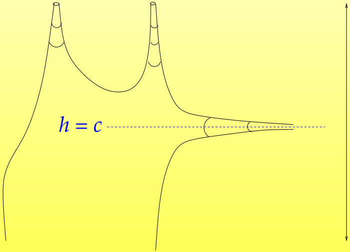
Morse lemma: *Let $h : \mathcal{V} \to \mathbb{R}$ be a proper (stratified) Morse function*[1]*. Then the (stratified) downward gradient flow pushes any contour down to an attachment cycle at a critical point.*

[1] Plus some other technical assumptions.

Properness

There are some technical assumptions too, but the worst one is that h is proper. In the picture on the left, h is proper.

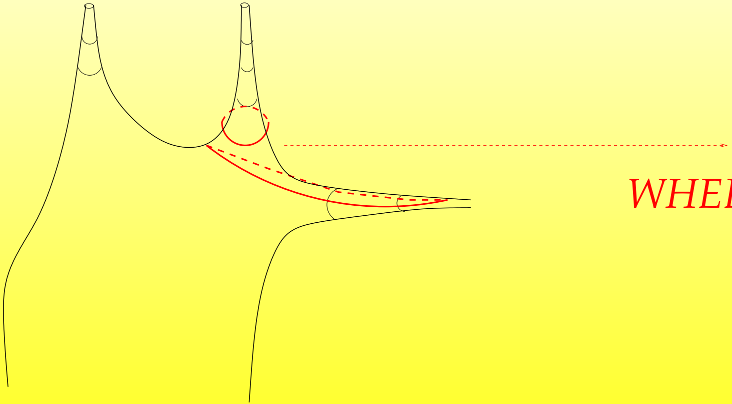




But in higher dimensions, the picture on the right is more typical. When $c \in (a, b)$, the inverse image of $h[a, b]$ is no longer compact. This means that the downward gradient flow could experience...

Shooting off to infinity

To infinity, and beyond!



WHEE !

The cycle goes to infinity before ever getting down to height c, and it never reaches a stationary phase point.

When can this happen?

This can happen. In some contexts it happens a lot!

For example, the algebraic function $\tilde{F}(x, y) = x/\sqrt{1 - x - y}$, is a diagonal of P/Q where

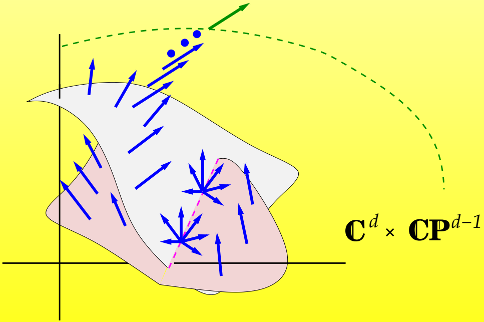
$$Q(x, y, z) = 2 + x + xy - z + 2xz + 2xyz + z^2x + z^2xy\,.$$

All downward flows on $\mathcal{V}_Q$ get forced to infinity.

This sort of obstruction can only occur when there is a critical point at infinity.

Critical points at infinity

A critical point at infinity in direction r is a point at infinity of the projective variety $\tilde{\mathcal{V}}$ which is the limit of points $z^{(n)}$ that are critical points for the height function $-r^{(n)} \cdot z$ where $r^{(n)} \to r$ as $n \to \infty$.



A critical point at ∞ is a point at infinity on the closure of the relation over $\mathcal{V}$:

z is critical in dir. r

Computability of critical points at infinity

Fortunately, computer algebra can detect critical points at infinity. The following algorithm produces an ideal identifying critical points at infinity in the given direction r.

Algorithm 1 (Find critical points at infinity)

code posted on Melczer's website

1. *Projectivize Q*
2. *Let I be the projectivized ideal for the critical point equations as functions of both z and r*
3. *Saturate I by the projectivizing variable*
4. *Set the projectivizing variable to zero*
5. *Substitute to specify the r variables*

The algorithm

What this algorithm does is to check for sequences of pairs $(z^{(n)}, r^{(n)})$ in the affine variety such that $z^{(n)} \to \infty$, $r^{(n)} \to r$, and z_n is a critical point for $h_{r^{(n)}}$.

Saturation ensures the approximating points $z^{(n)}$ are not already at infinity; setting the saturating variable to zero then restricts to limit points of these that are indeed at infinity. The result is a projective point at infinity which is a limit point of points critical in directions converging to r.

This may be taken as a definition of a critical point at infinity in direction r.

Consequences of Morse theory

This algorithm effectively takes care of Question #1,

> *How can we be certain the contour can be deformed to hang from a critical point?*

and leads to the following result.

Theorem 2

If there are no critical points at infinity, then "Morse theory works." In other words, cycles can be pushed down (deformed) until they hit stratified critical points for h. Furthermore, a cycle can be pushed past a critical point unless there is a topological obstruction: the cycle projects to something nonzero in the attachment homology there.

V: Some effective algorithms

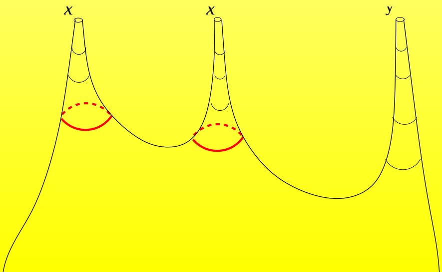
Which point(s)?

Question # 2 asked how to compute which critical point or points "catch" the contour:

Find the critical point(s) σ on which the pushed down contour will hang.

We have effective algorithms for this only in some cases.

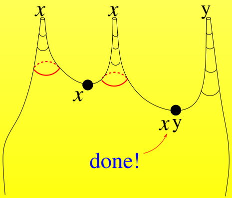
In this picture, for example, can you tell?



Two variables, $\mathcal{V}$ is smooth

Algorithm 3 (Find dominant critical point in 2D)

1. *Order the saddles by height.*
2. *Beginning with the highest, follow the two ascent paths. Each path must end at a pole or another saddle.*
3. *If both paths to a point marked x, then mark the saddle as x and continue; do similarly with a double y.*
4. *If one goes to x and one goes to y, you are done: output that saddle, as well as any of equal height that also go to x and y.*



Bi-colored supertrees

This algorithm handles, for example, the generating function for *bi-colored supertrees*, whose generating function has denominator

$$Q(x,y) := x^5y^2 + 2x^2y - 2x^3y + 4y + x - 2 \ .$$

Critical points in the main diagonal direction, from highest to lowest, are

$$\begin{aligned} \sigma_1 &:= \left(1+\sqrt{5}, \frac{3-\sqrt{5}}{16}\right) \\ \sigma_2 &:= \left(2, \frac{1}{8}\right) \\ \sigma_3 &:= \left(1-\sqrt{5}, \frac{3+\sqrt{5}}{16}\right) \end{aligned}$$

The algorithm shows the contour bypasses σ_1 and settles at σ_2.

Explanation

What this algorithm really does is to compute a cell complex representation of $\mathcal{V}$.

Zero-cells are critical points, including those on the axes (height $+\infty$) and a compactifying point at $-\infty$.

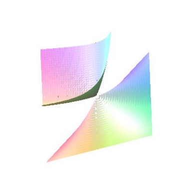
One-cells are ascent paths.

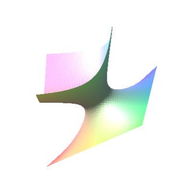
Problem: Compute the cell decomposition when $d = 3$, adding 2-cells in such a way that we can compute which saddles are topological obstacles.

There is much more to be said about why the 2-D algorithm works, but no time, so ask me later on, if interested.

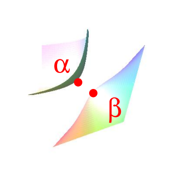
A more difficult case

Consider the GRZ denominator
$Q(x, y, z, w) = 1 - x - y - z - w + C\, x\, y\, z\, w\,.$





In the subcritical case $C < 27$, there are real critical points $(\alpha, \alpha, \alpha, \alpha)$ and $(\beta, \beta, \beta, \beta)$ with $\alpha < 1/3 < \beta$. As C increases through 27, these merge and split into two complex conjugate critical points.



We would like to prove the topological identity $\mathcal{C} \sim \mathcal{C}_\alpha + \mathcal{C}_*$, where $\mathcal{C}_\alpha$ is the saddle point contour through $(\alpha, \alpha, \alpha, \alpha)$ and $\mathcal{C}_*$ is a contour whose height does not exceed that of the next highest critical point $(\beta, \beta, \beta, \beta)$.

This topological fact would follow from a geometric proposition, namely that $(\alpha, \alpha, \alpha, \alpha)$ is *minimal*, meaning that $\mathcal{V}$ intersects the polydisk of radius 2α only at this one point.

This is in principle decidable by real algebraic geometry. However in 4 complex variables = 8 real variables, Maple computations typically will not halt.

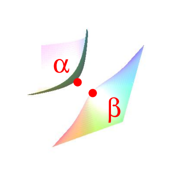
We would like to take advantage of the symmetric nature of Q, but unfortunately, Gröbner basis computations do not handle symmetric functions particularly well.

Symmetric multilinear case

Theorem 4

Suppose Q is symmetric and multilinear. Let q be the univariate polynomial $q(z) := Q(z, \ldots, z)$ and suppose z is a minimal modulus root of q. Then $z^\dagger := (z, \ldots, z)$ is a minimal point, meaning that $z^\dagger$ is a critical point in the diagonal direction and the set of minimal points in the diagonal direction is the intersection of the critical point set with the torus $\{z : |z_j| \leq |z|, 1 \leq j \leq d\}$.

As a consequence, the points $(\alpha, \alpha, \alpha, \alpha)$ are all minimal when $C \leq 27$ and the two conjugate complex points are minimal when $C > 27$.

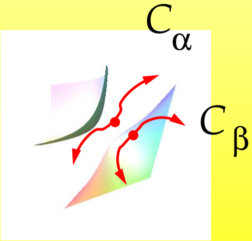


Higher order terms

As a consequence of Theorem 2 "Morse theory works in the absence of critical points at infinity," the cycle $\mathcal{C}$ is *in principle* completely described as a sum of cycles hanging from critical points. This is better than just pushing $\mathcal{C}$ down to the dominant critical point and showing that the topological leftovers live strictly further down.

In pictures, writing $\mathcal{C} = \mathcal{C}_\alpha + \mathcal{C}_\beta$ is better than knowing only:

$\mathcal{C} = \mathcal{C}_\alpha + \mathcal{C}_*$ where $\mathcal{C}_*$ has maximum height less than the height of α.



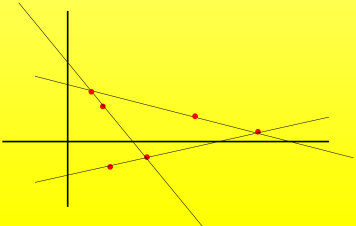
A complete decomposition of $\mathcal{C}$ in this manner would lead to a complete asymptotic transseries for the coefficients, and also allow one to compute behavior as $C \to 27$.

The product-linear case

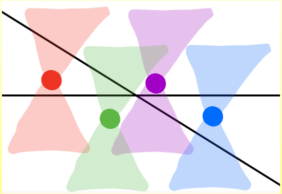
One case where we have an algorithm for a complete topological description if $\mathcal{C}$ is when Q is the product of real linear functions

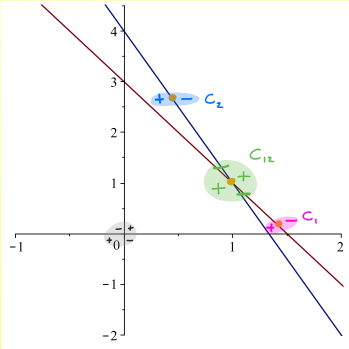
$$Q(z) = \prod_{j=1}^{m} 1 - b^{(j)} \cdot z \, .$$

The variety $\mathcal{V}$ is an *affine hyperplane arrangement*. Each flat has precisely one critical point. Pictured: in two variables the flats are just lines and intersection points. Each intersection is critical. Each line has precisely one critical point, which occurs in the orthant on which the line is bounded.



The *imaginary fibers* over chambers of the real arrangement form a homology basis.





Morse theory tells us that the *linking tori* form another basis. These are the alternating sums of imaginary fibers at each critical point.

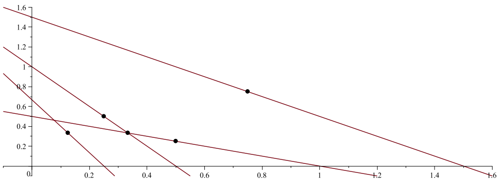
In the fiber basis, $\mathcal{C}$ is the alternating sum at the origin (not a linking torus because linking tori occur only at critical points).

Theorem 5

In the linking torus basis $\{\gamma_\sigma : \sigma$ *critical*$\}$

$$\mathcal{C} = \sum_\sigma \delta_\sigma \gamma_\sigma$$

where $\delta_\sigma = 1$ *if r is in the positive hull of the normal vectors to the hyperplanes meeting at* σ *and zero otherwise.*



The missing dot indicates that $\delta_\sigma = 0$ when σ is the left-most intersection.

Exact sum of stationary phase integrals

The linking tori are so named because the Cauchy integral over γ_σ is already in stationary phase. Thus,

$$(2\pi i)^d a_r = \sum_\sigma \delta_\sigma \int_{\gamma_\sigma} \omega \, .$$

Furthermore,

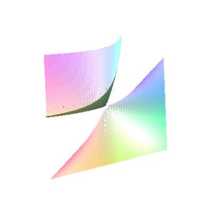
- The set of critical points σ is effectively computable via the critical point equations
- The coefficients δ_σ are zero or one and effectively computed via Theorem 5
- The integrals are automatically computable in terms of the coefficients $b^{(j)}$ of the linear factors of Q.
  code posted on Melczer's website

VI: Ad hoc computations

4-variable critical GRZ function

Let $Q(x, y, z, w) = 1 - x - y - z - w + C\,xyzw$ be the 4-variable GRZ denominator. The critical value is $C = 27$.

Our analysis of the coefficients of $1/Q$ is in some sense just a case study, but it gives a couple of very important lessons.



From diagonal analysis, one can establish that the diagonal coefficients $a_{n,n,n,n}$ grow at rate 9^n at criticality, whereas they grow at rate $\theta(C)^n$ when $C \neq 27$, with $\theta(C) \to 81$ as $C \to 27$.

In other words, the exponential growth rate experiences a sudden drop exactly at criticality.

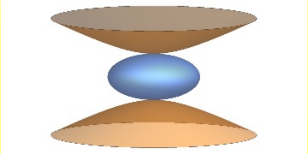
Topological explanation

The first interesting thing is the topological explanation.

Taking a residue gives a 3-dimensional intersection cycle. Imagine we rotate so that the tangent cone at the singular point is $x^2 - y^2 - z^2 - w^2$ with the x-coordinate pointing upward.

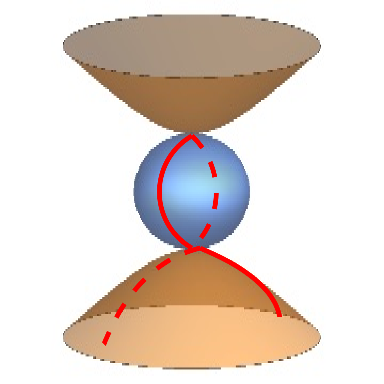
Perturbing Q to $Q + \epsilon$ resolves the cone into a hyperboloid, with two critical points.

The intersection cycle in hard to draw but it lives in this union of conics, where the two nodes are the two critical points for the perturbed variety.



Vanishing cycle

For the perturbed variety, the intersection cycle is the union of a sphere, living inside the sphere pictured, and a hyperboloid, hanging from the lower of the two nodes, and living in the downward hyperbola pictured.

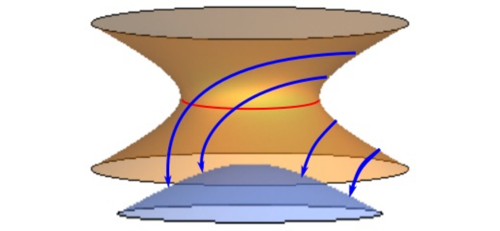


As $\epsilon \to 0$, the sphere shrinks to a point. Dimensional analysis shows that the integral goes to zero.

The hyperboloid is not what it looks like, but rather a double cover in opposite directions.

Double cover

The hyperboloid originates in the one-sheeted hyperboloid part of the cycle and can then be folded down as shown. Locally, where the tangent cone approximation is good, the hyperboloid can be made to cancel itself exactly, resulting in a cycle whose highest point is lower than the lower node.



Checking that there are no critical points at infinity, Morse theory then implies that this contour can be pushed down to the next critical point, yielding exponential order 9^n.

Using topology to rigorize numerics

A second interesting feature is the use of topology to rigorize numerics. Numeric homotopy continuation methods for univariate D-finite functions get you this far and no farther:

Lemma 6

$a_{n,n,n,n} \sim Cn^{-3/2}9^n$, *with* C *determined to arbitrary precision.*

With topological methods we can determine C and extend to a neighborhood of the diagonal.

Theorem 7

$$a_{nnnn} = 3\sqrt{27}\sqrt[4]{72}\frac{\cos(n\Psi + \ell)}{(4\pi n)^{3/2}}9^n + O(n^{-5/2}9^n)$$

in a neighborhood of the diagonal.

Proof of Theorem 7

1. Topological methods show the cycle can be pushed below the singular point.
2. There remain one conjugate pair of critical points.
3. Theorem 2 shows that smooth point asymptotics must hold there, with undetermined integer multiplicity.
4. Combined with Lemma 6, the integer is shown to be 3.

Note the interplay between:

Morse theoretic methods (steps 1 and 3) and

Computer algebra methods (Lemma 6): diagonal extraction and homotopy continuation methods for D-finite functions.

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