

# Computer Algebra for Lattice Path Combinatorics

Alin Bostan

*Inria*  
informatics mathematics

**AofA**

**CIRM, Luminy, France**

**June 27, 2019**

**Enumerative Combinatorics:** science of counting

Area of mathematics primarily concerned with counting discrete objects.

▷ Main outcome: theorems

**Computer Algebra:** effective mathematics

Area of computer science primarily concerned with the algorithmic manipulation of algebraic objects.

▷ Main outcome: algorithms

**Computer Algebra** for **Enumerative Combinatorics**

Today: **Algorithms** for proving **Theorems** on **Lattice Paths Combinatorics**.

## An (innocent looking) combinatorial question

Let  $\mathcal{S} = \{\uparrow, \leftarrow, \searrow\}$ . An  $\mathcal{S}$ -walk is a path in  $\mathbb{Z}^2$  using only steps from  $\mathcal{S}$ . Show that, for any integer  $n$ , the following quantities are equal:

(i) number  $a_n$  of  $n$ -steps  $\mathcal{S}$ -walks confined to the upper half plane  $\mathbb{Z} \times \mathbb{N}$  that start and finish at the origin  $(0,0)$  (*excursions*);

(ii) number  $b_n$  of  $n$ -steps  $\mathcal{S}$ -walks confined to the quarter plane  $\mathbb{N}^2$  that start at the origin  $(0,0)$  and finish on the diagonal of  $\mathbb{N}^2$  (*diagonal walks*).

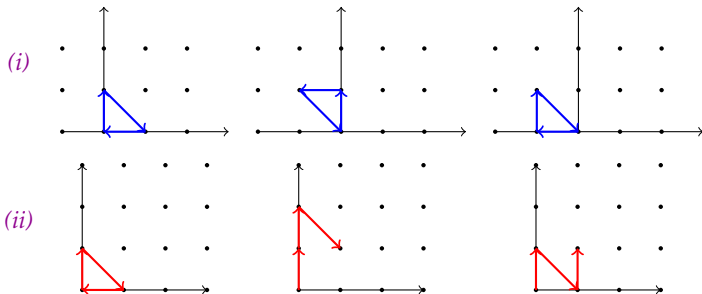
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For instance, for  $n = 3$ , this common value is  $a_3 = b_3 = 3$ :



Teaser 1: This “exercise” is non-trivial

Teaser 2: ...but it can be solved using **Computer Algebra**

Teaser 3: ...by two robust and efficient algorithmic techniques,  
**Guess-and-Prove** and **Creative Telescoping**

## Why care about counting walks?

Many objects from the *real world* can be encoded by walks:

- probability theory (*voting*, games of chance, branching processes, ...)
- discrete mathematics (permutations, trees, words, urns, ...)
- statistical physics (Ising model, ...)
- operations research (queueing theory, ...)

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7<sup>TH</sup> INTERNATIONAL CONFERENCE ON  
LATTICE PATH COMBINATORICS AND APPLICATIONS



Siena, Italy July 4-7, 2010

HOME	<b>TOPICS to be covered include</b> (but are not limited to) :	
Photo	Lattice path enumeration	Random walks
Program	Plane Partitions	Non parametric statistical inference
Proceedings	Young tableaux	Discrete distributions and urn models
Submission	q-calculus	Queueing theory
Important dates	Orthogonal polynomials	Analysis of algorithms
Participants		Graph Theory and Applications
General information		Self-dual codes and unimodular lattices
		Bijections between paths and other combinatoric structures

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21 > 25

**Lattice Path Combinatorics and Interactions**  
**Marches aléatoires, combinatoire et interactions**  
**CONFERENCE**

Cyril Banderier (CNRS, Université Paris 13)

Jehanne Dousse (CNRS, Université Lyon 1)

Enrica Duchi (Université Paris Diderot)

Christian Krattenthaler (TU Wien)

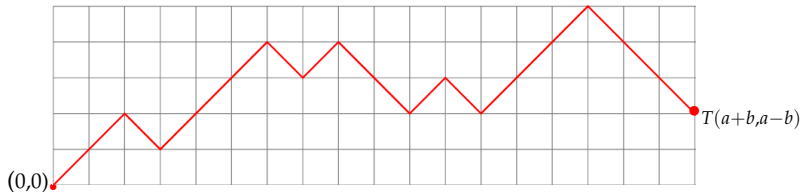
Michael Wallner (Université de Bordeaux)

<https://conferences.cirm-math.fr/2021-calendar.html>



CALCUL DES PROBABILITÉS. — *Solution d'un problème;*  
par M. J. BERTRAND.

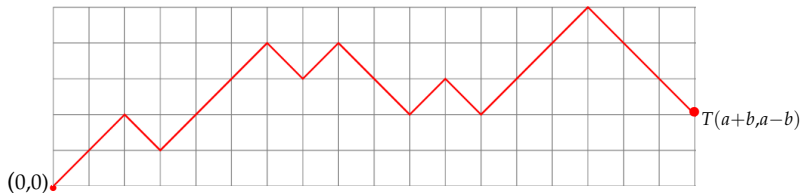
« On suppose que deux candidats A et B soient soumis à un scrutin de ballottage. Le nombre des votants est  $\mu$ . A obtient  $m$  suffrages et est élu, B en obtient  $\mu - m$ . On demande la probabilité pour que, pendant le dépouillement du scrutin, le nombre des voix de A ne cesse pas une seule fois de surpasser celles de son concurrent.



## Computation of probabilities – Solution of a problem by J. Bertrand

Suppose that candidates  $A$  and  $B$  are running in an election. If  $a$  votes are cast for  $A$  and  $b$  votes are cast for  $B$ , where  $a > b$ , then the probability that  $A$  stays ahead of  $B$  throughout the counting of the ballots is  $(a - b)/(a + b)$ .

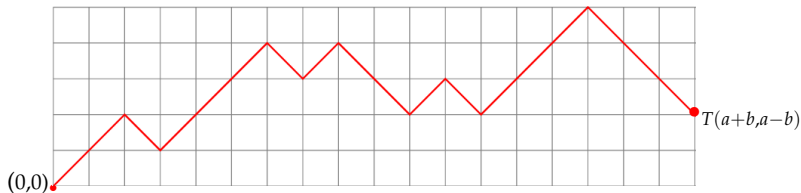
**Lattice path reformulation:** find the number of paths with  $a$  upsteps  $\nearrow$  and  $b$  downsteps  $\searrow$  that start at the origin and never touch the  $x$ -axis back again



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**Lattice path reformulation:** find the number of paths with  $a - 1$  upsteps ↗ and  $b$  downsteps ↘ that start at  $(1, 1)$  and never touch the  $x$ -axis



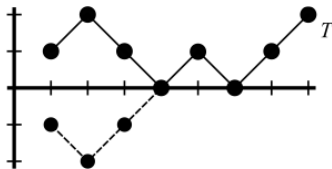
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**Reflection principle [Aebly, 1923]:** paths in  $\mathbb{N}^2$  from  $(1, 1)$  to  $T(a + b, a - b)$  that do touch the  $x$ -axis are in bijection with paths in  $\mathbb{Z}^2$  from  $(1, -1)$  to  $T$



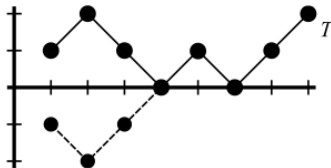
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$$\text{Answer: } \underbrace{\binom{a+b-1}{a-1}}_{\text{paths in } \mathbb{Z}^2 \text{ from } (1, 1) \text{ to } T} - \underbrace{\binom{a+b-1}{b-1}}_{\text{paths in } \mathbb{Z}^2 \text{ from } (1, -1) \text{ to } T}$$

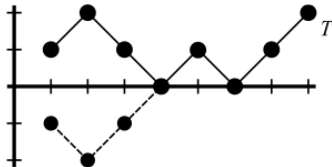
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**Answer:**  $\underbrace{(\text{paths in } \mathbb{Z}^2 \text{ from } (1, 1) \text{ to } T) - (\text{paths in } \mathbb{Z}^2 \text{ from } (1, -1) \text{ to } T)}$

$$\binom{a + b - 1}{a - 1} - \binom{a + b - 1}{b - 1} = \frac{a - b}{a + b} \binom{a + b}{a}$$

Lot of recent activity; many recent contributors:

Arquès, Bacher, Banderier, Bernardi, Bostan, Bousquet-Mélou, Budd, Chyzak, Cori, Courtiel, Denisov, Dreyfus, Du, Duchon, Dulucq, Duraj, Fayolle, Fisher, Flajolet, Fusy, Garbit, Gessel, Gouyou-Beauchamps, Guttmann, Guy, Hardouin, van Hoeij, Hou, Iasnogorodski, Johnson, Kauers, Kenyon, Koutschan, Krattenthaler, Kreweras, Kurkova, Malyshev, Melczer, Miller, Mishna, Niederhausen, Pech, Petkovšek, Prellberg, Raschel, Rechnitzer, Roques, Sagan, Salvy, Sheffield, Singer, Viennot, Wachtel, Wang, Wilf, D. Wilson, M. Wilson, Yatchak, Yeats, Zeilberger, ...

etc.

...but it is still a very hot topic

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~~Specific question  
Ad hoc solution~~

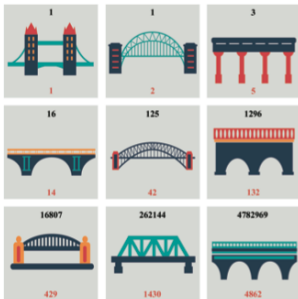


**Systematic approach**



DISCRETE MATHEMATICS AND ITS APPLICATIONS

# HANDBOOK OF ENUMERATIVE COMBINATORICS



Edited by  
**Miklós Bóna**

 **CRC Press**  
Taylor & Francis Group  
A CHAPMAN & HALL BOOK

## Chapter 10

### *Lattice Path Enumeration*

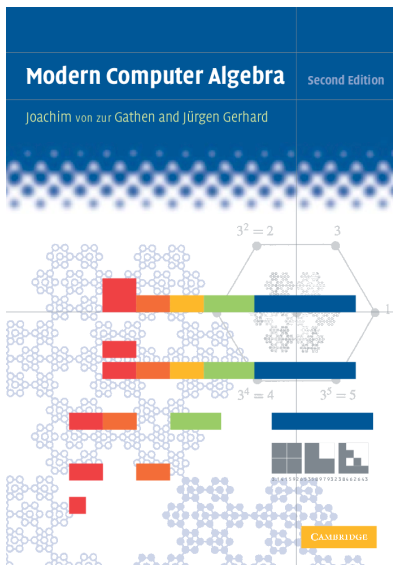
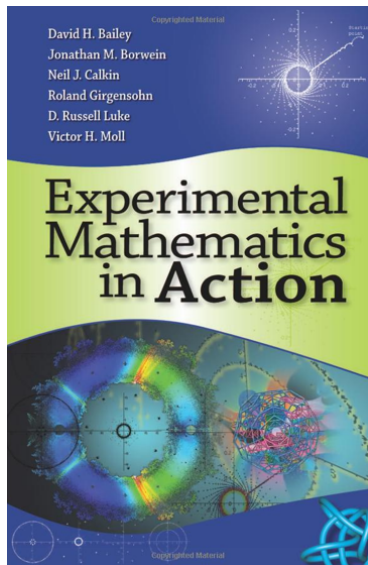
Christian Krattenthaler

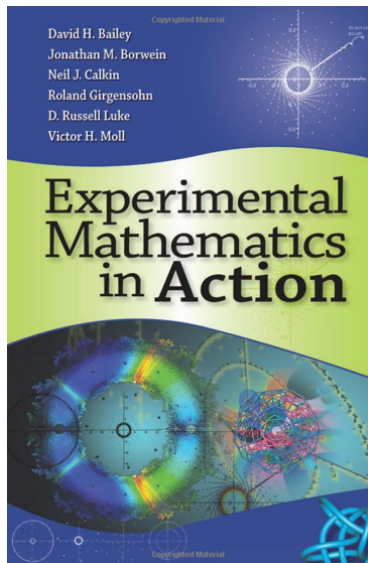
Universität Wien

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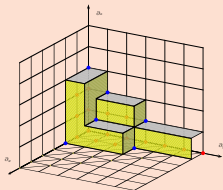
# Our approach: Experimental Mathematics using Computer Algebra





## Algorithmes Efficaces en Calcul Formel

Alin BOSTAN  
Frédéric CHYZAK  
Marc GIUSTI  
Romain LEBRETON  
Grégoire LECERF  
Bruno SALVY  
Éric SCHOST



▷ Nearest-neighbor walks in the quarter plane:

$\mathcal{S}$ -walks in  $\mathbb{N}^2$ : starting at  $(0,0)$  and using steps in a *fixed* subset  $\mathcal{S}$  of

$$\{\swarrow, \leftarrow, \nearrow, \uparrow, \nearrow, \rightarrow, \searrow, \downarrow\}$$

▷ Counting sequence  $q_{\mathcal{S}}(n)$ : number of  $\mathcal{S}$ -walks of length  $n$

▷ Generating function:

$$Q_{\mathcal{S}}(t) = \sum_{n=0}^{\infty} q_{\mathcal{S}}(n)t^n \in \mathbb{Z}[[t]]$$

▷ Nearest-neighbor walks in the quarter plane:

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▷ Counting sequence  $q_{\mathcal{S}}(i, j; n)$ : number of walks of length  $n$  ending at  $(i, j)$

▷ Complete generating function (with “catalytic” variables  $x, y$ ):

$$Q_{\mathcal{S}}(x, y; t) = \sum_{i, j, n=0}^{\infty} q_{\mathcal{S}}(i, j; n) x^i y^j t^n \in \mathbb{Z}[[x, y, t]]$$

Entire books dedicated to small step walks in the quarter plane!

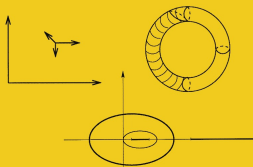
**Applications of Mathematics**  
Stochastic Modelling and Applied Probability

40

Guy Fayolle  
Roudolf Iasnogorodski  
Vadim Malyshev

## Random Walks in the Quarter-Plane

Algebraic Methods,  
Boundary Value Problems  
and Applications



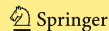
Probability Theory and Stochastic Modelling 40

Guy Fayolle  
Roudolf Iasnogorodski  
Vadim Malyshev

# Random Walks in the Quarter Plane

Algebraic Methods, Boundary Value  
Problems, Applications to Queueing  
Systems and Analytic Combinatorics

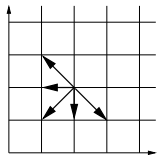
*Second Edition*



Among the  $2^8$  step sets  $\mathcal{S} \subseteq \{-1, 0, 1\}^2 \setminus \{(0, 0)\}$ , some are:

## Small-step models of interest

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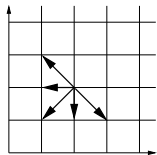


trivial,

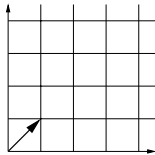


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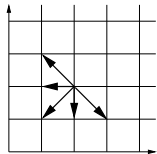
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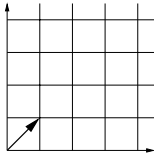
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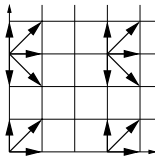
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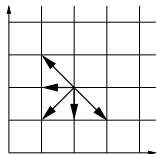
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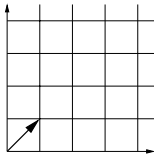
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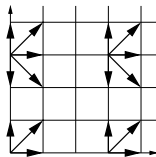
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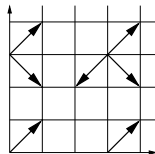
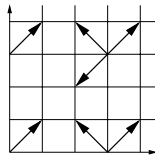
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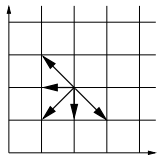
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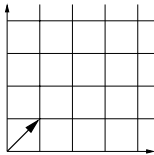
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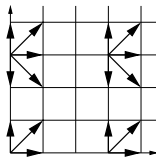
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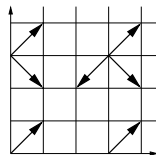
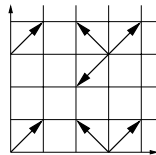
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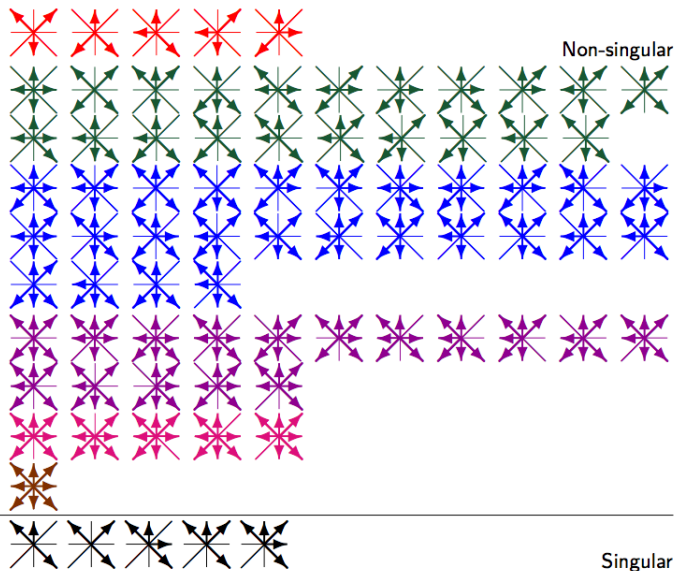
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One is left with [79 interesting distinct models](#).

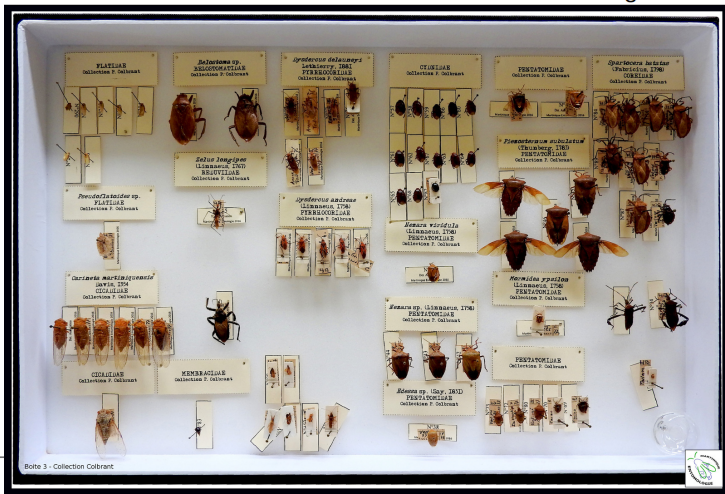
# The 79 small steps models of interest



# Task: classify their generating functions!



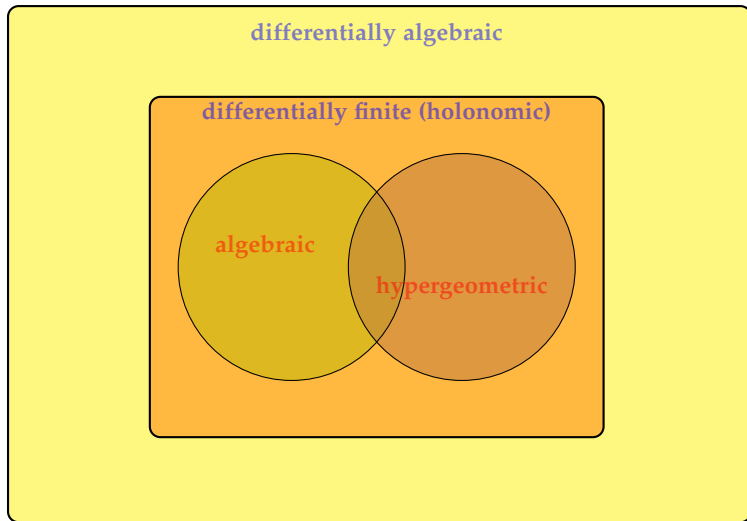
Non-singular

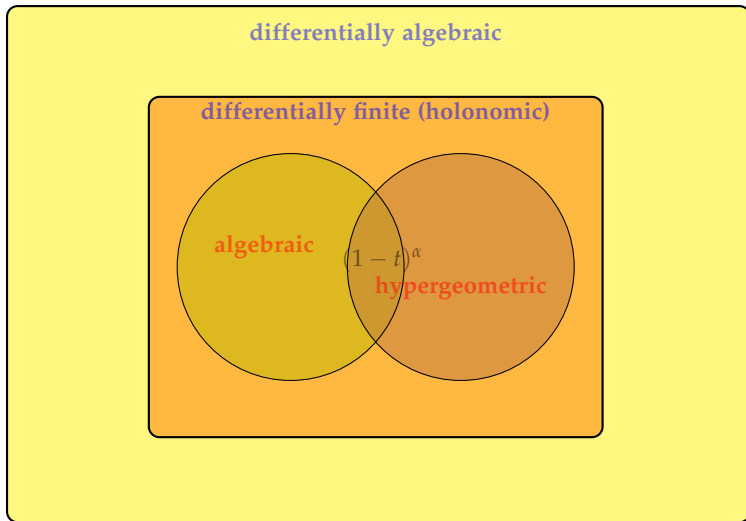


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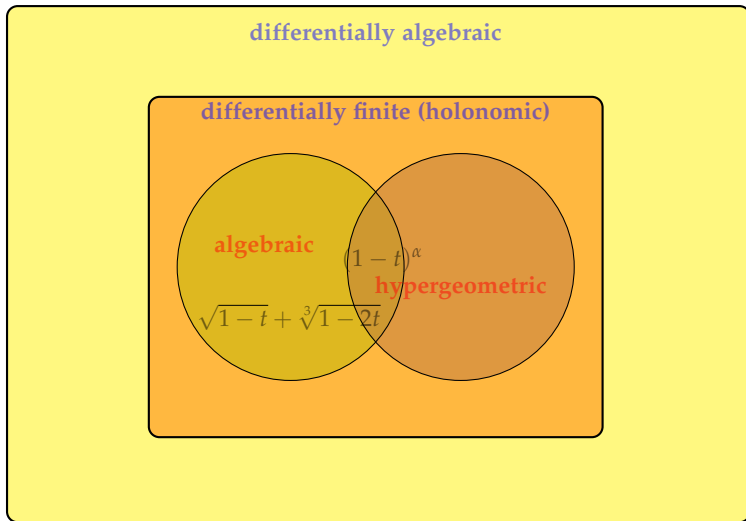


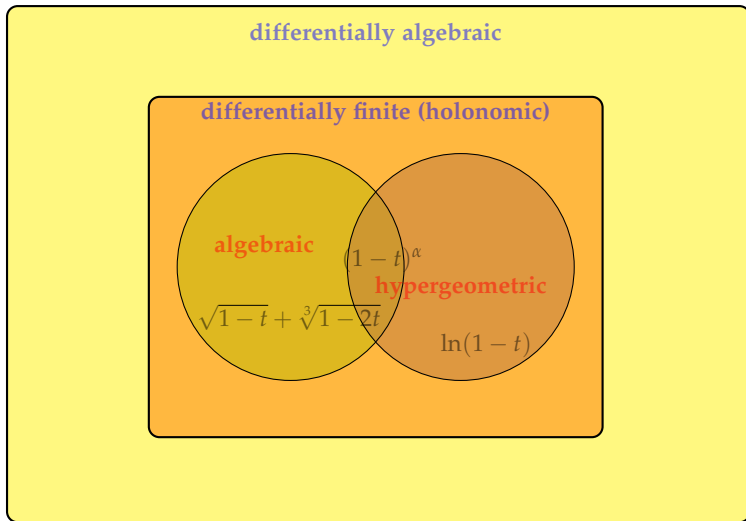
Singular

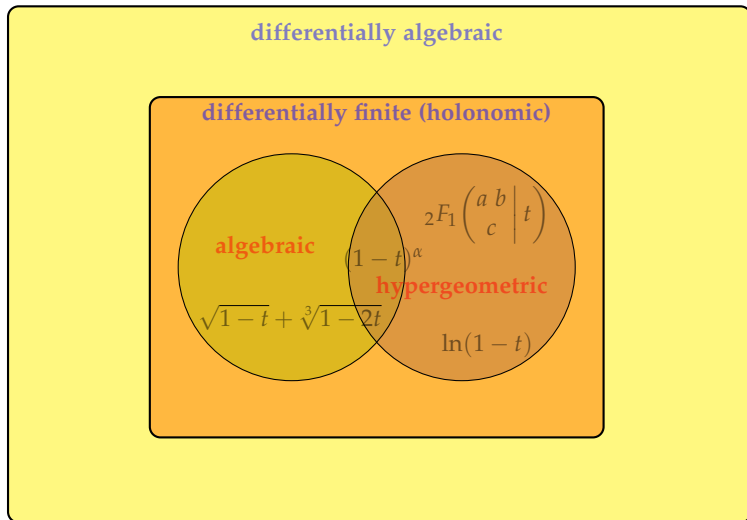






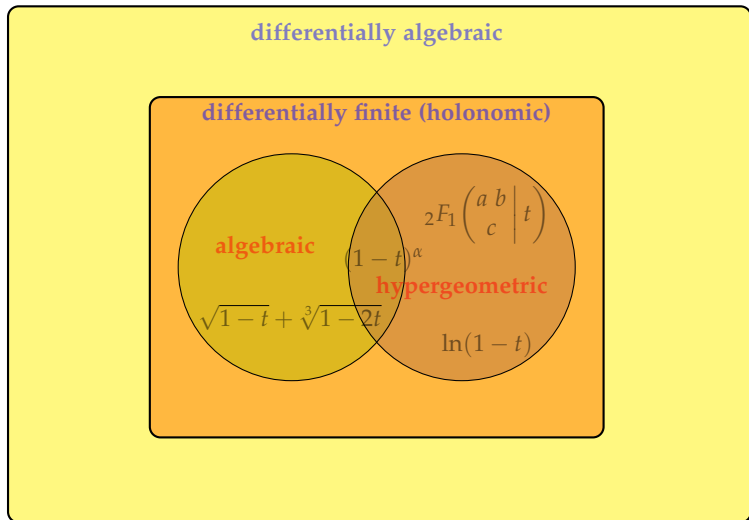






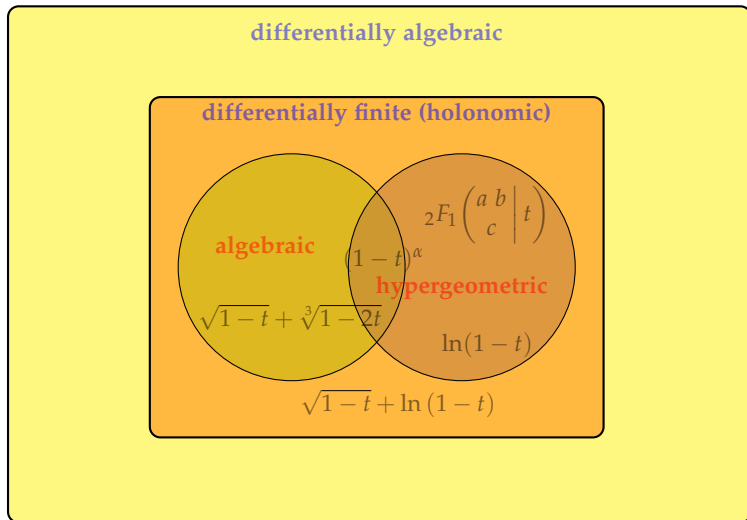
$${}_2F_1\left(\begin{matrix} a & b \\ c \end{matrix} \middle| t\right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{t^n}{n!}, \quad \text{where } (a)_n = a(a+1) \cdots (a+n-1).$$

# Classification criterion: properties of generating functions



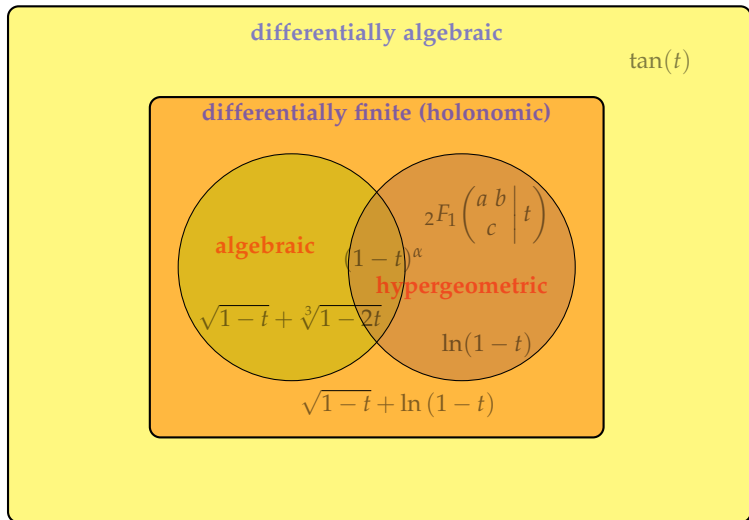
E.g.,  $(1-t)^\alpha = {}_2F_1\left(\begin{matrix} -\alpha & 1 \\ 1 \end{matrix} \middle| t\right)$ ,  $\ln(1-t) = -t \cdot {}_2F_1\left(\begin{matrix} 1 & 1 \\ 2 \end{matrix} \middle| t\right) = -\sum_{n=1}^{\infty} \frac{t^n}{n}$

# Classification criterion: properties of generating functions



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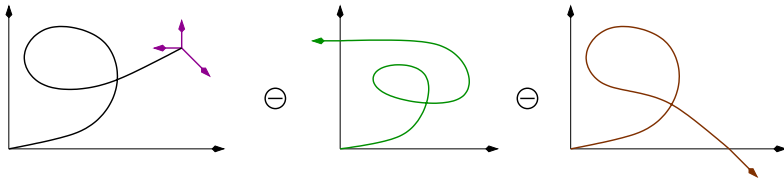
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# Algebraic reformulation of main task: solving a functional equation

Generating function:  $Q(x, y) \equiv Q(x, y; t) = \sum_{i, j, n=0}^{\infty} q(i, j; n) x^i y^j t^n \in \mathbb{Z}[[x, y, t]]$

Recursive construction yields the *kernel equation*

$$Q(x, y) = 1 + t \left( y + \frac{1}{x} + x \frac{1}{y} \right) Q(x, y) - t \frac{1}{x} Q(0, y) - tx \frac{1}{y} Q(x, 0)$$

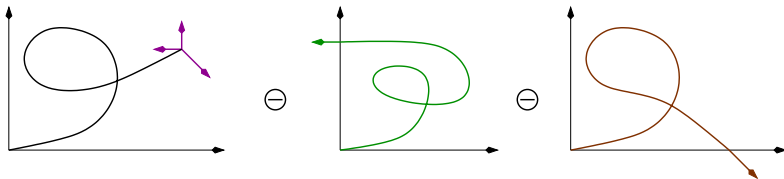


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New task: Solve this functional equation!

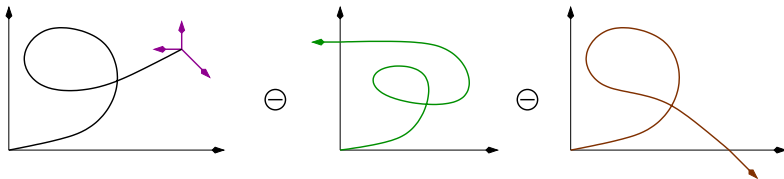


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**New task:** For the other models – solve 78 similar equations!

# “Special” models of walks in the quarter plane


Dyck: 


Motzkin: 


Pólya: 

Kreweras: 

Gessel: 

Gouyou-Beauchamps: 

King walks: 

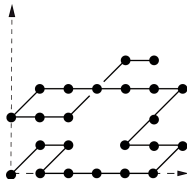
Tandem walks: 



- $g(n)$  = number of  $n$ -steps  $\{\nearrow, \swarrow, \leftarrow, \rightarrow\}$ -walks in  $\mathbb{N}^2$   
1, 2, 7, 21, 78, 260, 988, 3458, 13300, 47880, ...

**Question:** What is the nature of the generating function

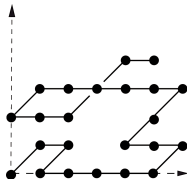
$$G(t) = \sum_{n=0}^{\infty} g(n) t^n ?$$



- $g(i, j; n)$  = number of  $n$ -steps  $\{\nearrow, \swarrow, \leftarrow, \rightarrow\}$ -walks in  $\mathbb{N}^2$  from  $(0, 0)$  to  $(i, j)$

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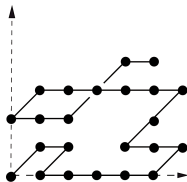
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**Theorem** [B., Kauers, 2010]

$G(x, y; t)$  is an algebraic function<sup>†</sup>.

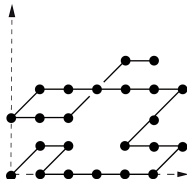
▷ computer-driven discovery/proof via *algorithmic Guess-and-Prove*

<sup>†</sup> Minimal polynomial  $P(G(x, y; t); x, y, t) = 0$  has  $> 10^{11}$  terms;  $\approx 30$  Gb (6 DVDs!)

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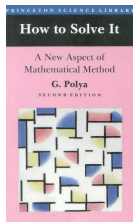
$$G(t) = \sum_{n=0}^{\infty} g(n) t^n ?$$



**Corollary** [B., Kauers, 2010] (former conjecture of Gessel's)

$$(3n + 1) g(2n) = (12n + 2) g(2n - 1) \text{ and } (n + 1) g(2n + 1) = (4n + 2) g(2n)$$

▷ computer-driven discovery/proof via *algorithmic Guess-and-Prove*



## *Guessing and Proving*

George Pólya

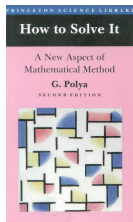


What is “scientific method”? Philosophers and non-philosophers have discussed this question and have not yet finished discussing it. Yet as a first introduction it can be described in three syllables:

**Guess and test.**

Mathematicians too follow this advice in their research although they sometimes refuse to confess it. They have, however, something which the other scientists cannot really have. For mathematicians the advice is

**First guess, then prove.**



## *Guessing and Proving*

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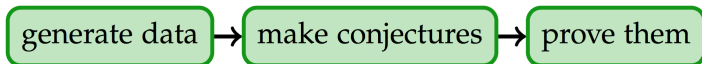


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## Guess-and-Prove: a toy example

**Question:** Find  $B_{i,j} :=$  the number of  $\{\rightarrow, \uparrow\}$ -walks in  $\mathbb{N}^2$  from  $(0,0)$  to  $(i,j)$

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- ① There are 2 ways to get to  $(i,j)$ , either from  $(i-1,j)$ , or from  $(i,j-1)$ :

$$B_{i,j} = B_{i-1,j} + B_{i,j-1}$$

- ② There is only one way to get to a point on an axis:  $B_{i,0} = B_{0,j} = 1$

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⋮

(I) Generate data:

1	7	28	84	210	462	924	
1	6	21	56	126	252	462	
1	5	15	35	70	126	210	
1	4	10	20	35	56	84	
1	3	6	10	15	21	28	
1	2	3	4	5	6	7	
1	1	1	1	1	1	1	...

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(II) Guess:

$$\begin{aligned} &\longrightarrow \dots \\ &\longrightarrow \frac{(i+1)(i+2)}{2} \\ &\longrightarrow i+1 \\ &\longrightarrow 1 \end{aligned}$$

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$$B_{i,j} \stackrel{?}{=} \frac{(i+j)!}{i!j!}$$

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(III) Prove: If

$C_{i,j} \stackrel{\text{def}}{=} \frac{(i+j)!}{i!j!}$ , then

$$\frac{C_{i-1,j}}{C_{i,j}} + \frac{C_{i,j-1}}{C_{i,j}} = \frac{i}{i+j} + \frac{j}{i+j} = 1$$

and  $C_{i,0} = C_{0,j} = 1$ .

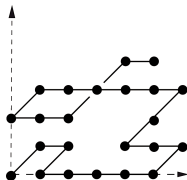
Thus  $B_{i,j} = C_{i,j}$

# Guess-and-Prove for Gessel walks

- $g(i, j; n)$  = number of  $n$ -steps  $\{\nearrow, \swarrow, \leftarrow, \rightarrow\}$ -walks in  $\mathbb{N}^2$  from  $(0, 0)$  to  $(i, j)$

**Question:** What is the nature of the generating function

$$G(x, y; t) = \sum_{i, j, n=0}^{\infty} g(i, j; n) x^i y^j t^n ?$$



**Answer:** [B., Kauers, 2010]  $G(x, y; t)$  is an algebraic function<sup>†</sup>.

**Approach:**

- ① **Generate data:** compute  $G$  to precision  $t^{1200}$  ( $\approx 1.5$  billion coeffs!)
- ② **Guess:** conjecture polynomial equations for  $G(x, 0; t)$  and  $G(0, y; t)$  (degree 24 each, coeffs. of degree (46, 56), with 80-bits digits coeffs.)
- ③ **Prove:** multivariate resultants of (very big) polynomials (30 pages each)

<sup>†</sup> Minimal polynomial  $P(G(x, y; t); x, y, t) = 0$  has  $> 10^{11}$  terms;  $\approx 30$  Gb (6 DVDs!)

## A typical Guess-and-Prove algorithmic proof

**Theorem** ["Gessel excursions are algebraic"]

$$g(t) := G(0,0; \sqrt{t}) = \sum_{n=0}^{\infty} \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n} (16t)^n \text{ is algebraic.}$$



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- 1 Find  $P$  such that  $P(t, g(t)) = 0 \pmod{t^{100}}$  by **(structured) linear algebra**.

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













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```
> P:=gfun:-listtoalgeq([seq(pochhammer(5/6,n)*pochhammer(1/2,n)/
  pochhammer(5/3,n)/pochhammer(2,n)*16^n, n=0..100)], g(t)):
> gfun:-diffeqtoec(gfun:-algeqtodiffeq(P[1], g(t)), g(t), r(n));
```

# Algorithmic classification of models with D-Finite $Q_{\mathcal{S}}(t) := Q_{\mathcal{S}}(1, 1; t)$

	OEIS	$\mathcal{S}$	Pol size	LDE size	Rec size		OEIS	$\mathcal{S}$	Pol size	LDE size	Rec size
1	A005566		—	(3, 4)	(2, 2)	13	A151275		—	(5, 24)	(9, 18)
2	A018224		—	(3, 5)	(2, 3)	14	A151314		—	(5, 24)	(9, 18)
3	A151312		—	(3, 8)	(4, 5)	15	A151255		—	(4, 16)	(6, 8)
4	A151331		—	(3, 6)	(3, 4)	16	A151287		—	(5, 19)	(7, 11)
5	A151266		—	(5, 16)	(7, 10)	17	A001006		(2, 2)	(2, 3)	(2, 1)
6	A151307		—	(5, 20)	(8, 15)	18	A129400		(2, 2)	(2, 3)	(2, 1)
7	A151291		—	(5, 15)	(6, 10)	19	A005558		—	(3, 5)	(2, 3)
8	A151326		—	(5, 18)	(7, 14)						
9	A151302		—	(5, 24)	(9, 18)	20	A151265		(6, 8)	(4, 9)	(6, 4)
10	A151329		—	(5, 24)	(9, 18)	21	A151278		(6, 8)	(4, 12)	(7, 4)
11	A151261		—	(4, 15)	(5, 8)	22	A151323		(4, 4)	(2, 3)	(2, 1)
12	A151297		—	(5, 18)	(7, 11)	23	A060900		(8, 9)	(3, 5)	(2, 3)

Equation sizes = (order, degree)

▷ Computerized discovery: enumeration + guessing [B., Kauers, 2009]
























# Algorithmic classification of models with D-Finite $Q_{\mathcal{S}}(t) := Q_{\mathcal{S}}(1, 1; t)$

	OEIS	$\mathcal{S}$	Pol size	LDE size	Rec size		OEIS	$\mathcal{S}$	Pol size	LDE size	Rec size
1	A005566		—	(3, 4)	(2, 2)	13	A151275		—	(5, 24)	(9, 18)
2	A018224		—	(3, 5)	(2, 3)	14	A151314		—	(5, 24)	(9, 18)
3	A151312		—	(3, 8)	(4, 5)	15	A151255		—	(4, 16)	(6, 8)
4	A151331		—	(3, 6)	(3, 4)	16	A151287		—	(5, 19)	(7, 11)
5	A151266		—	(5, 16)	(7, 10)	17	A001006		(2, 2)	(2, 3)	(2, 1)
6	A151307		—	(5, 20)	(8, 15)	18	A129400		(2, 2)	(2, 3)	(2, 1)
7	A151291		—	(5, 15)	(6, 10)	19	A005558		—	(3, 5)	(2, 3)
8	A151326		—	(5, 18)	(7, 14)						
9	A151302		—	(5, 24)	(9, 18)	20	A151265		(6, 8)	(4, 9)	(6, 4)
10	A151329		—	(5, 24)	(9, 18)	21	A151278		(6, 8)	(4, 12)	(7, 4)
11	A151261		—	(4, 15)	(5, 8)	22	A151323		(4, 4)	(2, 3)	(2, 1)
12	A151297		—	(5, 18)	(7, 11)	23	A060900		(8, 9)	(3, 5)	(2, 3)

Equation sizes = (order, degree)

- ▷ Computerized discovery: enumeration + guessing [B., Kauers, 2009]
- ▷ 1–22: DF confirmed by human proofs in [Bousquet-Mélou, Mishna, 2010]
- ▷ 23: DF confirmed by a human proof in [B., Kurkova, Raschel, 2017]
- ▷ All: explicit eqs. proved via CA [B., Chyzak, van Hoeij, Kauers, Pech, 2017]
























# Algorithmic classification of models with D-Finite $Q_{\mathcal{S}}(t) := Q_{\mathcal{S}}(1, 1; t)$

	OEIS	$\mathcal{S}$	algebraic?	asymptotics		OEIS	$\mathcal{S}$	algebraic?	asymptotics
1	A005566		N	$\frac{4}{\pi} \frac{4^n}{n}$	13	A151275		N	$\frac{12\sqrt{30}}{\pi} \frac{(2\sqrt{6})^n}{n^2}$
2	A018224		N	$\frac{2}{\pi} \frac{4^n}{n}$	14	A151314		N	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi} \frac{(2C)^n}{n^2}$
3	A151312		N	$\frac{\sqrt{6}}{\pi} \frac{6^n}{n}$	15	A151255		N	$\frac{24\sqrt{2}}{\pi} \frac{(2\sqrt{2})^n}{n^2}$
4	A151331		N	$\frac{8}{3\pi} \frac{8^n}{n}$	16	A151287		N	$\frac{2\sqrt{2}A^{7/2}}{\pi} \frac{(2A)^n}{n^2}$
5	A151266		N	$\frac{1}{2} \sqrt{\frac{3}{\pi}} \frac{3^n}{n^{1/2}}$	17	A001006		Y	$\frac{3}{2} \sqrt{\frac{3}{\pi}} \frac{3^n}{n^{3/2}}$
6	A151307		N	$\frac{1}{2} \sqrt{\frac{5}{2\pi}} \frac{5^n}{n^{1/2}}$	18	A129400		Y	$\frac{3}{2} \sqrt{\frac{3}{\pi}} \frac{6^n}{n^{3/2}}$
7	A151291		N	$\frac{4}{3\sqrt{\pi}} \frac{4^n}{n^{1/2}}$	19	A005558		N	$\frac{8}{\pi} \frac{4^n}{n^2}$
8	A151326		N	$\frac{2}{\sqrt{3\pi}} \frac{6^n}{n^{1/2}}$	$A = 1 + \sqrt{2}, B = 1 + \sqrt{3}, C = 1 + \sqrt{6}, \lambda = 7 + 3\sqrt{6}, \mu = \sqrt{\frac{4\sqrt{6}-1}{19}}$				
9	A151302		N	$\frac{1}{3} \sqrt{\frac{5}{2\pi}} \frac{5^n}{n^{1/2}}$	20	A151265		Y	$\frac{2\sqrt{2}}{\Gamma(1/4)} \frac{3^n}{n^{3/4}}$
10	A151329		N	$\frac{1}{3} \sqrt{\frac{7}{3\pi}} \frac{7^n}{n^{1/2}}$	21	A151278		Y	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(1/4)} \frac{3^n}{n^{3/4}}$
11	A151261		N	$\frac{12\sqrt{3}}{\pi} \frac{(2\sqrt{3})^n}{n^2}$	22	A151323		Y	$\frac{\sqrt{2}3^{3/4}}{\Gamma(1/4)} \frac{6^n}{n^{3/4}}$
12	A151297		N	$\frac{\sqrt{3}B^{7/2}}{2\pi} \frac{(2B)^n}{n^2}$	23	A060900		Y	$\frac{4\sqrt{3}}{3\Gamma(1/3)} \frac{4^n}{n^{2/3}}$

► Computerized discovery: convergence acceleration + LLL [B., Kauers, '09]



# Algorithmic classification of models with D-Finite $Q_{\mathcal{S}}(t) := Q_{\mathcal{S}}(1, 1; t)$

	OEIS	$\mathcal{S}$	algebraic?	asymptotics		OEIS	$\mathcal{S}$	algebraic?	asymptotics
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2	A018224		N	$\frac{2}{\pi} \frac{4^n}{n}$	14	A151314		N	$\frac{\sqrt{6}\lambda\mu C^{5/2}}{5\pi} \frac{(2C)^n}{n^2}$
3	A151312		N	$\frac{\sqrt{6}}{\pi} \frac{6^n}{n}$	15	A151255		N	$\frac{24\sqrt{2}}{\pi} \frac{(2\sqrt{2})^n}{n^2}$
4	A151331		N	$\frac{8}{3\pi} \frac{8^n}{n}$	16	A151287		N	$\frac{2\sqrt{2}A^{7/2}}{\pi} \frac{(2A)^n}{n^2}$
5	A151266		N	$\frac{1}{2} \sqrt{\frac{3}{\pi}} \frac{3^n}{n^{1/2}}$	17	A001006		Y	$\frac{3}{2} \sqrt{\frac{3}{\pi}} \frac{3^n}{n^{3/2}}$
6	A151307		N	$\frac{1}{2} \sqrt{\frac{5}{2\pi}} \frac{5^n}{n^{1/2}}$	18	A129400		Y	$\frac{3}{2} \sqrt{\frac{3}{\pi}} \frac{6^n}{n^{3/2}}$
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8	A151326		N	$\frac{2}{\sqrt{3\pi}} \frac{6^n}{n^{1/2}}$	$A = 1 + \sqrt{2}, B = 1 + \sqrt{3}, C = 1 + \sqrt{6}, \lambda = 7 + 3\sqrt{6}, \mu = \sqrt{\frac{4\sqrt{6}-1}{19}}$				
9	A151302		N	$\frac{1}{3} \sqrt{\frac{5}{2\pi}} \frac{5^n}{n^{1/2}}$	20	A151265		Y	$\frac{2\sqrt{2}}{\Gamma(1/4)} \frac{3^n}{n^{3/4}}$
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12	A151297		N	$\frac{\sqrt{3}B^{7/2}}{2\pi} \frac{(2B)^n}{n^2}$	23	A060900		Y	$\frac{4\sqrt{3}}{3\Gamma(1/3)} \frac{4^n}{n^{2/3}}$

- ▷ Computerized discovery: convergence acceleration + LLL [B., Kauers, '09]
- ▷ Asympt. confirmed by human proofs via ACSV in [Melczer, Wilson, 2016]
- ▷ Transcendence proofs via CA [B., Chyzak, van Hoeij, Kauers, Pech, 2017]

**Theorem** [B., Chyzak, van Hoeij, Kauers, Pech, 2017]

Let  $\mathcal{S}$  be one of the models 1–19. Then

- $Q_{\mathcal{S}}(t)$  is expressible using iterated integrals of  ${}_2F_1$  expressions.
- $Q_{\mathcal{S}}(t)$  is transcendental, except for  $\mathcal{S} = \begin{array}{c} \uparrow \\ \swarrow \searrow \\ \cdot \end{array}$  and  $\mathcal{S} = \begin{array}{c} \uparrow \downarrow \\ \swarrow \searrow \\ \cdot \end{array}$ .

**Theorem** [B., Chyzak, van Hoeij, Kauers, Pech, 2017]

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**Example** (King walks in the quarter plane, A151331)

$$Q_{\begin{smallmatrix} \swarrow \uparrow \\ \searrow \downarrow \end{smallmatrix}}(t) = \frac{1}{t} \int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\frac{3}{2} \mid \frac{3}{2} \mid \frac{16x(1+x)}{(1+4x)^2}\right) dx$$

$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \dots$$

**Theorem** [B., Chyzak, van Hoeij, Kauers, Pech, 2017]

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- $Q_{\mathcal{S}}(t)$  is expressible using iterated integrals of  ${}_2F_1$  expressions.
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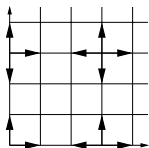
**Example** (King walks in the quarter plane, A151331)

$$Q_{\begin{smallmatrix} \uparrow \downarrow \\ \downarrow \uparrow \end{smallmatrix}}(t) = \frac{1}{t} \int_0^t \frac{1}{(1+4x)^3} \cdot {}_2F_1\left(\frac{3}{2} \mid \frac{3}{2} \mid \frac{16x(1+x)}{(1+4x)^2}\right) dx$$

$$= 1 + 3t + 18t^2 + 105t^3 + 684t^4 + 4550t^5 + 31340t^6 + 219555t^7 + \dots$$

- ▷ Computer-driven discovery and proof; no human proof yet.
- ▷ Proof uses: (1) kernel method + (2) **creative telescoping**.

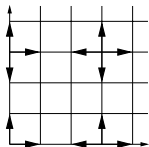
## (1) Kernel method [Bousquet-Mélou, Mishna, 2010]



The kernel  $J = 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$  is **invariant** under the change of  $(x, y)$  into elements of

$$\mathcal{G}_{\mathcal{S}} := \left\{ (x, y), \left( \frac{1}{x}, y \right), \left( \frac{1}{x}, \frac{1}{y} \right), \left( x, \frac{1}{y} \right) \right\}$$

## (1) Kernel method [Bousquet-Mélou, Mishna, 2010]



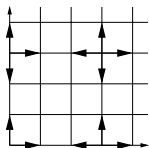
The kernel  $J = 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$  is invariant under the change of  $(x, y)$  into elements of

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Kernel equation:

$$J(x, y; t)xyQ(x, y; t) = xy - txQ(x, 0; t) - tyQ(0, y; t)$$

# (1) Kernel method [Bousquet-Mélou, Mishna, 2010]



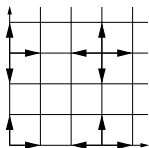
The kernel  $J = 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$  is invariant under the change of  $(x, y)$  into elements of

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Kernel equation:

$$\begin{aligned} J(x, y; t)xyQ(x, y; t) &= xy - txQ(x, 0; t) - tyQ(0, y; t) \\ - J(x, y; t)\frac{1}{x}yQ\left(\frac{1}{x}, y; t\right) &= -\frac{1}{x}y + t\frac{1}{x}Q\left(\frac{1}{x}, 0; t\right) + tyQ(0, y; t) \end{aligned}$$

# (1) Kernel method [Bousquet-Mélou, Mishna, 2010]



The kernel  $J = 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$  is invariant under the change of  $(x, y)$  into elements of

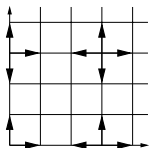
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Kernel equation:

$$\begin{aligned} J(x, y; t)xyQ(x, y; t) &= xy - txQ(x, 0; t) - tyQ(0, y; t) \\ - J(x, y; t)\frac{1}{x}yQ\left(\frac{1}{x}, y; t\right) &= -\frac{1}{x}y + t\frac{1}{x}Q\left(\frac{1}{x}, 0; t\right) + tyQ(0, y; t) \\ J(x, y; t)\frac{1}{x}\frac{1}{y}Q\left(\frac{1}{x}, \frac{1}{y}; t\right) &= \frac{1}{x}\frac{1}{y} - t\frac{1}{x}Q\left(\frac{1}{x}, 0; t\right) - t\frac{1}{y}Q\left(0, \frac{1}{y}; t\right) \end{aligned}$$



# (1) Kernel method [Bousquet-Mélou, Mishna, 2010]



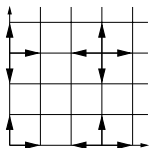
The kernel  $J = 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$  is invariant under the change of  $(x, y)$  into elements of

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Kernel equation:

$$\begin{aligned} J(x, y; t)xyQ(x, y; t) &= xy - txQ(x, 0; t) - tyQ(0, y; t) \\ - J(x, y; t)\frac{1}{x}yQ\left(\frac{1}{x}, y; t\right) &= -\frac{1}{x}y + t\frac{1}{x}Q\left(\frac{1}{x}, 0; t\right) + tyQ(0, y; t) \\ J(x, y; t)\frac{1}{x}\frac{1}{y}Q\left(\frac{1}{x}, \frac{1}{y}; t\right) &= \frac{1}{x}\frac{1}{y} - t\frac{1}{x}Q\left(\frac{1}{x}, 0; t\right) - t\frac{1}{y}Q\left(0, \frac{1}{y}; t\right) \\ - J(x, y; t)x\frac{1}{y}Q\left(x, \frac{1}{y}; t\right) &= -x\frac{1}{y} + txQ(x, 0; t) + t\frac{1}{y}Q\left(0, \frac{1}{y}; t\right) \end{aligned}$$

# (1) Kernel method [Bousquet-Mélou, Mishna, 2010]



The kernel  $J = 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$  is invariant under the change of  $(x, y)$  into elements of

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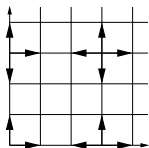
Kernel equation:

$$\begin{aligned} J(x, y; t)xyQ(x, y; t) &= xy - txQ(x, 0; t) - tyQ(0, y; t) \\ - J(x, y; t)\frac{1}{x}yQ\left(\frac{1}{x}, y; t\right) &= -\frac{1}{x}y + t\frac{1}{x}Q\left(\frac{1}{x}, 0; t\right) + tyQ(0, y; t) \\ J(x, y; t)\frac{1}{x}\frac{1}{y}Q\left(\frac{1}{x}, \frac{1}{y}; t\right) &= \frac{1}{x}\frac{1}{y} - t\frac{1}{x}Q\left(\frac{1}{x}, 0; t\right) - t\frac{1}{y}Q\left(0, \frac{1}{y}; t\right) \\ - J(x, y; t)x\frac{1}{y}Q\left(x, \frac{1}{y}; t\right) &= -x\frac{1}{y} + txQ(x, 0; t) + t\frac{1}{y}Q\left(0, \frac{1}{y}; t\right) \end{aligned}$$

Summing up yields the orbit equation:

$$\sum_{\theta \in \mathcal{G}} (-1)^{\theta} \theta(xy Q(x, y; t)) = \frac{xy - \frac{1}{x}y + \frac{1}{x}\frac{1}{y} - x\frac{1}{y}}{J(x, y; t)}$$

# (1) Kernel method [Bousquet-Mélou, Mishna, 2010]



The kernel  $J = 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$  is invariant under the change of  $(x, y)$  into elements of

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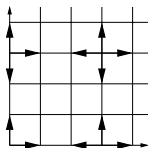
Kernel equation:

$$\begin{aligned} J(x, y; t)xyQ(x, y; t) &= xy - txQ(x, 0; t) - tyQ(0, y; t) \\ - J(x, y; t)\frac{1}{x}yQ\left(\frac{1}{x}, y; t\right) &= -\frac{1}{x}y + t\frac{1}{x}Q\left(\frac{1}{x}, 0; t\right) + tyQ(0, y; t) \\ J(x, y; t)\frac{1}{x}\frac{1}{y}Q\left(\frac{1}{x}, \frac{1}{y}; t\right) &= \frac{1}{x}\frac{1}{y} - t\frac{1}{x}Q\left(\frac{1}{x}, 0; t\right) - t\frac{1}{y}Q\left(0, \frac{1}{y}; t\right) \\ - J(x, y; t)x\frac{1}{y}Q\left(x, \frac{1}{y}; t\right) &= -x\frac{1}{y} + txQ(x, 0; t) + t\frac{1}{y}Q\left(0, \frac{1}{y}; t\right) \end{aligned}$$

Taking positive parts yields:

$$[x^>y^>] \sum_{\theta \in \mathcal{G}} (-1)^{\theta} \theta(xyQ(x, y; t)) = [x^>y^>] \frac{xy - \frac{1}{x}y + \frac{1}{x}\frac{1}{y} - x\frac{1}{y}}{J(x, y; t)}$$

# (1) Kernel method [Bousquet-Mélou, Mishna, 2010]



The kernel  $J = 1 - t \cdot \sum_{(i,j) \in \mathcal{S}} x^i y^j = 1 - t \left( x + \frac{1}{x} + y + \frac{1}{y} \right)$  is invariant under the change of  $(x, y)$  into elements of

$$\mathcal{G}_{\mathcal{S}} := \left\{ (x, y), \left( \frac{1}{x}, y \right), \left( \frac{1}{x}, \frac{1}{y} \right), \left( x, \frac{1}{y} \right) \right\}$$

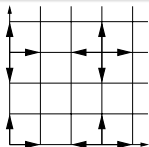
Kernel equation:

$$\begin{aligned} J(x, y; t)xyQ(x, y; t) &= xy - txQ(x, 0; t) - tyQ(0, y; t) \\ - J(x, y; t)\frac{1}{x}yQ\left(\frac{1}{x}, y; t\right) &= -\frac{1}{x}y + t\frac{1}{x}Q\left(\frac{1}{x}, 0; t\right) + tyQ(0, y; t) \\ J(x, y; t)\frac{1}{x}\frac{1}{y}Q\left(\frac{1}{x}, \frac{1}{y}; t\right) &= \frac{1}{x}\frac{1}{y} - t\frac{1}{x}Q\left(\frac{1}{x}, 0; t\right) - t\frac{1}{y}Q\left(0, \frac{1}{y}; t\right) \\ - J(x, y; t)x\frac{1}{y}Q\left(x, \frac{1}{y}; t\right) &= -x\frac{1}{y} + txQ(x, 0; t) + t\frac{1}{y}Q\left(0, \frac{1}{y}; t\right) \end{aligned}$$

Summing up and taking positive parts yields:

$$xyQ(x, y; t) = [x > y] \frac{xy - \frac{1}{x}y + \frac{1}{x}\frac{1}{y} - x\frac{1}{y}}{J(x, y; t)}$$

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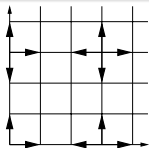
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$$\text{GF} = \text{PosPart} \left( \frac{\text{OS}}{\text{kernel}} \right) = \oint \text{RatFrac}$$

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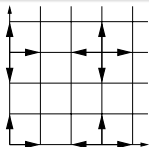
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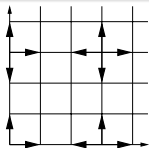
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▷ Argument works if  $\text{OS} \neq 0$ : algebraic version of the reflection principle

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▷ **Creative Telescoping** finds a differential equation for  $\text{GF} = \int \text{RatFrac}$



## (2) Creative Telescoping

“An algorithmic toolbox for multiple sums and integrals with parameters”

**Example [Apéry 1978]:**  $A_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2$  satisfies the recurrence

$$(n+1)^3 A_{n+1} + n^3 A_{n-1} = (2n+1)(17n^2 + 17n + 5)A_n.$$

▷ Key fact used to prove that  $\zeta(3) := \sum_{n \geq 1} \frac{1}{n^3} \approx 1.202056903 \dots$  is irrational.

### 1. Journées Arithmétiques de Marseille-Luminy, June 1978

The board of programme changes informed us that R. Apéry (Caen) would speak Thursday, 14.00 “Sur l’irrationalité de  $\zeta(3)$ .” Though there had been earlier rumours of his claiming a proof, scepticism was general. The lecture tended to strengthen this view to rank disbelief. Those who listened casually, or who were afflicted with being non-Francophone, appeared to hear only a sequence of unlikely assertions.

### 7. ICM '78, Helsinki, August 1978

Neither Cohen nor I had been able to prove  $\textcircled{5}$  or  $\textcircled{5}$  in the intervening 2 months. After a few days of fruitless effort the specific problem was mentioned to Don Zagier (Bonn), and with irritating speed he showed that indeed the sequence  $\{b'_n\}$  satisfies the recurrence (4). This more or less broke the dam and  $\textcircled{5}$  and  $\textcircled{5}$  were quickly conquered. Henri Cohen addressed a very well-attended meeting at 17.00 on Friday, August 18 in the language of the majority, proving  $\textcircled{5}$  and explaining how this implied the

[Van der Poorten, 1979: “A proof that Euler missed”]

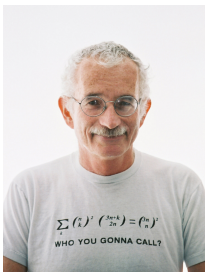
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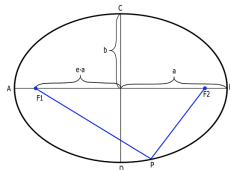
[Zeilberger, 1990: “The method of creative telescoping”]

## (2) Creative Telescoping

“An algorithmic toolbox for multiple sums and integrals with parameters”

**Example [Euler, 1733]:** Perimeter of an ellipse of eccentricity  $e$ , semi-major axis 1

$$p(e) = 4 \int_0^1 \sqrt{\frac{1 - e^2 u^2}{1 - u^2}} du = 4 \oint \frac{du dv}{1 - \frac{1 - e^2 u^2}{(1 - u^2) v^2}}$$



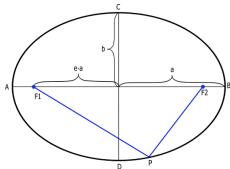
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**Principle:** Find algorithmically

$$\left( (e - e^3) \partial_e^2 + (1 - e^2) \partial_e + e \right) \cdot \left( \frac{1}{1 - \frac{1 - e^2 u^2}{(1 - u^2) v^2}} \right) =$$
$$\partial_u \left( - \frac{e(-1 - u + u^2 + u^3) v^2 (-3 + 2u + v^2 + u^2 (-2 + 3e^2 - v^2))}{(-1 + v^2 + u^2 (e^2 - v^2))^2} \right)$$
$$+ \partial_v \left( \frac{2e(-1 + e^2) u (1 + u^3) v^3}{(-1 + v^2 + u^2 (e^2 - v^2))^2} \right)$$

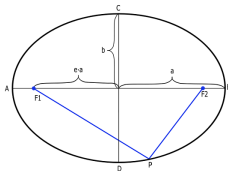
▷ Conclusion:  $(e - e^3) \cdot p''(e) + (1 - e^2) \cdot p'(e) + e \cdot p(e) = 0.$

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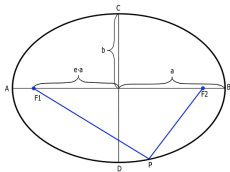
▷ Conclusion:  $p(e) = \frac{\pi}{2} \cdot {}_2F_1 \left( -\frac{1}{2}, \frac{1}{2} \mid e^2 \right) = 2\pi - \frac{\pi}{2}e^2 - \frac{3\pi}{32}e^4 - \dots$

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▷ Drawback: Size(certificate)  $\gg$  Size(telescopier).

## (2) 4G Creative Telescoping

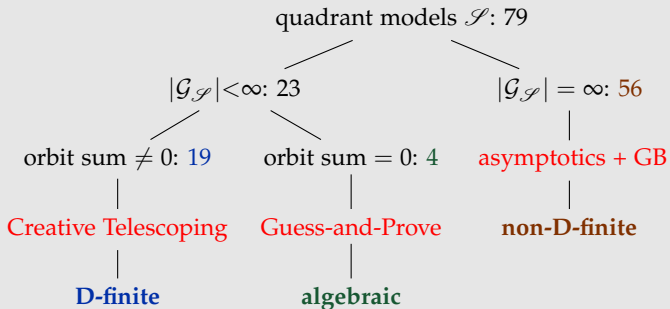
Algorithm for the integration of rational functions [B., Lairez, Salvy, 2013]

- **Input:**  $R(e, \mathbf{x})$  a rational function in  $e$  and  $\mathbf{x} = x_1, \dots, x_n$ .
- **Output:** A linear ODE  $T(e, \partial_e)y = 0$  satisfied by  $y(e) = \int R(e, \mathbf{x})d\mathbf{x}$ .
- **Complexity:**  $\mathcal{O}(D^{8n+2})$ , where  $D = \deg R$ .
- **Output size:**  $T$  has order  $\leq D^n$  in  $\partial_e$  and degree  $\leq D^{3n+2}$  in  $e$ .

- ▷ Avoids the (costly) computation of **certificates**, of size  $\Omega(D^{n^2}/2)$ .
- ▷ Previous algorithms: complexity (at least) doubly exponential in  $n$ .
- ▷ Very efficient in practice.

# Summary: classification of walks with small steps in $\mathbb{N}^2$

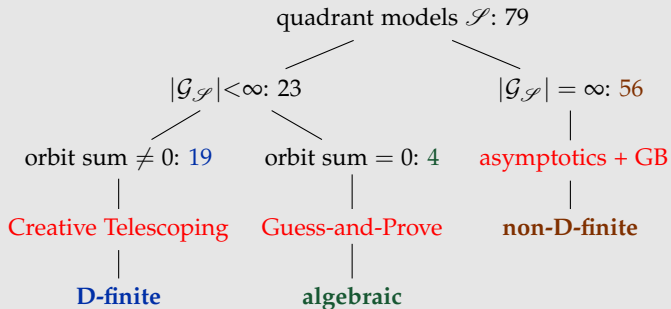
$Q_{\mathcal{S}}$  is D-finite  $\iff$  a certain group  $\mathcal{G}_{\mathcal{S}}$  is finite (!)





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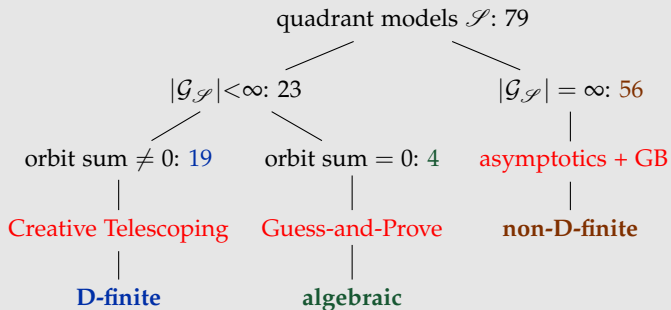
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▷ Many contributors (2010–2019): B., Bousquet-Mélou, Chyzak, van Hoeij, Kauers, Kurkova, Mishna, Pech, Raschel, Salvy

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- ▷ Many contributors (2010–2019): B., Bousquet-Mélou, Chyzak, van Hoeij, Kauers, Kurkova, Mishna, Pech, Raschel, Salvy
- ▷ Proofs use **various tools**: algebra, complex analysis, probability theory, computer algebra, etc.



Enumerative Combinatorics and Computer Algebra enrich one another



Classification of  $Q(x, y; t)$  **fully completed** for 2D small step walks



**Robust algorithmic** methods, based on efficient algorithms:

- **Guess-and-Prove**
- **Creative Telescoping**



Brute-force and/or use of naive algorithms = **hopeless**.

E.g. size of algebraic equations for  $G(x, y; t) \approx 30\text{Gb}$ .



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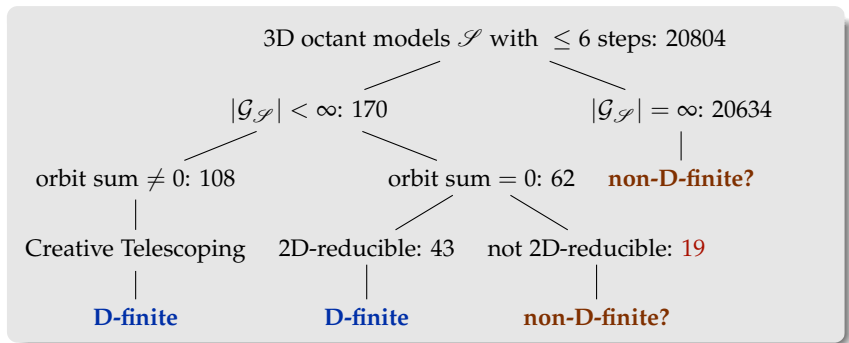
Lack of “purely human” proofs for some results.



Many beautiful open questions for 2D models with **repeated** or **large** steps, and in **dimension  $> 2$** .

## Beyond dimension 2: walks with small steps in $\mathbb{N}^3$

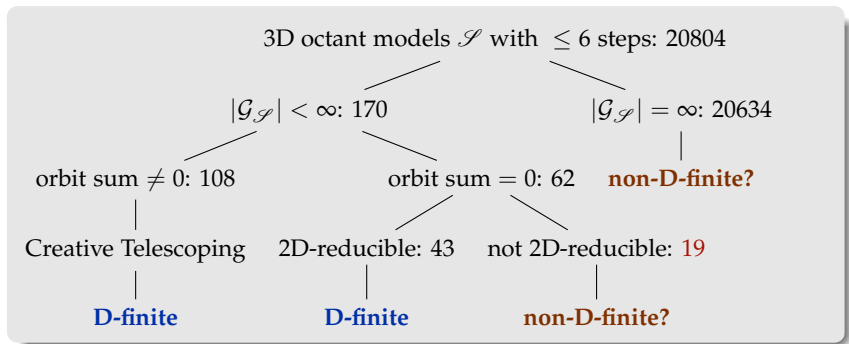
▷  $2^{3^3-1} \approx 67$  million models, of which  $\approx 11$  million inherently 3D



[B., Bousquet-Mélou, Kauers, Melczer, 2016] + [Du, Hou, Wang, 2017];  
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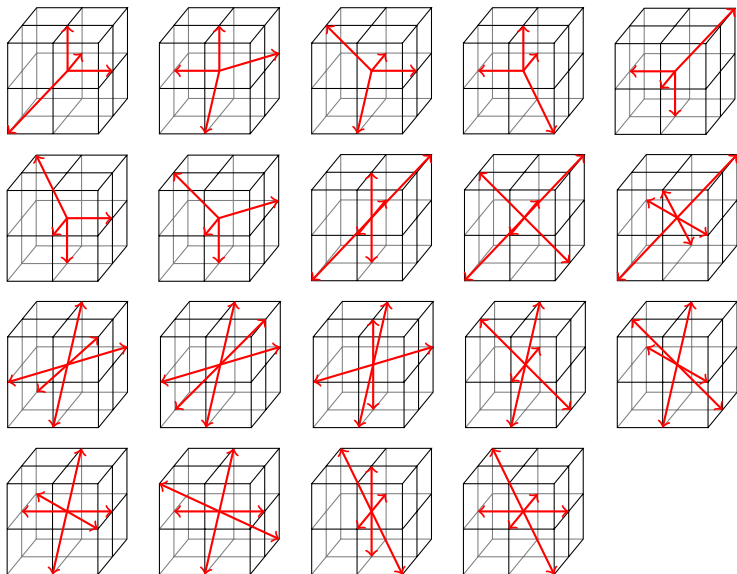


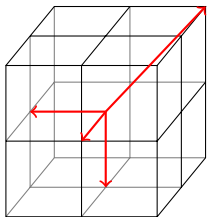
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Question: differential finiteness  $\iff$  finiteness of the group?

Answer: probably no

# 19 mysterious 3D-models





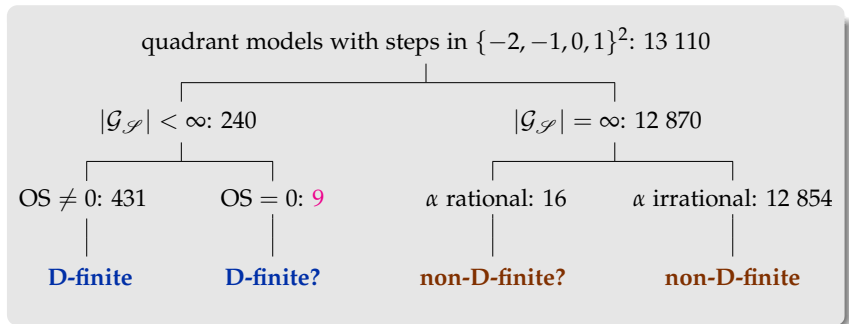
Two different computations suggest:

$$k_{4n} \approx C \cdot 256^n / n^{3.3257570041744\dots},$$

so excursions are very probably transcendental  
(and even non-D-finite)



# Beyond small steps: Walks in $\mathbb{N}^2$ with large steps




[B., Bousquet-Mélou, Melczer, 2018]

Question: differential finiteness  $\iff$  finiteness of the group?

Answer: ?


## Two challenging models with large steps

**Conjecture 1** [B., Bousquet-Mélou, Melczer, 2018]

For the model  the excursions generating function  $Q(0,0;t^{1/2})$  equals

$$\frac{1}{3t} - \frac{1}{6t} \cdot \left( \frac{1-12t}{(1+36t)^{1/3}} \cdot {}_2F_1 \left( \begin{matrix} \frac{1}{6} & \frac{2}{3} \\ 1 \end{matrix} \middle| \frac{108t(1+4t)^2}{(1+36t)^2} \right) + \right. \\ \left. \sqrt{1-12t} \cdot {}_2F_1 \left( \begin{matrix} -\frac{1}{6} & \frac{2}{3} \\ 1 \end{matrix} \middle| \frac{108t(1+4t)^2}{(1-12t)^2} \right) \right).$$

**Conjecture 2** [B., Bousquet-Mélou, Melczer, 2018]

For the model  the excursions generating function  $Q(0,0;t)$  equals

$$\frac{(1-24U+120U^2-144U^3)(1-4U)}{(1-3U)(1-2U)^{3/2}(1-6U)^{9/2}},$$

where  $U = t^4 + 53t^8 + 4363t^{12} + \dots$  is the unique series in  $\mathbb{Q}[[t]]$  satisfying

$$U(1-2U)^3(1-3U)^3(1-6U)^9 = t^4(1-4U)^4.$$

- Automatic classification of restricted lattice walks, with M. Kauers. *Proceedings FPSAC*, 2009.
- The complete generating function for Gessel walks is algebraic, with M. Kauers. *Proceedings of the American Mathematical Society*, 2010.
- Explicit formula for the generating series of diagonal 3D Rook paths, with F. Chyzak, M. van Hoeij and L. Pech. *Séminaire Lotharingien de Combinatoire*, 2011.
- Non-D-finite excursions in the quarter plane, with K. Raschel and B. Salvy. *Journal of Combinatorial Theory A*, 2014.
- On 3-dimensional lattice walks confined to the positive octant, with M. Bousquet-Mélou, M. Kauers and S. Melczer. *Annals of Comb.*, 2016.
- A human proof of Gessel's lattice path conjecture, with I. Kurkova, K. Raschel, *Transactions of the American Mathematical Society*, 2017.
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