The combinatorics colliding bullets

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The colliding bullet problem

One shoots n bullets at times $1, 2, \ldots, n$ in the same direction.

- So, ignoring interactions, the trajectories are identical
- If two bullets collide, they both annihilate and disappear
- The speeds are $V_1, V_2, \ldots, V_n$ iid uniform rv on $[0, 1]$.

Questions

- What is the probability p_n that all bullets pair-wise annihiliate ?
- What is the distribution of the number of surviving bullets ?

"Ponder this" blog at IBM's monthly puzzles:

- Compute p_{20} rounded to the 10th decimal

Model: $(V_i)_{i \geq 1}$ iid rv uniform on $[0, 1]$.
For $i \geq 1$ at time i, shoot a bullet with speed V_i
Two bullets annihilate if they collide
Question: Do we have $\mathbf{P}(\exists i : V_i \text{ survives}) > 0$?

Backyard experimental study

A recursive argument from the web

General idea: Find two bullets that collide

- Remove them from the problem
- Proceed with the remaining bullets

Consider the fastest bullet:

with proba $\frac{1}{n}$: it is shot first and it survives

with proba $1 - \frac{1}{n}$: it is not first and some two bullets must collide
remove them and induct

So, the number X_n of surviving bullets satisfies the recurrence relation:

with proba $\frac{1}{n}$: add one to X_{n-1}

with proba $1 - \frac{1}{n}$: return X_{n-2}

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Hold on... not so fast:

- Need to verify the induction hypotheses...
- and they are clearly not satisfied!

A combinatorial model and a simple distribution

Bullets with arbitrary speeds and delays:

$(v_1, v_2, \ldots, v_n)$ vector of speeds

$(\delta_1, \delta_2, \ldots, \delta_{n-1})$ vector of time delays

(σ, τ) uniform on $\mathfrak{S}_n \times \mathfrak{S}_{n-1}$

For $1 \leq i \leq n$ set $V_i = v_{\sigma(i)}$ and $T_i = \sum_{i=2}^{n} \delta_{\tau(i-1)}$

At time T_i shoot a bullet with speed V_i

A distribution: $\mathbf{q}_n$ such that $X_n \sim \mathbf{q}_n$ for each $n \geq 0$, then

$X_0 = 0$ and

$X_n \overset{d}{=} B_n(1 + X_{n-1}) + (1 - B_n)X_{n-2}$

where $(B_i)_{i \geq 1}$ are independent and $B_n \sim \text{Ber}(1/n)$

Can be verified by exhaustive search for small values of n...

More related problems

Sorted bullet flock $(V_i)_{i\geq 1}$ iid rv uniform on $[0,1]$ Set $L_0 = \varnothing$

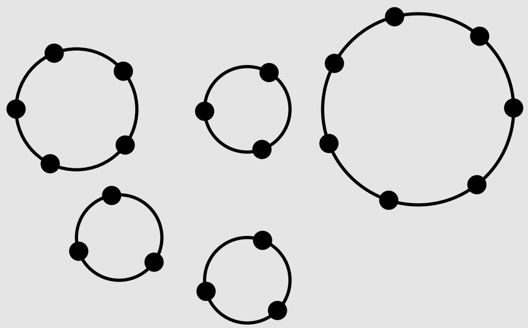
$$L_i = \begin{cases} L_{i-1} \cup \{V_i\} & \text{if } V_i \leq \min(L_{i-1}) \\ L_{i-1} \setminus \{\min L_{i-1}\} & \text{if } V_i > \min(L_{i-1}) \end{cases}$$

Odd cycles in permutations:

σ a uniformly random permutation on $\mathfrak{S}_n$

See σ as a directed graph on $[n]$

Y_n the number of cycles of odd length



Note : these are only two of many different models

Results

Theorem: for all previous models, the distribution of the parameter of interest is $\mathbf{q}_n$.

Remarks: i) The distribution is independent of $(\mathbf{V}, \mathbf{\Delta})$ provided the model is well-defined

ii) If the delays are not permuted, the result is in general false

iii) The distribution of the set of surviving bullets does depend on $\mathbf{V}$

Corollary: $\mathbf{q}_{2n}(0) = \prod_{i=1}^{n} \left(1 - \frac{1}{2i}\right)$

$\mathbf{q}_{2n}(0) \sim cn^{-1/2}$ as $n \to \infty$

Proposition. if $X_n \sim \mathbf{q}_n$ then

$$\frac{X_n - \frac{1}{2}\log n}{\sqrt{\frac{1}{2}\log n}} \xrightarrow[n\to\infty]{d} \mathcal{N}(0,1)$$

Back to the bullet problem

Recall: the combinatorial version:

- Speeds and delays are arbitrary
- Both are uniformly permuted, independently

Speed vector $\mathbf{V} = (V_1, V_2, \ldots, V_n)$

Delay vector $\boldsymbol{\Delta} = (\Delta_1, \Delta_2, \ldots, \Delta_{n-1})$

The pair $(\mathbf{V}, \boldsymbol{\Delta})$ is called the *parameter*

A given pair (σ, τ) of permutations is a *configuration*

$V_i^\sigma = V_{\sigma(i)}$

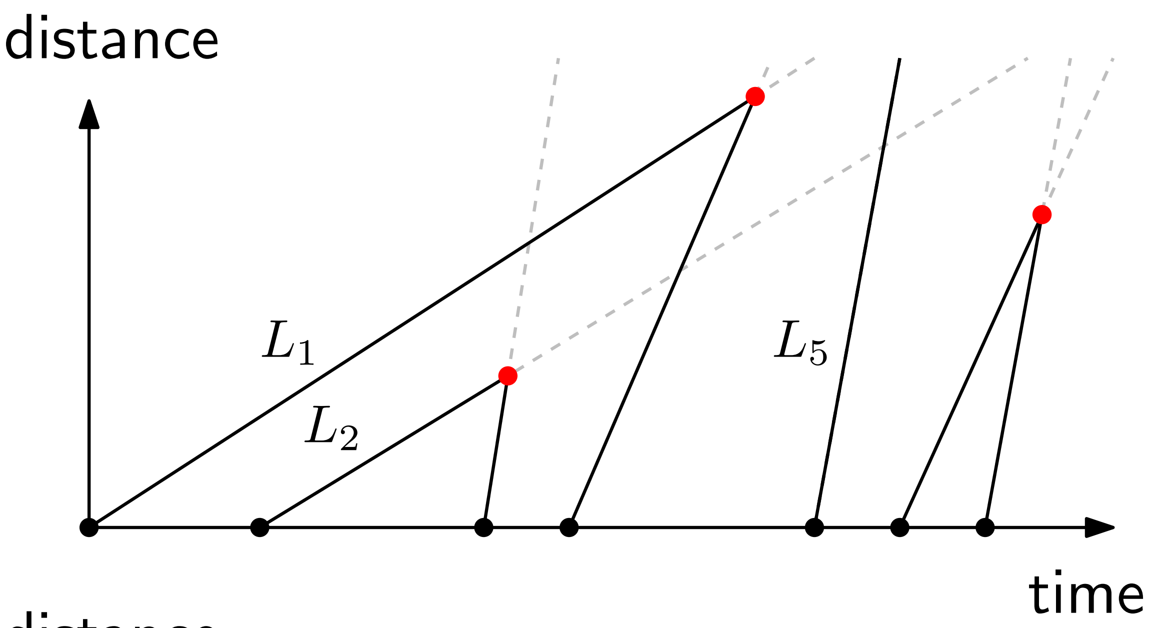
$T_1^\tau = 0$ and $T_i^\tau = \sum_{j=1}^{i-1} \Delta_{\tau(j)}$

A geometric representation

Space-time diagram:

actual trajectories

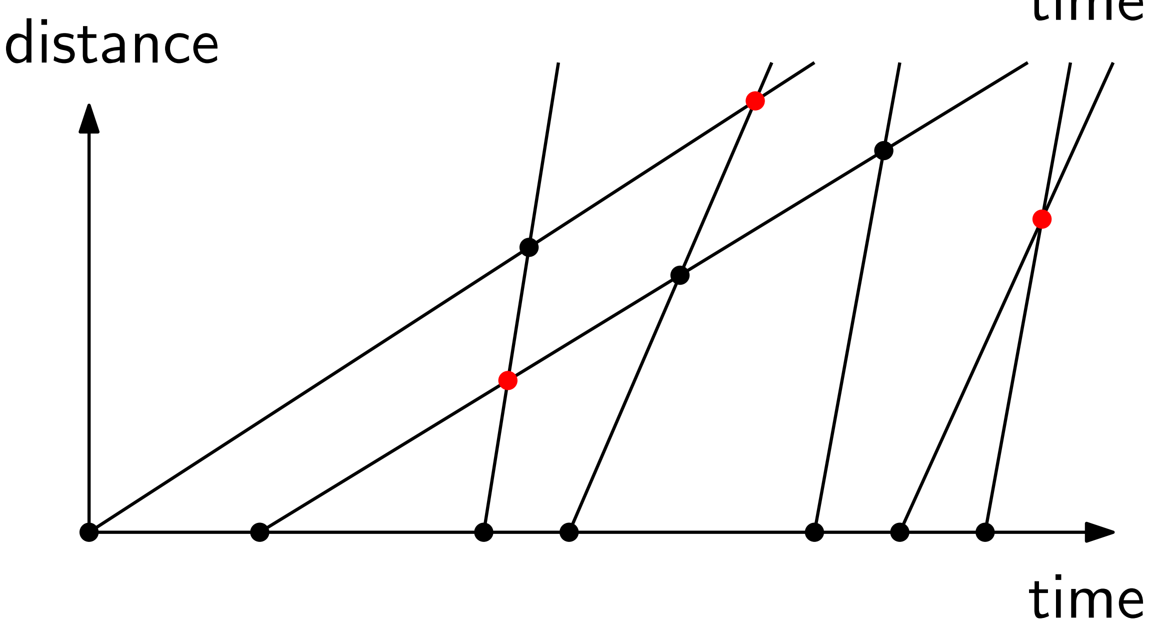
$$L_i = L_i(\mathbf{V}, \boldsymbol{\Delta}, \sigma, \tau)$$



Virtual space-time diagram:

extended trajectories

$$\overline{L_i} = \overline{L_i}(\mathbf{V}, \boldsymbol{\Delta}, \sigma, \tau)$$



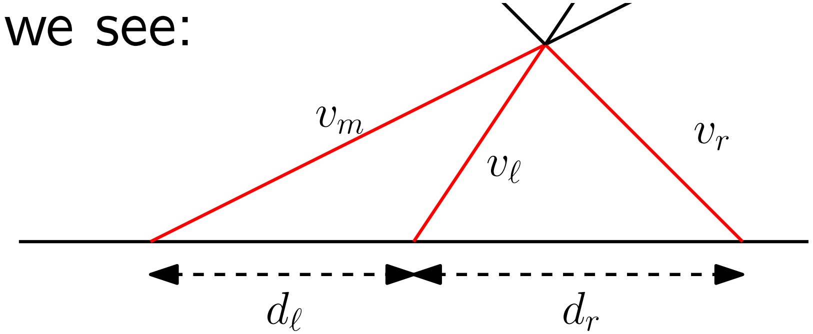
Note: depends on the parameter $(\mathbf{V}, \boldsymbol{\Delta})$ and the configuration (σ, τ)

Is it even well-defined ?

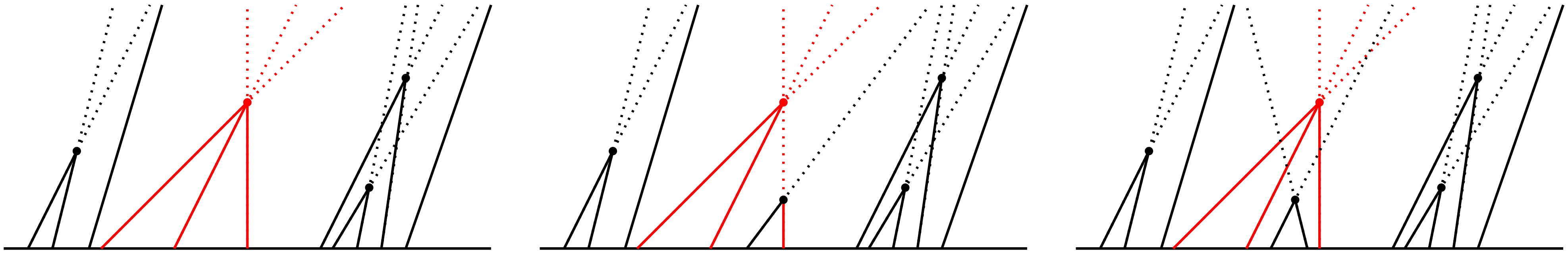
Generic parameter: $(\mathbf{V}, \mathbf{\Delta})$ such that there does not exist a configuration (σ, τ) such that 3 extended trajectories meet at the same point

Note: If $(\mathbf{V}, \mathbf{\Delta})$ is generic, the distribution $\mathbf{P}_{\mathbf{V},\mathbf{\Delta}}$ is well-defined

Singular parameter There exists a pattern $\pi = (v_m, v_\ell, v_r, d_\ell, d_r)$ such that, for some configuration, we see:



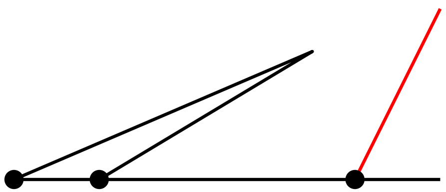
A few possibilities when looking at the actual trajectories



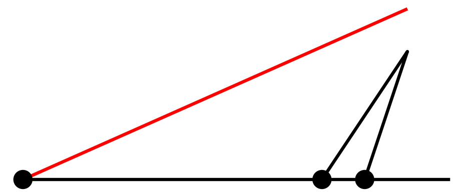
Picturing the difficulties in the recurrence

1) Difficult to identify a canonical pair of colliding bullets

the fastest may not catch up
with the previous one

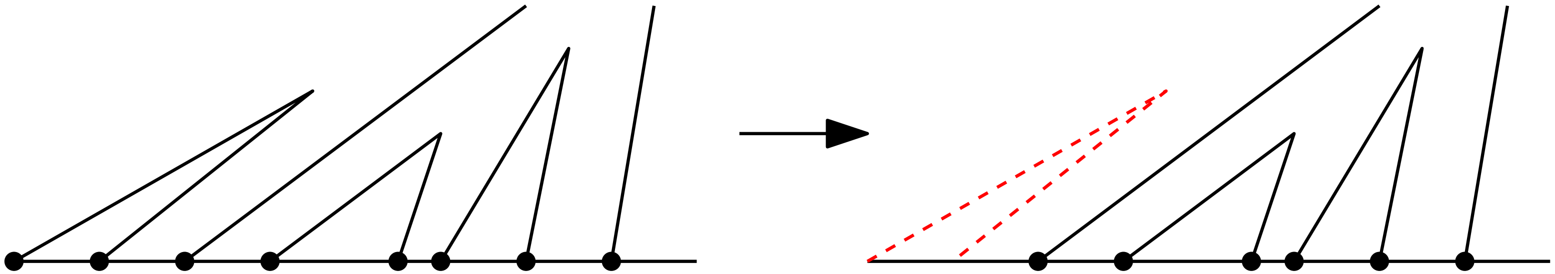


the slowest may be be caught
by the next one



⇒ **difficult to verify the inductive assumptions**

2) Removing bullets a priori conditions the configurations



⇒ **in general the assumptions may actually just be false...**

A bit of hope...

One case in which the recurrence kind of works: when $\min \mathbf{V} = 0$

- either the bullet is shot last and survives
- or it does collide with the next one

  remove these two

  except for the hole created, the rest is unconditioned

But: **1)** This is only one step of the recurrence

2) The delays are modified

Strategy: prove that

- changing the delays does not alter the distribution
- changing the speeds does not alter the distribution
- then, can inductively put the min speed to 0 and "fill up" holes

Topological colliding scheme

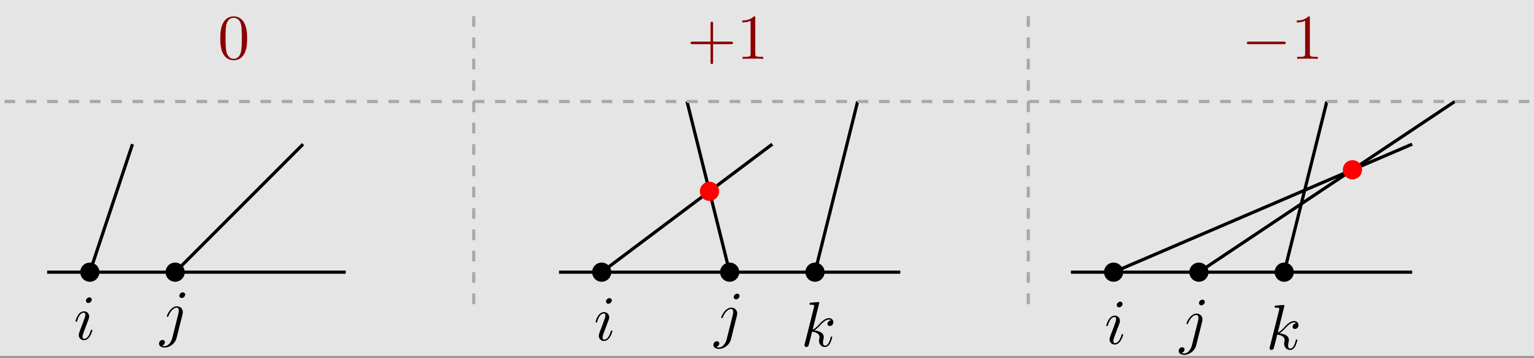
Idea: have a convenient *function* that encodes everything

For each $(\mathbf{V}, \boldsymbol{\Delta})$ the TCS is a function $\Gamma_{\mathbf{V},\boldsymbol{\Delta}}(\sigma, \tau, i, j, k)$ of 5 var.

- where (σ, τ) is a configuration

  $1 \leq i < j$ and $k \neq i, j$

  $\Gamma_{\mathbf{V},\boldsymbol{\Delta}}(\sigma, \tau, i, j, k) \in \{-1, 0, +1\}$
- that is given by the rule: $\Gamma_{\mathbf{V},\boldsymbol{\Delta}}(\sigma, \tau, i, j, k)$ is



Observations:

- $\Gamma_{\mathbf{V},\boldsymbol{\Delta}}$ determines $\mathbf{P}_{\mathbf{V},\boldsymbol{\Delta}}$
- $\Gamma_{\mathbf{V},\boldsymbol{\Delta}}$ is locally constant if $(\mathbf{V}, \boldsymbol{\Delta})$ is generic

Lowering the minimum speed

Partition the parameter space $\mathcal{D} = \mathbb{R}_+^n \times \mathbb{R}_+^{n-1}$ into "cells"

a cell is a maximal open set of $\mathcal{D}$ where $(\mathbf{V}, \mathbf{\Delta}) \mapsto \Gamma_{\mathbf{V},\mathbf{\Delta}}$ is constant

What happens when we lower the minimum speed:

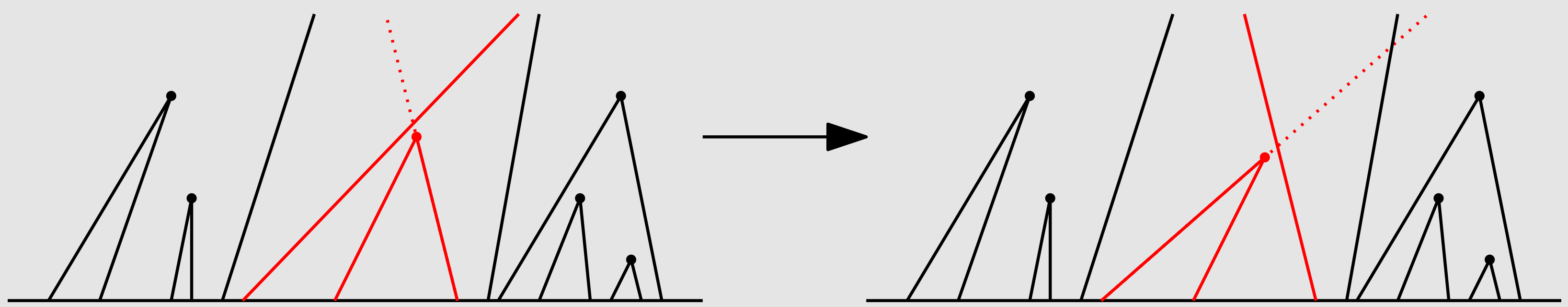
As long as $(\mathbf{V}, \mathbf{\Delta})$ stays in the same "cell"

the distribution $\mathbf{P}_{\mathbf{V},\mathbf{\Delta}}$ is unchanged (since $\Gamma_{\mathbf{V},\mathbf{\Delta}}$ is unchanged!)

May only change when changing cell (and $\Gamma_{\mathbf{V},\mathbf{\Delta}}$ does change!)

One changes cell precisely at a singular parameter

Controlling what happens when "crossing a singular point"

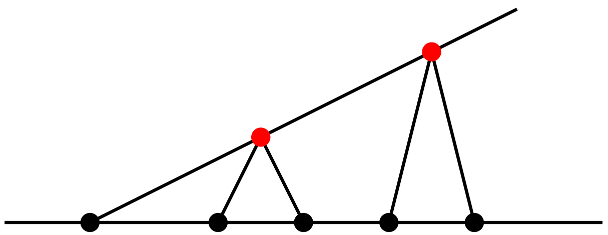


Simplifications / Reductions

1) Can restrict to crossings of “simple” critical points:

only 3 speeds are involved in a critical pattern

no configuration looks like

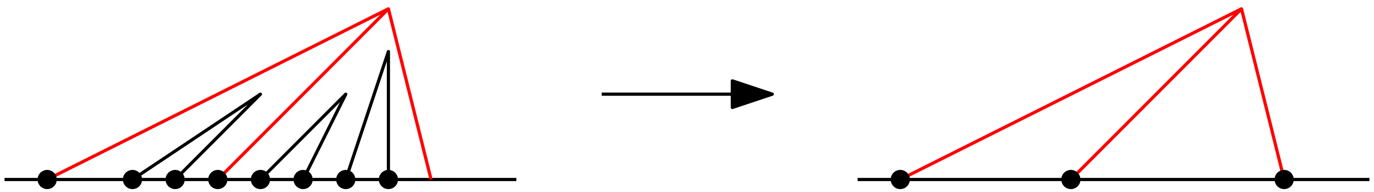


2) When accounting for the changes in number of surviving bullets:

i. only need to deal with configurations that are critical

ii. only need to deal with configurations that contain a ”realized” critical pattern (made of actual trajectories!)

iii. only need to deal with minimal critical patterns (induction!)

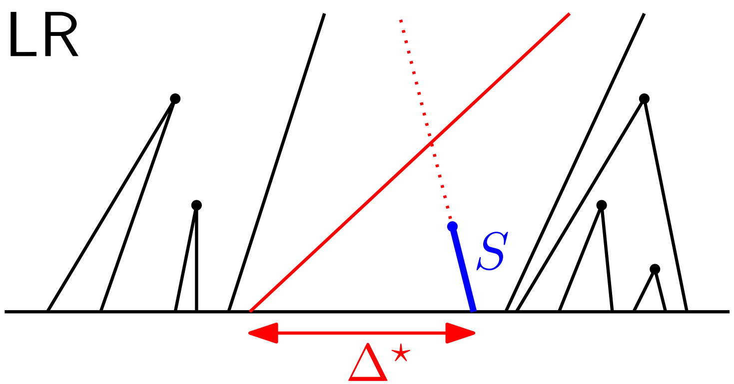


Counting configurations with constraints

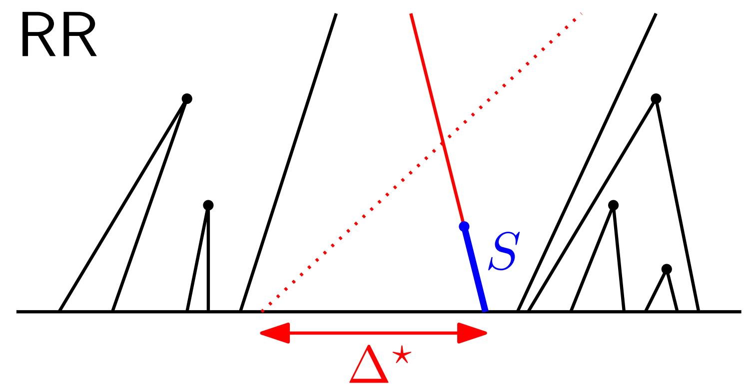
There are: two distinguished speeds: $V_{\min}$ and V_r

one distinguished delay $\Delta^\star$

a "special" segment S and a constraint set $A \subset \mathbb{N}$



$V_{\min}$ is shot V_r is not shot



$V_{\min}$ is not shot V_r is shot

LR$(\mathbf{V}, \mathbf{\Delta}, |S|, A, k)$ and RR$(\mathbf{V}, \mathbf{\Delta}, |S|, A, k)$ are the sets of configurations such that the number of bullets touching the special segment S lies in the set A there are eventually k survivors

Remark: in practice, only need $A = \{0\}$, $A = \mathbb{Z}_+$, and $A = \mathbb{Z}_+ \setminus \{0\}$

Rolling up sleeves

Prove by induction that $\mathcal{P}_n = \mathcal{P}_n^{(1)} \cap \mathcal{P}_n^{(2)} \cap \mathcal{P}_n^{(3)}$ holds

$$\mathcal{P}_n^{(1)} : \quad \left\{ \text{ for any generic parameter } (\mathbf{V}_n, \boldsymbol{\Delta}_{n-1}) \in \mathcal{G}_n \text{ we have } \mathbf{P}_{\mathbf{V}_n, \boldsymbol{\Delta}_{n-1}} = \mathbf{q}_n \right\}$$

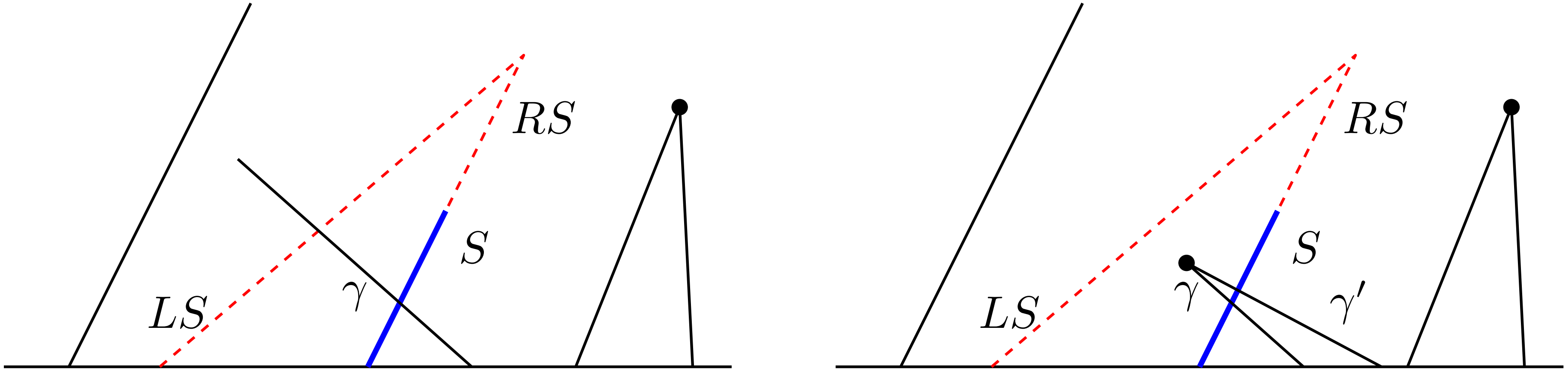
$$\mathcal{P}_n^{(2)} : \quad \left\{ \begin{array}{c} \text{for the set } A \text{ being either } \{0\}, \mathbb{Z}_+, \text{ or } \mathbb{Z}_+ \setminus \{0\} \text{ we have} \\ \text{for all } (\mathbf{V}_n, \boldsymbol{\Delta}_{n-1}) \in \mathcal{G}_n, \text{for all } s, \text{ and for all } k \geq 0 \\ |\mathsf{LR}(\mathbf{V}_n, \boldsymbol{\Delta}_{n-1}, s, A, k)| = |\mathsf{RR}(\mathbf{V}_n, \boldsymbol{\Delta}_{n-1}, s, A, k)| \end{array} \right\}$$

$$\mathcal{P}_n^{(3)} : \quad \left\{ \begin{array}{c} \text{there exists a map } g_n : \mathbb{Z}_+ \to \mathbb{Z}_+ \text{ such that} \\ \text{for any } (\mathbf{V}_n, \boldsymbol{\Delta}_{n-1}) \in \mathcal{G}_n \text{ and for all } k \geq 0 \text{ we have} \\ |\mathsf{LR}(\mathbf{V}_n, \boldsymbol{\Delta}_{n-1}, 0, \{0\}, k)| = |\mathsf{RR}(\mathbf{V}_n, \boldsymbol{\Delta}_{n-1}, 0, \{0\}, k)| = g_n(k) \end{array} \right\}.$$

Note: did not manage to "untangle" the three parts

What kind of cases do we have to treat

An example: If $A = \mathbb{Z}_+ \setminus \{0\}$



Notation:

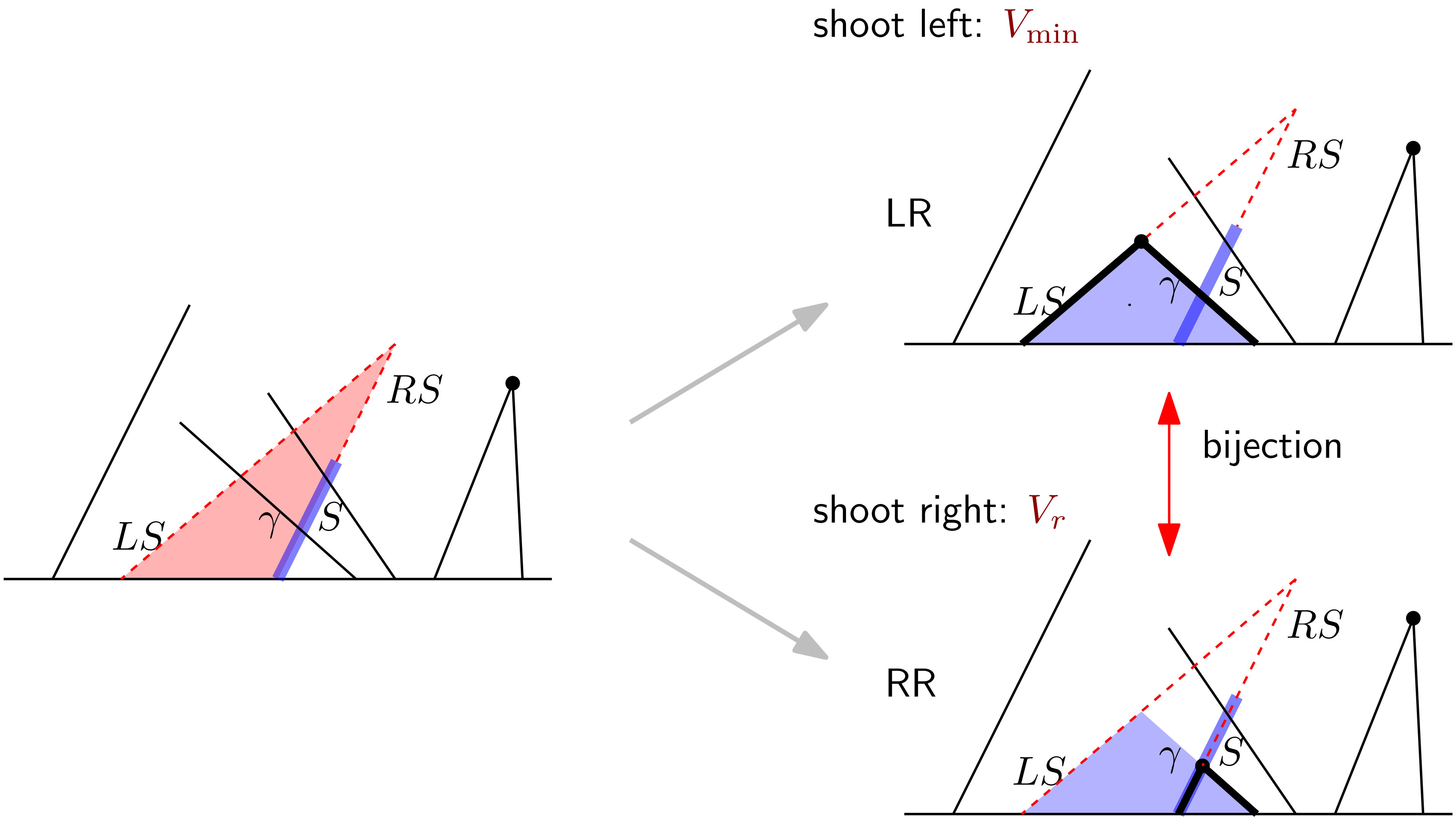
LS the left side of the triangle, with speed $V_{\min}$

RS the right side of the triangle, with speed V_r

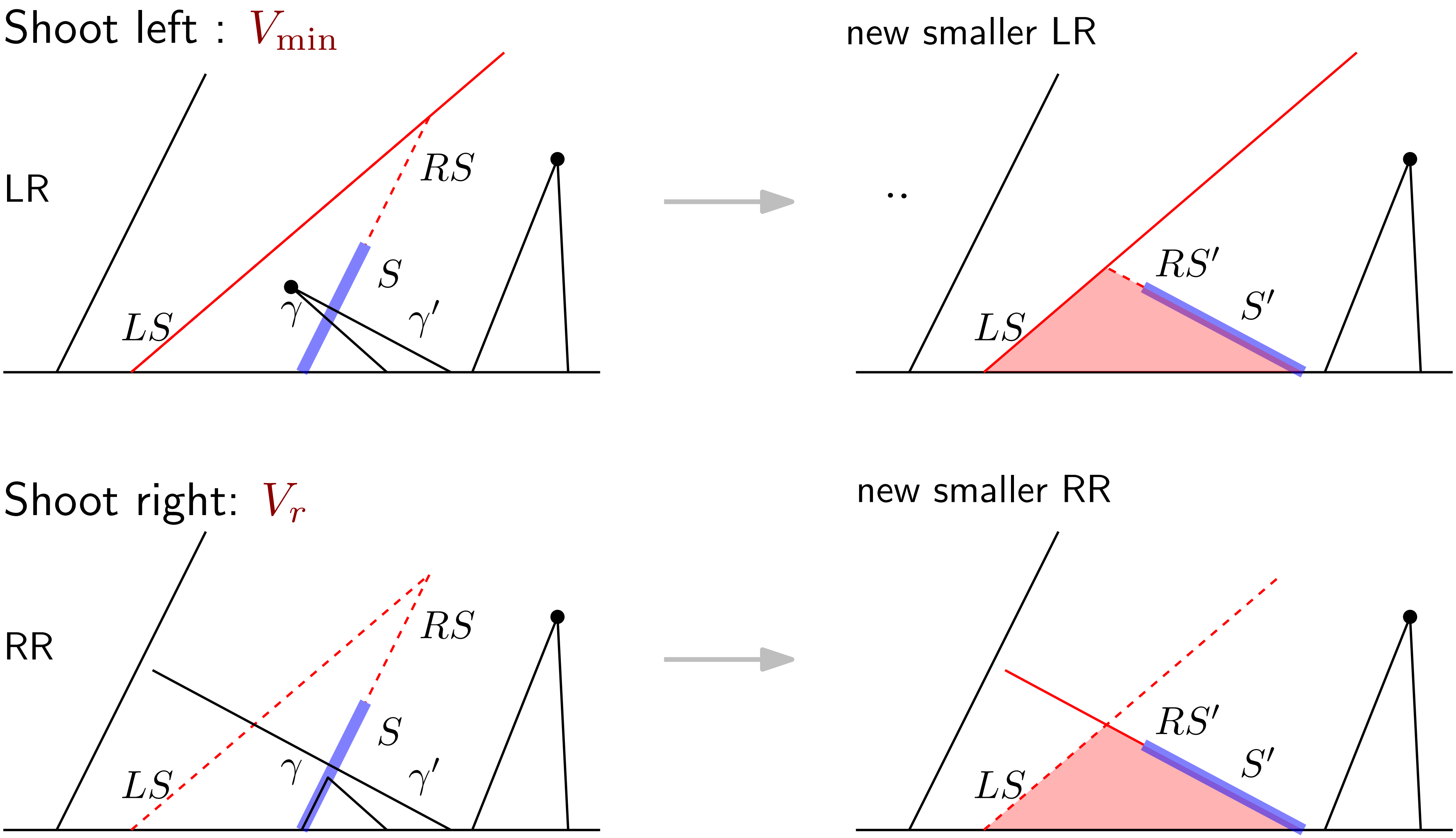
γ the lowest trajectory hitting S

γ' the trajectory hitting γ inside the triangle, if any

γ is not hit before reaching LS



γ is hit before reaching LS



THANK YOU!

The easy problems - I

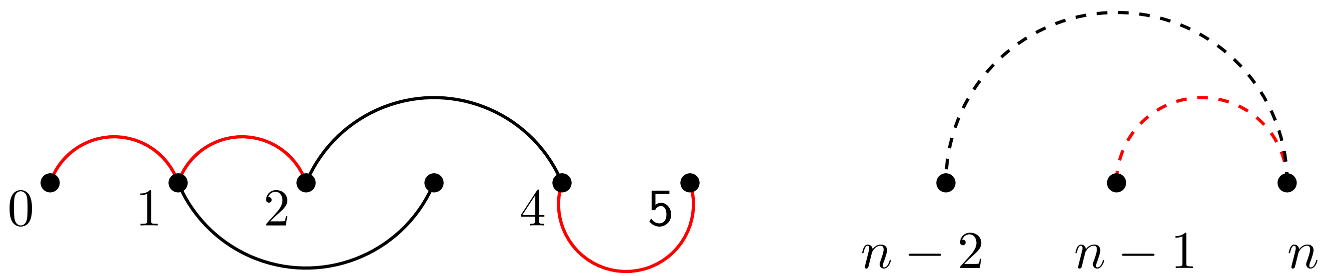
One can "easily" justify that the recurrence relation indeed holds:

- **Two-step tree on $\mathbb{Z}_+$:** Trivial

  For every $i \geq 1$ there is one single directed path between i and 0

  First step from n is to ancestor $a \in \{n-1, n-2\}$ wp $\frac{1}{n}$ or $1 - \frac{1}{n}$

  The red distance $d(a, 0)$ is independent with the correct distribution



- **Sorted bullet flock:** Only depends on the ranks of the speeds

  Consider slowest bullet, say it is shot at time i

  Does not influence bullets shot at times $1, 2, \ldots, i-1$

  Is hit by the bullet shot at time $i+1$, unless $i = n$

  Thus, does not influence bullets shot at times $i+2, \ldots, n$ if any

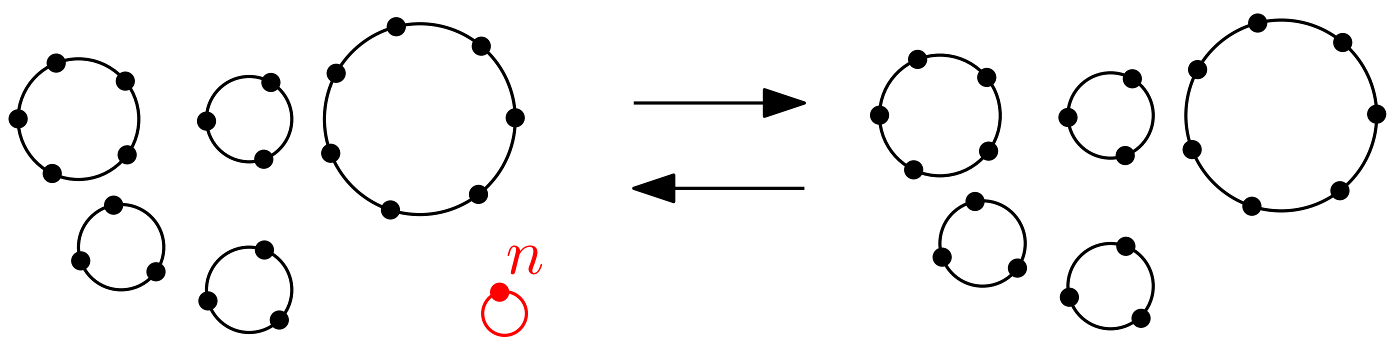
The easy problems - II

- **Odd cycles in permutations** Decompose permutations of $[n]$

Consider n, and remove it (reconnecting if necessary)

i) either it is in its own cycle $n \to n$: *remove it*

one single way to put it back in (by adding the cycle n)



ii) or it is not alone in its cycle: *remove it together with its image*

exactly $n-1$ ways to put it back in (by splitting an arc)