Metastability for the contact process on evolving scale-free networks

Peter Mörters

joint work with

Emmanuel Jacob (ENS Lyon)
Amitai Linker (Universidad de Chile)

Aim of the project

Motivation: We would like to understand how processes on large complex networks can be affected by time-variability of the network.

Aim of the project

Motivation: We would like to understand how processes on large complex networks can be affected by time-variability of the network.

This talk: Results obtained for the contact process on scale-free networks.

Aim of the project

Motivation: We would like to understand how processes on large complex networks can be affected by time-variability of the network.

This talk: Results obtained for the contact process on scale-free networks.

The contact process is a model for the spread of an infection on a finite graph.

Aim of the project

Motivation: We would like to understand how processes on large complex networks can be affected by time-variability of the network.

This talk: Results obtained for the contact process on scale-free networks.

The contact process is a model for the spread of an infection on a finite graph.

- Every vertex can either be infected or healthy.

Aim of the project

Motivation: We would like to understand how processes on large complex networks can be affected by time-variability of the network.

This talk: Results obtained for the contact process on scale-free networks.

The contact process is a model for the spread of an infection on a finite graph.

- Every vertex can either be infected or healthy.
- An infected vertex infects each of its neighbours at a fixed rate $\lambda > 0$.

Aim of the project

Motivation: We would like to understand how processes on large complex networks can be affected by time-variability of the network.

This talk: Results obtained for the contact process on scale-free networks.

The contact process is a model for the spread of an infection on a finite graph.

- Every vertex can either be infected or healthy.
- An infected vertex infects each of its neighbours at a fixed rate $\lambda > 0$.
- An infected vertex recovers with a fixed rate one.

Aim of the project

Motivation: We would like to understand how processes on large complex networks can be affected by time-variability of the network.

This talk: Results obtained for the contact process on scale-free networks.

The contact process is a model for the spread of an infection on a finite graph.

- Every vertex can either be infected or healthy.
- An infected vertex infects each of its neighbours at a fixed rate $\lambda > 0$.
- An infected vertex recovers with a fixed rate one.
- Once recovered, a vertex is again susceptible to infection by its neighbours.

The contact process

After a random finite extinction time T_{ext} all vertices become healthy and remain so forever. Starting the process with all vertices infected we ask how large is the extinction time?

The contact process

After a random finite extinction time $T_{\rm ext}$ all vertices become healthy and remain so forever. Starting the process with all vertices infected we ask how large is the extinction time?

Fast extinction: For sufficiently small infection rates $0 < \lambda < \lambda_c$ the expected extinction time is at most polynomial in the number N of vertices in the network.

The contact process

After a random finite extinction time T_{ext} all vertices become healthy and remain so forever. Starting the process with all vertices infected we ask how large is the extinction time?

Fast extinction: For sufficiently small infection rates $0 < \lambda < \lambda_c$ the expected extinction time is at most polynomial in the number N of vertices in the network.

Slow extinction: For all $\lambda > 0$ with high probability the extinction time is at least exponential in the number N of vertices in the network.

Figure: Schematic energy landscape for fast and slow extinction.
Slow extinction is due to metastability.

Scale-free networks

A feature of many networks is that they are (at least approximately) scale-free, which means that for very large N and large k,

$$\text{proportion of nodes of degree } k \approx k^{-\tau},$$

for some positive power law exponent τ.

Scale-free networks

A feature of many networks is that they are (at least approximately) scale-free, which means that for very large N and large k,

$$\text{proportion of nodes of degree } k \approx k^{-\tau},$$

for some positive power law exponent τ.

Easiest model: The vertex set is $\{1, \ldots, N\}$ with small indices indicating large strength. Every pair of vertices connects independently and the probability of connecting the ith and jth indexed vertex in the network of size N is

$$p_{i,j} = \frac{1}{N}\, p(i/N, j/N) \wedge 1,$$

for the two paradigmatic kernels

- Factor kernel $p(x, y) = \beta\, x^{-\gamma} y^{-\gamma}$,
- Preferential attachment kernel $p(x, y) = \beta\, (x \wedge y)^{-\gamma} (x \vee y)^{\gamma - 1}$

where $\beta > 0$ and $\gamma \in (0, 1)$ are the parameters of the model.

Scale-free networks

A feature of many networks is that they are (at least approximately) scale-free, which means that for very large N and large k,

$$\text{proportion of nodes of degree } k \approx k^{-\tau},$$

for some positive power law exponent τ.

Easiest model: The vertex set is $\{1, \dots, N\}$ with small indices indicating large strength. Every pair of vertices connects independently and the probability of connecting the ith and jth indexed vertex in the network of size N is

$$p_{i,j} = \frac{1}{N}\, p(i/N, j/N) \wedge 1,$$

for the two paradigmatic kernels

- Factor kernel $p(x, y) = \beta\, x^{-\gamma} y^{-\gamma}$,
- Preferential attachment kernel $p(x, y) = \beta\, (x \wedge y)^{-\gamma} (x \vee y)^{\gamma-1}$

where $\beta > 0$ and $\gamma \in (0, 1)$ are the parameters of the model.

In both cases the power law exponent is $\tau = 1 + \frac{1}{\gamma}$.

Scale-free networks

Easiest model: The vertex set is $\{1, \ldots, N\}$ with small indices indicating large strength. Every pair of vertices connects independently and the probability of connecting the ith and jth indexed vertex in the network of size N is

$$p_{i,j} = \frac{1}{N}\, p(i/N, j/N) \wedge 1,$$

for the two paradigmatic kernels

- Factor kernel $p(x, y) = \beta\, x^{-\gamma} y^{-\gamma}$,
- Preferential attachment kernel $p(x, y) = \beta\, (x \wedge y)^{-\gamma} (x \vee y)^{\gamma - 1}$

where $\beta > 0$ and $\gamma \in (0, 1)$ are the parameters of the model.

In both cases the power law exponent is $\tau = 1 + \frac{1}{\gamma}$.

Classical result: For all values of τ the contact process shows slow extinction. Proved by Chatterjee and Durrett (2009) for the factor kernel.

Our evolving scale-free network model

Our evolving scale-free network model

We look at the following evolving network $(\mathscr{G}_t)_{t\geq 0}$.

Our evolving scale-free network model

We look at the following evolving network $(\mathscr{G}_t)_{t\geq 0}$.

- at all times the vertex set is given as $\{1, \ldots, N\}$.

Our evolving scale-free network model

We look at the following evolving network $(\mathscr{G}_t)_{t\geq 0}$.

- at all times the vertex set is given as $\{1, \ldots, N\}$.
- $\mathscr{G}_0$ is formed by independently connecting every pair $\{i, j\}$ with probability

$$p_{i,j} = \frac{1}{N}\, p(i/N, j/N).$$

Our evolving scale-free network model

We look at the following evolving network $(\mathscr{G}_t)_{t\geq 0}$.

- at all times the vertex set is given as $\{1,\ldots,N\}$.
- $\mathscr{G}_0$ is formed by independently connecting every pair $\{i,j\}$ with probability

$$p_{i,j} = \frac{1}{N}\, p(i/N, j/N).$$

- The network evolves by vertex updating:

Our evolving scale-free network model

We look at the following evolving network $(\mathscr{G}_t)_{t\geq 0}$.

- at all times the vertex set is given as $\{1,\dots,N\}$.
- $\mathscr{G}_0$ is formed by independently connecting every pair $\{i,j\}$ with probability

$$p_{i,j} = \frac{1}{N}\, p(i/N, j/N).$$

- The network evolves by vertex updating:
  - Every vertex has a clock which strikes after an exponential time with parameter $\kappa > 0$.

Our evolving scale-free network model

We look at the following evolving network $(\mathscr{G}_t)_{t\geq 0}$.

- at all times the vertex set is given as $\{1, \ldots, N\}$.
- $\mathscr{G}_0$ is formed by independently connecting every pair $\{i, j\}$ with probability

$$p_{i,j} = \frac{1}{N}\, p(i/N, j/N).$$

- The network evolves by vertex updating:
  - Every vertex has a clock which strikes after an exponential time with parameter $\kappa > 0$.
  - When it strikes, say for vertex i, all adjacent edges are removed, and

Our evolving scale-free network model

We look at the following evolving network $(\mathscr{G}_t)_{t\geq 0}$.

- at all times the vertex set is given as $\{1,\ldots,N\}$.
- $\mathscr{G}_0$ is formed by independently connecting every pair $\{i,j\}$ with probability

$$p_{i,j} = \frac{1}{N}\, p(i/N, j/N).$$

- The network evolves by vertex updating:
  - Every vertex has a clock which strikes after an exponential time with parameter $\kappa > 0$.
  - When it strikes, say for vertex i, all adjacent edges are removed, and
  - new edges $i \leftrightarrow j$ are formed with probability $p_{i,j}$, independently for every $j \in \{1,\ldots,N\} \setminus \{i\}$.

Our evolving scale-free network model

We look at the following evolving network $(\mathscr{G}_t)_{t\geq 0}$.

- at all times the vertex set is given as $\{1,\dots,N\}$.
- $\mathscr{G}_0$ is formed by independently connecting every pair $\{i,j\}$ with probability

$$p_{i,j} = \frac{1}{N}\, p(i/N, j/N).$$

- The network evolves by vertex updating:
  - Every vertex has a clock which strikes after an exponential time with parameter $\kappa > 0$.
  - When it strikes, say for vertex i, all adjacent edges are removed, and
  - new edges $i \leftrightarrow j$ are formed with probability $p_{i,j}$, independently for every $j \in \{1,\dots,N\} \setminus \{i\}$.
- Note that $\mathscr{G}_t \overset{d}{=} \mathscr{G}_0$ for all $t > 0$.

The contact process on evolving scale-free networks

Theorem: Jacob, M (2015)

Consider the contact process on the evolving network $(\mathscr{G}_t)_{t\geq 0}$ with factor kernel.

(a) If $\tau < 4$ (or equivalently $\gamma > 1/3$), then for all $\lambda > 0$ there exists $c > 0$ such that, uniformly in $N > 0$,

$$\mathbb{P}(T_{\rm ext} \leq e^{cN}) \leq e^{-cN}.$$

(b) If $\tau > 4$ (or equivalently $\gamma < 1/3$), then there exists a parameter $\lambda_{\rm c} > 0$ such that, for all $\lambda < \lambda_{\rm c}$, there exists $C > 0$ such that, uniformly in $N > 0$,

$$\mathbb{E}[T_{\rm ext}] \leq CN^{\gamma} \log N.$$

The contact process on evolving scale-free networks

Theorem: Jacob, M (2015)

Consider the contact process on the evolving network $(\mathscr{G}_t)_{t \geq 0}$ with factor kernel.

(a) If $\tau < 4$ (or equivalently $\gamma > 1/3$) then we have slow extinction.

(b) If $\tau > 4$ (or equivalently $\gamma < 1/3$), then there exists a parameter $\lambda_{\rm c} > 0$ such that, for all $\lambda < \lambda_{\rm c}$, there exists $C > 0$ such that, uniformly in $N > 0$,

$$\mathbb{E}[T_{\rm ext}] \leq CN^{\gamma} \log N.$$

The contact process on evolving scale-free networks

Theorem: Jacob, M (2015)

Consider the contact process on the evolving network $(\mathscr{G}_t)_{t\geq 0}$ with factor kernel.

(a) If $\tau < 4$ (or equivalently $\gamma > 1/3$) then we have slow extinction.

(b) If $\tau > 4$ (or equivalently $\gamma < 1/3$) then we have fast extinction.

The contact process on evolving scale-free networks

Theorem: Jacob, M (2015)

Consider the contact process on the evolving network $(\mathscr{G}_t)_{t\geq 0}$ with factor kernel.

(a) If $\tau < 4$ (or equivalently $\gamma > 1/3$) then we have slow extinction.

(b) If $\tau > 4$ (or equivalently $\gamma < 1/3$) then we have fast extinction.

Observation:

- Time-variability has made fast extinction possible, but only if $\tau > 4$.

The contact process on evolving scale-free networks

Theorem: Jacob, M (2015)

Consider the contact process on the evolving network $(\mathscr{G}_t)_{t\geq 0}$ with factor kernel.

(a) If $\tau < 4$ (or equivalently $\gamma > 1/3$) then we have slow extinction.

(b) If $\tau > 4$ (or equivalently $\gamma < 1/3$) then we have fast extinction.

Observation:

- Time-variability has made fast extinction possible, but only if $\tau > 4$.
- This is also different from the mean-field prediction of Pastor-Sattoras and Vespignani (2001) who find fast extinction for $\tau > 3$, which is the value at which there is a transition in the network topology.

Heuristic explanation

For a suitable $a(\lambda) \downarrow 0$ the most powerful vertices with index in $\{1, \ldots, a(\lambda)N\}$ are called stars.

Heuristic explanation

For a suitable $a(\lambda) \downarrow 0$ the most powerful vertices with index in $\{1, \ldots, a(\lambda)N\}$ are called stars.

Mean field calculation:
The infection can be sustained on the set of stars if

$$\lambda \int_0^{a(\lambda)} \int_0^{a(\lambda)} p(x, y)\, dx\, dy \approx a(\lambda)$$

Heuristic explanation

For a suitable $a(\lambda) \downarrow 0$ the most powerful vertices with index in $\{1, \ldots, a(\lambda)N\}$ are called stars.

Mean field calculation:
The infection can be sustained on the set of stars if

$$\lambda \int_0^{a(\lambda)} \int_0^{a(\lambda)} x^{-\gamma} y^{-\gamma} \, dx \, dy \approx a(\lambda)$$

Heuristic explanation

For a suitable $a(\lambda) \downarrow 0$ the most powerful vertices with index in $\{1, \dots, a(\lambda)N\}$ are called stars.

Mean field calculation:
The infection can be sustained on the set of stars if

$$\lambda \, a(\lambda)^{2-2\gamma} \approx a(\lambda)$$

Heuristic explanation

For a suitable $a(\lambda) \downarrow 0$ the most powerful vertices with index in $\{1, \ldots, a(\lambda)N\}$ are called stars.

Mean field calculation:
The infection can be sustained on the set of stars if

$$\lambda \, a(\lambda)^{1-2\gamma} \approx 1$$

which can be achieved if $\gamma > \frac{1}{2}$ or, equivalently, $\tau < 3$.

Heuristic explanation

For a suitable $a(\lambda) \downarrow 0$ the most powerful vertices with index in $\{1, \ldots, a(\lambda)N\}$ are called stars.

Mean field calculation:

The infection can be sustained on the set of stars if

$$\lambda \int_0^{a(\lambda)} \int_0^{a(\lambda)} p(x, y)\, dx\, dy \approx a(\lambda)$$

which can be achieved if $\gamma > \frac{1}{2}$ or, equivalently, $\tau < 3$.

Vertices of degree $k \gg \lambda^{-2}$ can keep the infection for longer:

Heuristic explanation

For a suitable $a(\lambda) \downarrow 0$ the most powerful vertices with index in $\{1, \dots, a(\lambda)N\}$ are called stars.

Mean field calculation:

The infection can be sustained on the set of stars if

$$\lambda \int_0^{a(\lambda)} \int_0^{a(\lambda)} p(x, y)\, dx\, dy \approx a(\lambda)$$

which can be achieved if $\gamma > \frac{1}{2}$ or, equivalently, $\tau < 3$.

Vertices of degree $k \gg \lambda^{-2}$ can keep the infection for longer:

- If they get infected they infect on average $k\lambda$ neighbours before recovery,

Heuristic explanation

For a suitable $a(\lambda) \downarrow 0$ the most powerful vertices with index in $\{1, \ldots, a(\lambda)N\}$ are called stars.

Mean field calculation:

The infection can be sustained on the set of stars if

$$\lambda \int_0^{a(\lambda)} \int_0^{a(\lambda)} p(x,y)\, dx\, dy \approx a(\lambda)$$

which can be achieved if $\gamma > \frac{1}{2}$ or, equivalently, $\tau < 3$.

Vertices of degree $k \gg \lambda^{-2}$ can keep the infection for longer:

- If they get infected they infect on average $k\lambda$ neighbours before recovery,
- with probability $\frac{\lambda^2 k}{\kappa + \lambda^2 k}$ they will be immediately reinfected by one of their infected neighbours,

Heuristic explanation

For a suitable $a(\lambda) \downarrow 0$ the most powerful vertices with index in $\{1, \ldots, a(\lambda)N\}$ are called stars.

Mean field calculation:

The infection can be sustained on the set of stars if

$$\lambda \int_0^{a(\lambda)} \int_0^{a(\lambda)} p(x,y)\, dx\, dy \approx a(\lambda)$$

which can be achieved if $\gamma > \frac{1}{2}$ or, equivalently, $\tau < 3$.

Vertices of degree $k \gg \lambda^{-2}$ can keep the infection for longer:

- If they get infected they infect on average $k\lambda$ neighbours before recovery,
- with probability $\frac{\lambda^2 k}{\kappa + \lambda^2 k}$ they will be immediately reinfected by one of their infected neighbours,
- if $\lambda^2 k \gg 1$ the infection stays alive for order $\lambda^2 k$ time units.

Heuristic explanation

For a suitable $a(\lambda) \downarrow 0$ the most powerful vertices with index in $\{1, \ldots, a(\lambda)N\}$ are called stars.

Mean field calculation:

The infection can be sustained on the set of stars if

$$\lambda \int_0^{a(\lambda)} \int_0^{a(\lambda)} p(x,y)\, dx\, dy \approx a(\lambda)$$

which can be achieved if $\gamma > \frac{1}{2}$ or, equivalently, $\tau < 3$.

Vertices of degree $k \gg \lambda^{-2}$ can keep the infection for longer:

- if $\lambda^2 k \gg 1$ the infection stays alive for order $\lambda^2 k$ time units.

Topology based calculation:

$$\lambda^3\, a(\lambda)^{-\gamma} \int_0^{a(\lambda)} \int_0^{a(\lambda)} p(x,y)\, dx\, dy \approx a(\lambda)$$

Heuristic explanation

For a suitable $a(\lambda) \downarrow 0$ the most powerful vertices with index in $\{1, \ldots, a(\lambda)N\}$ are called stars.

Mean field calculation:

The infection can be sustained on the set of stars if

$$\lambda \int_0^{a(\lambda)} \int_0^{a(\lambda)} p(x,y)\, dx\, dy \approx a(\lambda)$$

which can be achieved if $\gamma > \frac{1}{2}$ or, equivalently, $\tau < 3$.

Vertices of degree $k \gg \lambda^{-2}$ can keep the infection for longer:

- if $\lambda^2 k \gg 1$ the infection stays alive for order $\lambda^2 k$ time units.

Topology based calculation:

$$\lambda^3\, a(\lambda)^{-\gamma} \int_0^{a(\lambda)} \int_0^{a(\lambda)} x^{-\gamma} y^{-\gamma}\, dx\, dy \approx a(\lambda)$$

Heuristic explanation

For a suitable $a(\lambda) \downarrow 0$ the most powerful vertices with index in $\{1, \ldots, a(\lambda)N\}$ are called stars.

Mean field calculation:
The infection can be sustained on the set of stars if

$$\lambda \int_0^{a(\lambda)} \int_0^{a(\lambda)} p(x, y)\, dx\, dy \approx a(\lambda)$$

which can be achieved if $\gamma > \frac{1}{2}$ or, equivalently, $\tau < 3$.

Vertices of degree $k \gg \lambda^{-2}$ can keep the infection for longer:

- if $\lambda^2 k \gg 1$ the infection stays alive for order $\lambda^2 k$ time units.

Topology based calculation:

$$\lambda^3\, a(\lambda)^{2-3\gamma} \approx a(\lambda)$$

Heuristic explanation

For a suitable $a(\lambda) \downarrow 0$ the most powerful vertices with index in $\{1, \ldots, a(\lambda)N\}$ are called stars.

Mean field calculation:
The infection can be sustained on the set of stars if

$$\lambda \int_0^{a(\lambda)} \int_0^{a(\lambda)} p(x,y)\,dx\,dy \approx a(\lambda)$$

which can be achieved if $\gamma > \frac{1}{2}$ or, equivalently, $\tau < 3$.

Vertices of degree $k \gg \lambda^{-2}$ can keep the infection for longer:

- if $\lambda^2 k \gg 1$ the infection stays alive for order $\lambda^2 k$ time units.

Topology based calculation:

$$\lambda^3\, a(\lambda)^{1-3\gamma} \approx 1$$

which can be achieved if $\gamma > \frac{1}{3}$ or, equivalently, $\tau < 4$.

Metastable Densities: Factor kernel

In the slow extinction case the density of infected vertices is likely to be maintained at a certain level up to the exponential survival time of the infection. Denoting

$$I_N(t) = \mathbb{E}\big[\text{ proportion of infected vertices at time } t\big]$$

we say that $\rho(\lambda)$ is the metastable density if, whenever t_N is going to infinity slower than exponentially, we have

$$\lim_{N\to\infty} I_N(t_N) = \rho(\lambda) > 0.$$

Metastable Densities: Factor kernel

In the slow extinction case the density of infected vertices is likely to be maintained at a certain level up to the exponential survival time of the infection. Denoting

$$I_N(t) = \mathbb{E}\big[\text{ proportion of infected vertices at time } t\big]$$

we say that $\rho(\lambda)$ is the metastable density if, whenever t_N is going to infinity slower than exponentially, we have

$$\lim_{N\to\infty} I_N(t_N) = \rho(\lambda) > 0.$$

Theorem: Jacob, Linker, M (2017)

Consider the contact process on the evolving network $(\mathscr{G}_t)_{t\geq 0}$ with factor kernel. Then, as $\lambda \downarrow 0$, the metastable density $\rho(\lambda)$ satisfies

$$\rho(\lambda) = \begin{cases} \lambda^{\frac{2}{3\gamma-1}+o(1)} & \text{if} \quad 1/3 < \gamma < 2/3 \quad \text{or} \quad 4 > \tau > 5/2, \\ \lambda^{\frac{\gamma}{2\gamma-1}+o(1)} & \text{if} \quad \gamma > 2/3 \quad \text{or} \quad \tau < 5/2. \end{cases}$$

Metastable Densities: Factor kernel

In the slow extinction case the density of infected vertices is likely to be maintained at a certain level up to the exponential survival time of the infection. Denoting

$$I_N(t) = \mathbb{E}\big[\text{ proportion of infected vertices at time } t\big]$$

we say that $\rho(\lambda)$ is the metastable density if, whenever t_N is going to infinity slower than exponentially, we have

$$\lim_{N\to\infty} I_N(t_N) = \rho(\lambda) > 0.$$

Theorem: Jacob, Linker, M (2017)

Consider the contact process on the evolving network $(\mathscr{G}_t)_{t\geq 0}$ with factor kernel. Then, as $\lambda \downarrow 0$, the metastable density $\rho(\lambda)$ satisfies

$$\rho(\lambda) = \begin{cases} \lambda^{\frac{2}{3\gamma-1}+o(1)} & \text{if} \quad 1/3 < \gamma < 2/3 \quad \text{or} \quad 4 > \tau > 5/2, \\ \lambda^{\frac{\gamma}{2\gamma-1}+o(1)} & \text{if} \quad \gamma > 2/3 \quad \text{or} \quad \tau < 5/2. \end{cases}$$

- At $\tau = 5/2$ a change in survival strategies happens.

Insight: A transition of time-scales

The transition occurs in the time-scale on which the infection spreads.

- $1/3 < \gamma < 2/3$: Delayed direct spreading

  Individual stars can survive recoveries through immediate reinfection by their neighbours and thus keep the infection on a time-scale of

  $$T_\lambda = \lambda^2 a(\lambda)^{-\gamma} = \lambda^{\frac{3\gamma-2}{3\gamma-1}} \gg 1.$$

  On this time-scale stars spread the infection to other stars thereby retaining a skeleton of infected stars in a set of infected vertices of density

  $$\lambda a(\lambda)^{1-\gamma} = \lambda^{\frac{2}{3\gamma-1}}.$$

- $2/3 < \gamma < 1$: Quick direct spreading

  The time-delay mechanism is no longer effective. Stars infect a sufficient number of other stars at time-scale of order one to retain a skeleton of infected stars in a set of infected vertices of density

  $$\lambda a(\lambda)^{1-\gamma} = \lambda^{\frac{\gamma}{2\gamma-1}}.$$

Metastable Densities: Preferential attachment kernel

The situation is quite different for preferential attachment kernels.

Theorem: Jacob, Linker, M (2017)

Consider the contact process on the evolving network $(\mathscr{G}_t)_{t\geq 0}$ with preferential attachment kernel.

(i) For all $0 < \gamma < 1$ there is slow extinction.

(ii) As $\lambda \downarrow 0$, the metastable density $\rho(\lambda)$ satisfies

$$\rho(\lambda) = \begin{cases} \lambda^{\frac{3-2\gamma}{\gamma}+o(1)} & \text{if} \quad \gamma < 3/5 \quad \text{or} \quad \tau > 8/3, \\ \lambda^{\frac{3-\gamma}{3\gamma-1}+o(1)} & \text{if} \quad \gamma > 3/5 \quad \text{or} \quad \tau < 8/3. \end{cases}$$

Metastable Densities: Preferential attachment kernel

The situation is quite different for preferential attachment kernels.

Theorem: Jacob, Linker, M (2017)

Consider the contact process on the evolving network $(\mathscr{G}_t)_{t\geq 0}$ with preferential attachment kernel.

(i) For all $0 < \gamma < 1$ there is slow extinction.

(ii) As $\lambda \downarrow 0$, the metastable density $\rho(\lambda)$ satisfies

$$\rho(\lambda) = \begin{cases} \lambda^{\frac{3-2\gamma}{\gamma}+o(1)} & \text{if} \quad \gamma < 3/5 \quad \text{or} \quad \tau > 8/3, \\ \lambda^{\frac{3-\gamma}{3\gamma-1}+o(1)} & \text{if} \quad \gamma > 3/5 \quad \text{or} \quad \tau < 8/3. \end{cases}$$

- Unlike in the case of factor kernels we do not have a fast extinction phase.

Metastable Densities: Preferential attachment kernel

The situation is quite different for preferential attachment kernels.

Theorem: Jacob, Linker, M (2017)

Consider the contact process on the evolving network $(\mathscr{G}_t)_{t\geq 0}$ with preferential attachment kernel.

(i) For all $0 < \gamma < 1$ there is slow extinction.

(ii) As $\lambda \downarrow 0$, the metastable density $\rho(\lambda)$ satisfies

$$\rho(\lambda) = \begin{cases} \lambda^{\frac{3-2\gamma}{\gamma}+o(1)} & \text{if} \quad \gamma < 3/5 \quad \text{or} \quad \tau > 8/3, \\ \lambda^{\frac{3-\gamma}{3\gamma-1}+o(1)} & \text{if} \quad \gamma > 3/5 \quad \text{or} \quad \tau < 8/3. \end{cases}$$

- Unlike in the case of factor kernels we do not have a fast extinction phase.
- At power law exponent $\tau = 8/3$ a change in survival strategies happens.

Insights: A transition of spreading mechanism

In the preferential attachment case time-delay always works. What changes is the mechanism how the infection spreads most effectively from star to star.

- $\gamma < 3/5$: Delayed direct spreading

  Individual stars can survive recoveries through immediate reinfection by their neighbours and thus keep the infection on a time-scale of $T_\lambda = \lambda^{-1} \gg 1$. On this time-scale stars spread the infection directly to other stars.

- $\gamma > 3/5$: Delayed indirect spreading

  Individual stars can survive recoveries through immediate reinfection by their neighbours and thus keep the infection on a time-scale

  $$T_\lambda = \lambda^2 a(\lambda)^{-\gamma} = \lambda^{\frac{2\gamma-2}{3\gamma-1}} \gg 1.$$

  On this time-scale stars infect other stars by infecting a large number of their neighbours, which pass the infection to other stars thereby retaining a skeleton of infected stars in a set of infected vertices of density

  $$\lambda a(\lambda)^{1-\gamma} = \lambda^{\frac{3-\gamma}{3\gamma-1}}.$$

Degree dependent update rates

By making the update rates of vertices dependent on the degree we get a more complete understanding of the phases. Let the update rate of the ith vertex be

$$\kappa(i) = \kappa \times \left(\frac{N}{i}\right)^{\gamma\eta},$$

for some $\eta \in \mathbb{R}$.

Degree dependent update rates

By making the update rates of vertices dependent on the degree we get a more complete understanding of the phases. Let the update rate of the ith vertex be

$$\kappa(i) = \kappa \times \left(\frac{N}{i}\right)^{\gamma\eta},$$

for some $\eta \in \mathbb{R}$. Then we have the following phase diagrams.

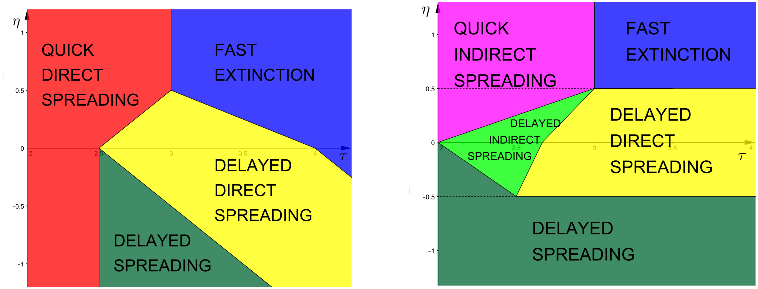

Figure: Phase diagrams interpolating between the mean-field case, for $\eta \uparrow \infty$, and the static case, for $\eta \downarrow -\infty$. For the factor kernel metatable densities in the static case are due to Mountford, Valesin, Yao (2013).

Edge updating with variable rates

We also study the case that all potential edges $\{i, j\}$ update with rate

$$\kappa(i,j) = \kappa \times \left(\left(\frac{N}{i}\right)^{\gamma\eta} + \left(\frac{N}{j}\right)^{\gamma\eta} \right),$$

for some $\eta \in \mathbb{R}$.

Edge updating with variable rates

We also study the case that all potential edges $\{i,j\}$ update with rate

$$\kappa(i,j) = \kappa \times \left(\left(\frac{N}{i}\right)^{\gamma\eta} + \left(\frac{N}{j}\right)^{\gamma\eta} \right),$$

for some $\eta \in \mathbb{R}$. Then we have the following phase diagrams.

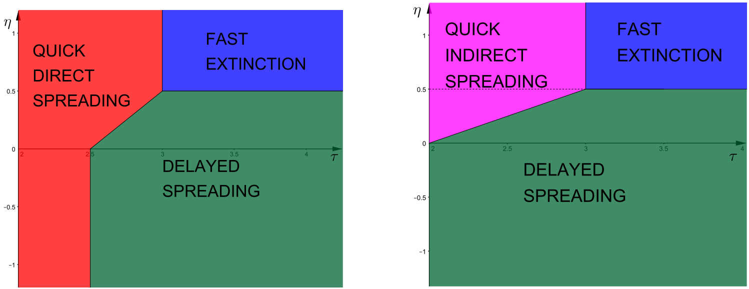

Figure: Phase diagrams for edge-updating scheme, factor kernel on the left, preferential attachment kernel on the right.

Method of proof: Existence of a fast extinction phase

Coupling with a mean-field model

Method of proof: Existence of a fast extinction phase

Coupling with a mean-field model

In the mean-field model every vertex can have three states *healthy*, *ready*, or *infected*. The state *ready* means that the vertex is infected but ready to recover.

Method of proof: Existence of a fast extinction phase

Coupling with a mean-field model

In the mean-field model every vertex can have three states *healthy*, *ready*, or *infected*. The state *ready* means that the vertex is infected but ready to recover.

- Recovery and update times are taken from the original process. For every pair $\{i, j\}$ of vertices there is a Poisson process of infection times with rate $\lambda p_{i,j}$.

Method of proof: Existence of a fast extinction phase

Coupling with a mean-field model

In the mean-field model every vertex can have three states *healthy*, *ready*, or *infected*. The state *ready* means that the vertex is infected but ready to recover.

- Recovery and update times are taken from the original process. For every pair $\{i, j\}$ of vertices there is a Poisson process of infection times with rate $\lambda p_{i,j}$.
- If at an infection time of the pair $\{i, j\}$ one of the vertices is not *healthy*, both become *infected*.

Method of proof: Existence of a fast extinction phase

Coupling with a mean-field model

In the mean-field model every vertex can have three states *healthy*, *ready*, or *infected*. The state *ready* means that the vertex is infected but ready to recover.

- Recovery and update times are taken from the original process. For every pair $\{i, j\}$ of vertices there is a Poisson process of infection times with rate $\lambda p_{i,j}$.
- If at an infection time of the pair $\{i, j\}$ one of the vertices is not *healthy*, both become *infected*.
- If at an update time the vertex is *infected*, it becomes *ready*.

Method of proof: Existence of a fast extinction phase

Coupling with a mean-field model

In the mean-field model every vertex can have three states *healthy*, *ready*, or *infected*. The state *ready* means that the vertex is infected but ready to recover.

- Recovery and update times are taken from the original process. For every pair $\{i,j\}$ of vertices there is a Poisson process of infection times with rate $\lambda p_{i,j}$.
- If at an infection time of the pair $\{i,j\}$ one of the vertices is not *healthy*, both become *infected*.
- If at an update time the vertex is *infected*, it becomes *ready*.
- If at a recovery time a vertex is *ready*, it becomes *healthy*.

Method of proof: Existence of a fast extinction phase

Coupling with a mean-field model

In the mean-field model every vertex can have three states *healthy*, *ready*, or *infected*. The state *ready* means that the vertex is infected but ready to recover.

- Recovery and update times are taken from the original process. For every pair $\{i, j\}$ of vertices there is a Poisson process of infection times with rate $\lambda p_{i,j}$.
- If at an infection time of the pair $\{i, j\}$ one of the vertices is not *healthy*, both become *infected*.
- If at an update time the vertex is *infected*, it becomes *ready*.
- If at a recovery time a vertex is *ready*, it becomes *healthy*.

It is possible to couple the original process to the mean-field model in such a way that, at every time $t > 0$, every vertex which is *infected* in the original model, is either *ready* or *infected* in the mean-field model.

Method of proof: Existence of a fast extinction phase

Coupling with a mean-field model

In the mean-field model every vertex can have three states *healthy*, *ready*, or *infected*. The state *ready* means that the vertex is infected but ready to recover.

- Recovery and update times are taken from the original process. For every pair $\{i, j\}$ of vertices there is a Poisson process of infection times with rate $\lambda p_{i,j}$.
- If at an infection time of the pair $\{i, j\}$ one of the vertices is not *healthy*, both become *infected*.
- If at an update time the vertex is *infected*, it becomes *ready*.
- If at a recovery time a vertex is *ready*, it becomes *healthy*.

It is possible to couple the original process to the mean-field model in such a way that, at every time $t > 0$, every vertex which is *infected* in the original model, is either *ready* or *infected* in the mean-field model.

Hence the extinction time in the mean-field model is a stochastic upper bound to the original extinction time.

Method of proof: Existence of a fast extinction phase

Extinction time in the mean-field model

If $\gamma < \frac{1}{3}$ and λ is small enough, the process

$$M(t) := \sum_{i=1}^{N} \mathbf{1}\{i \textit{ ready} \text{ at time } t\}\, s_1(i) + \sum_{i=1}^{N} \mathbf{1}\{i \textit{ infected} \text{ at time } t\}\, s_2(i)$$

with

$$s_1(i) = \left(\frac{N}{i}\right)^{2\gamma} \qquad s_2(i) = s_1(i) + \left(\frac{N}{i}\right)^{\gamma},$$

satisfies

$$\frac{1}{dt}\mathbb{E}\big[M(t+dt) - M(t)\big|\mathscr{F}_t\big] \leq -2c\, N^{-\gamma} M(t).$$

Method of proof: Existence of a fast extinction phase

Extinction time in the mean-field model

If $\gamma < \frac{1}{3}$ and λ is small enough, the process

$$M(t) := \sum_{i=1}^{N} \mathbf{1}\{i \textit{ ready} \text{ at time } t\}\, s_1(i) + \sum_{i=1}^{N} \mathbf{1}\{i \textit{ infected} \text{ at time } t\}\, s_2(i)$$

with

$$s_1(i) = \left(\frac{N}{i}\right)^{2\gamma} \qquad\qquad s_2(i) = s_1(i) + \left(\frac{N}{i}\right)^{\gamma},$$

satisfies

$$\frac{1}{dt}\mathbb{E}\big[M(t+dt) - M(t)\big|\mathscr{F}_t\big] \leq -2c\, N^{-\gamma} M(t).$$

We introduce $Z(t) = \log(M(t)+1) + cN^{-\gamma}t$, and get

$$\frac{1}{dt}\mathbb{E}\big[Z(t+dt) - Z(t)\big|\mathscr{F}_t\big] \leq 0.$$

Method of proof: Existence of a fast extinction phase

Extinction time in the mean-field model

If $\gamma < \frac{1}{3}$ and λ is small enough, the process

$$M(t) := \sum_{i=1}^{N} \mathbf{1}\{i \textit{ ready} \text{ at time } t\}\, s_1(i) + \sum_{i=1}^{N} \mathbf{1}\{i \textit{ infected} \text{ at time } t\}\, s_2(i)$$

with

$$s_1(i) = \left(\frac{N}{i}\right)^{2\gamma} \qquad s_2(i) = s_1(i) + \left(\frac{N}{i}\right)^{\gamma},$$

satisfies

$$\frac{1}{dt}\mathbb{E}\big[M(t+dt) - M(t)\big|\mathscr{F}_t\big] \le -2c\, N^{-\gamma} M(t).$$

We introduce $Z(t) = \log(M(t)+1) + cN^{-\gamma}t$, and get

$$\frac{1}{dt}\mathbb{E}\big[Z(t+dt) - Z(t)\big|\mathscr{F}_t\big] \le 0.$$

Hence $(Z(t))_{0\le t<T_{\text{ext}}}$ is a positive supermartingale, and we deduce

$$\mathbb{E}\,T_{\text{ext}} = c^{-1}N^{\gamma}\mathbb{E}[Z(T_{\text{ext}})] \le c^{-1}N^{\gamma}\mathbb{E}Z(0) = \mathcal{O}(N^{\gamma}\log N).$$

Thank you very much for your attention!