# Non-Gaussian local limit properties for pattern statistics in rational stochastic models

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[GLV18] Proc. DCFS 2018, 20 Int. Conf. Descriptional Complexity of Formal Systems, LNCS 10952.

[GLV19] Proc. DLT 2019, 23rd Int. Conf. Developments of Language Theory, LNCS 11647.

# Pattern Statistics

finite alphabet $A$

pattern $a \in A$,
$w \in A^+$, $|w| = m$
$R \subseteq A^*$, finite, regular, ...

random text $x \in A^+$, $|x| = n$
stochastic model (Bernoullian, Markovian,.., rational)

$$O_n = \sharp\{\text{occurrences of pattern in } x\} \qquad \text{(positions)}$$

$$O_n \in \{0, 1, \ldots, n\}$$

Goals: - asymptotic properties of $\{O_n\}$
- $E(O_n)$, $var(O_n)$, ... ;
- limit distributions, $Pr\{O_n \leq z\} \sim \cdots$
- local limit laws, $Pr\{O_n = k\} \sim \cdots$

[Guibas-Odlyzko 78, 81], [Regnier-Szpankowski 96, 98],
[Nicodeme-Salvy-Flajolet 02], [Flajolet-Szpankowski-Vallée 06], ...

# Rational models

[BCGL03, BCGL06, DGL04, GL06]

Rational formal series over $\{a, b\}$

$r : \{a, b\}^* \to \mathbb{R}_+, \quad r = \sum_{w \in \{a,b\}^*} r(w)w$

defined by a f.s. automaton with weights in $\mathbb{R}_+$

$\Rightarrow L = \{x \in \{a, b\}^* \mid r(x) > 0\}$ is regular

Symbol statistics ($\forall n \in \mathbb{N}$)

$$Y_n = |x|_a \quad \text{indipendent r.v.}$$

$x \in \{a, b\}^n$ with $\Pr(x) = \frac{r(x)}{\sum_{w \in \{a,b\}^n} r(w)}$

$\Rightarrow \; Y_n \in \{0, 1, \ldots, n\}$, $Y_n = Y_n(r)$

Special cases:

- $r = \chi_L \quad \Rightarrow \quad x \in L \cap \{a, b\}^n$ under uniform dist.
- Markovian source $(\pi, P)$ with set of states $A$

$$\pi' P^n e = \sum_{w \in \{a,b\}^n} r(w) = 1, \; \forall n \in \mathbb{N}$$

# Regular pattern + Markovian source $\subsetneq$ Rational models

[Nicodeme-Salvy-Flajolet 02, BCGL03]

1) For any:
- $R \subseteq A^*$: regular language (pattern)
- $(\pi, P)$: Markovian source over $A$

$\Rightarrow \exists r \in \mathbb{R}_+^{\text{rat}}\langle\langle a, b\rangle\rangle$ s.t.

$O_n(\pi, P, R)$ and $Y_n(r)$ have the same distribution

i.e. $\forall k = 0, ..., n$, $\Pr(O_n = k) = \Pr(Y_n = k)$.

2) The opposite of 1) does not hold:

$$\sum_{n=0}^{+\infty}\sum_{k=0}^{n} \Pr(O_n = k)x^k y^n \text{ is a rational function} \quad \forall(\pi, P, R)$$

while for the rational model

$$\sum_{n=0}^{+\infty}\sum_{k=0}^{n} \Pr(Y_n = k)x^k y^n \text{ is not rational in general.}$$

# Formal notions

Linear representation for $r \in \mathbb{R}_+^{\mathrm{rat}}\langle\langle a, b\rangle\rangle$

$(\xi, A, B, \eta)$:

- $\xi, \eta \in \mathbb{R}_+^m$, weights of initial and final states
- $A, B \in \mathbb{R}_+^{m\times m}$ $(\neq 0)$, weights of $a$- and $b$-transitions

$A, B$ generate a morphism $\mu : \{a, b\}^* \to \mathbb{R}_+^{m\times m}$

$$\mu(a) = A, \mu(b) = B$$

$$\mu(a_1 a_2 \cdots a_n) = \mu(a_1)\mu(a_2)\cdots\mu(a_n)$$

Rational series $r \in \mathbb{R}_+^{\mathrm{rat}}\langle\langle a, b\rangle\rangle$ defined by $(\xi, A, B, \eta)$:

for every $x = a_1 a_2 \cdots a_n \in \{a, b\}^*$

$$r(x) = \xi'\mu(x)\eta = \xi'\mu(a_1)\mu(a_2)\cdots\mu(a_n)\eta$$

## Probability Measure over $\{a,b\}^n : \forall x \in \{a,b\}^n$

$$\Pr(x) = \frac{r(x)}{\sum_{w\in\{a,b\}^n} r(w)} = \frac{\xi'\mu(w)\eta}{\xi'(A+B)^n\eta}$$

special case: If $r = \chi_L$ where $L \subseteq \{a,b\}^*$ is regular, then

$$\Pr(x) = \begin{cases} \frac{1}{\sharp(L\cap\{a,b\}^n)} & \text{if } x \in L \\ 0 & \text{otherwise} \end{cases}$$

## Symbol statistics

$$Y_n = |x|_a, \qquad \text{where } \Pr(x) = \frac{r(x)}{\sum_{w\in\{a,b\}^n} r(w)}$$

$$Y_n \in \{0,1,...,n\} \text{ is a r.v.,} \qquad Y_n = Y_n(r)$$

## Distribution of $Y_n$

$$\Pr(Y_n = k) = \frac{\sum_{x\in\{a,b\}^n, |x|_a=k} r(x)}{\sum_{w\in\{a,b\}^n} r(w)} = \\ = \frac{[z^k]\xi'(Az+B)^n\eta}{\xi'(A+B)^n\eta}, \qquad \text{for } k = 0, \ldots, n.$$

## Characteristic function of $Y_n$

$$\Psi_n(t) = \frac{\xi'(Ae^{it}+B)^n\eta}{\xi'(A+B)^n\eta} = \frac{h_n(it)}{h_n(0)}$$

$\Longrightarrow$ asymptotic behaviour of $\{Y_n\}_n$

## Case analysis according to $M = A + B$ [BCGL03, DGL04, BCGL06]

- primitive — Gaussian
- dominant bicomponent — Gaussian
- equipotent bicomponent comm.
  - $E_1 \neq E_2$ — uniform
  - $E_1 = E_2, V_1 \neq V_2$ — normal Mixture
  - $E_1 = E_2, V_1 = V_2$ — Gaussian

# Primitive rational models

[BCGL03, BCGL06]

$(\xi, A, B, \eta)$ with primitive $M = A + B,\ A \neq 0 \neq B$

Main parameters: $\lambda, \beta, \gamma$

$\lambda > 0$ main eigenvalue of $M$

$v, u \in \mathbb{R}_+^m$ left - right normalized eigenvectors $(\lambda)$

$v'M = \lambda v',\ Mu = \lambda u,\ v'u = 1,\ v, u > 0$

$\beta = \dfrac{v'Au}{\lambda}$ mean constant, $0 < \beta < 1$

$\gamma > 0$ variance constant,

$\gamma = \beta - \beta^2 + \frac{v'ADAu}{\lambda^2},\ D \in \mathbb{R}^{m \times m} \sim >$ other eigen.

Results:

$$E(Y_n) = \beta n + c + o(1), \qquad c \in \mathbb{R}$$

$$var(Y_n) = \gamma n + O(1)$$

$$\frac{Y_n - \beta n}{\sqrt{\gamma n}} \longrightarrow \mathcal{N}(0,1) \qquad \text{in dist.}$$

# Local limit law (in the primitive case)

[BCGL06, GLV19]

Aperiodicity Condition :

$G$ labelled graph with $a$-label weights $A_{ij}$
$b$-label weights $B_{ij}$

$$d = \mathrm{GCD}\{|C_1|_a - |C_2|_a : i \overset{C_1}{\rightsquigarrow} i, i \overset{C_2}{\rightsquigarrow} i, |C_1| = |C_2|\}$$

$(A, B)$ aperiodic if $d = 1$

Theorem If $M$ primitive, $A \neq 0 \neq B$, $(A, B)$ aperiodic, then

$$\left| \sqrt{n} \Pr(Y_n = k) - \frac{e^{-\frac{(k-\beta n)^2}{2\gamma n}}}{\sqrt{2\pi\gamma}} \right| = O\left(n^{-1/2}\right)$$

uniformly for every $k \in \{0, 1, \ldots, n\}$.

# Bicomponent models (with communication)

[dGL04]

Model formed by two communicating irreducible components $(\xi, A, B, \eta)$ of size $m_1 + m_2$ such that

$$\xi' = (\xi'_1, \xi'_2), \quad A = \left[\begin{array}{c|c} A_1 & A_0 \\ \hline 0 & A_2 \end{array}\right], \quad B = \left[\begin{array}{c|c} B_1 & B_0 \\ \hline 0 & B_2 \end{array}\right], \quad \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$$

$(\xi_1, A_1, B_1, \eta_1)$ of size $m_1$, $(\xi_2, A_2, B_2, \eta_2)$ of size $m_2$

$$M = \left[\begin{array}{c|c} A_1 + B_1 & A_0 + B_0 \\ \hline 0 & A_2 + B_2 \end{array}\right]$$

Hypotheses:

1) $A_1 + B_1$ and $A_2 + B_2$ irreducible with $\lambda_1$, $\lambda_2$ main eigenvalues

2) $A_0 + B_0 \neq 0$ and $\xi_1 \neq 0 \neq \eta_2$ communication cond. $1 \rightsquigarrow 2$

$\Rightarrow$ various cases depending on: $\lambda_1 \stackrel{?}{=} \lambda_2$, $\beta_1 \stackrel{?}{=} \beta_2$, $\gamma_1 \stackrel{?}{=} \gamma_2$

# Dominant bicomponent models: $\lambda_1 \neq \lambda_2$

Hp: $\lambda_1 > \lambda_2$, $M_1$ primitive, $A_1 \neq 0 \neq B_1$ $(\Rightarrow\ 0 < \beta_1 < 1,\ 0 < \gamma_1)$

$$E(Y_n) \sim \beta_1 n, \qquad var(Y_n) \sim \gamma_1 n, \qquad \frac{Y_n - \beta_1 n}{\sqrt{\gamma_1 n}} \xrightarrow{d} \mathcal{N}(0,1)$$

[dGL04]

Theorem Under the same hps, if $(A_1, B_1)$ aperiodic then

$$\left| \sqrt{n} \Pr(Y_n = k) - \frac{e^{-\frac{(k-\beta_1 n)^2}{2\gamma_1 n}}}{\sqrt{2\pi\gamma_1}} \right| = O\left(n^{-1/2}\right)$$

uniformly for every $k \in \{0, 1, \ldots, n\}$.

[GLV19]

# Equipotent bicomponent models with different $\beta$'s

Hp: $\lambda_1 = \lambda_2$, $\beta_1 \neq \beta_2$,
$M_1, M_2$ primitive, $A_j \neq 0 \neq B_j$ $(j = 1, 2)$

$$E(Y_n) = \frac{\beta_1 + \beta_2}{2} n + O(1),$$

$$var(Y_n) = \frac{(\beta_1 - \beta_2)^2}{12} n^2 + O(n)$$

$$\frac{Y_n}{n} \xrightarrow{d} U \quad \text{(convergence in dist. to a uniform r.v.)}$$

$$\text{where } f_U(x) = \begin{cases} 0 & \text{if } x \leq b_1 \\ \frac{1}{b_2 - b_1} & \text{if } b_1 < x \leq b_2 \\ 0 & \text{if } b_2 < x \end{cases}$$

$$\text{with } b_1 = \min\{\beta_1, \beta_2\}, b_2 = \max\{\beta_1, \beta_2\}$$

# Local limit law of uniform type

[GLV19]

$$f_U(x) = \begin{cases} 0 & \text{if } x \le b_1 \\ \frac{1}{b_2 - b_1} & \text{if } b_1 < x \le b_2 \\ 0 & \text{if } b_2 < x \end{cases}, \quad \text{where } \begin{cases} b_1 = \min\{\beta_1, \beta_2\}, \\ b_2 = \max\{\beta_1, \beta_2\} \end{cases}$$

## Theorem

Under the same hps $\begin{cases} \lambda_1 = \lambda_2, \beta_1 \neq \beta_2 \\ M_1, M_2 \text{ primitive}, A_j \neq 0 \neq B_j,\ j = 1, 2 \end{cases}$

if $(A_1, B_1), (A_2, B_2)$ aperiodic then

$$|n \Pr(Y_n = k) - f_U(x)| = O\left(\frac{(\log n)^{3/2}\, \tau_n}{\sqrt{n}}\right)$$

for all $k = k(n) \in [n]$ s.t. $k/n \to x$ , $x \in \mathbb{R}$, $\beta_1 \neq x \neq \beta_2$;
where $\{\tau_n\} \subset \mathbb{R} : \tau_n \to +\infty$, $\tau_n = o(\log \log n)$ (arbitrarily slow)

## Example

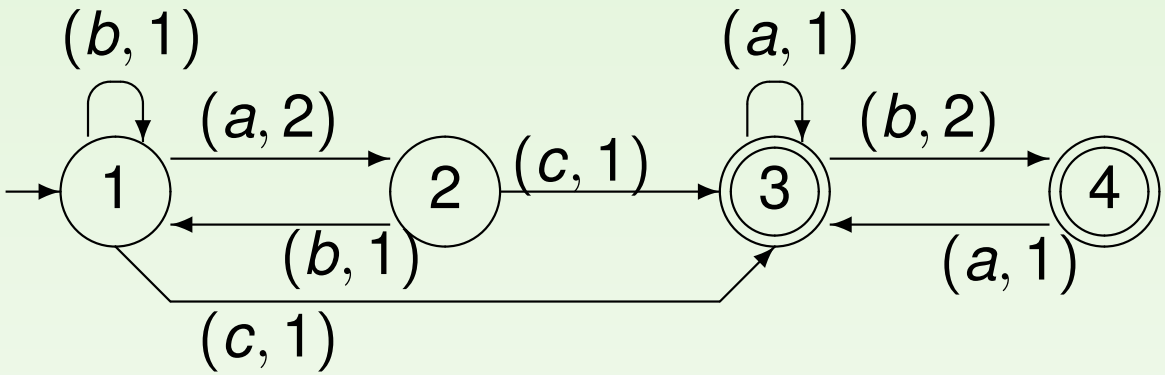


Figure: $\lambda_1 = \lambda_2 = 2$, $1/3 = \beta_1 \neq \beta_2 = 2/3$ (equipotent)

$L = \{x \in \{a, b\}^* \mid aa \notin x\}\ c\ \{y \in \{a, b\}^* \mid bb \notin y\}$
$M_1 = M_2$, $A_1 \neq A_2$, $(A_1, B_1), (A_2, B_2)$ aperiodic

$\Rightarrow$ $Y_n/n$ has local limit of uniform type

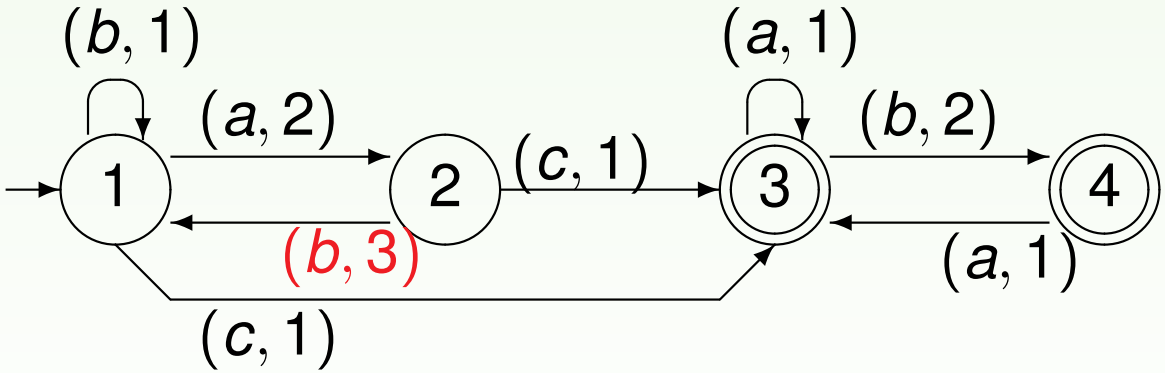


Figure: $\lambda_1 = 3, \lambda_2 = 2$ $\Rightarrow$ $Y_n/n$ has local limit of Gaussian type

# Proof (outline)

$$p_n(k) := \Pr(Y_n = k), \quad k = 0, 1, ..., n$$

Characteristic function

$$\Psi_n(t) = \sum_{j=0}^{n} p_n(j) e^{itj}$$

Inversion formula

$$p_n(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Psi_n(t) e^{-itk} dt$$

Ingredients: a) Saddle Point Method

b) analysis of $\Psi_n(t)$ in $[-\pi, \pi]$

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a) Split the interval $[-\pi, \pi]$ into 3 sets:

$$\underbrace{\{t \in \mathbb{R} : c < |t| \leq \pi\}}_{(I)}, \quad \underbrace{\{t \in \mathbb{R} : n^{-q} < |t| \leq c\}}_{(II)}, \quad \underbrace{[-n^{-q}, n^{-q}]}_{(III)}$$

for suitable $0 < c < 1$, $\frac{1}{3} < q < \frac{1}{2}$.

# Proof (outline)

($I$) From aperiodicity condition

$$\int_{c<|t|\leq\pi} \Psi_n(t)e^{-itk}dt = O(\varepsilon^n) \qquad 0 \leq \varepsilon < 1$$

($II$)

$$\int_{n^{-q}<|t|\leq c} \Psi_n(t)e^{-itk}dt = O(e^{-an^{1-2q}}) \qquad a > 0, \frac{1}{3} < q < \frac{1}{2}$$

($III$)

$$\int_{|t|\leq n^{-q}} \Psi_n(t)e^{-itk}dt = \frac{2\pi}{n} f_U(x) + o\left(\frac{1}{n}\right)$$

$f_U$ : PDF of $U \sim \mathcal{U}(\beta_1, \beta_2)$

Of interest here: details of claim ($III$)

$$\Psi_n(t) = \frac{e^{i\beta_1 tn - \frac{\gamma_1}{2}t^2 n} - e^{i\beta_2 tn - \frac{\gamma_2}{2}t^2 n}}{it(\beta_1 - \beta_2)n} + \cdots \qquad |t| \leq n^{-q}$$

# Equipotent bicomponent models with equal $\beta$'s and different $\gamma$'s

Hp: $\lambda_1 = \lambda_2$, $\beta_1 = \beta_2$, $\gamma_1 \neq \gamma_2$,
$M_1$, $M_2$ primitive, $A_j \neq 0 \neq B_j$ $(j = 1, 2)$

Set: $\beta = \beta_1 = \beta_2$, $\gamma = \frac{\gamma_1 + \gamma_2}{2}$
r.v. $T$ mixture of $\mathcal{N}(0, s)$ with
variance $s$ uniformly distributed between $\frac{\gamma_1}{\gamma}$ and $\frac{\gamma_2}{\gamma}$.

$$f_T(x) = \frac{\gamma}{\gamma_2 - \gamma_1} \int_{\frac{\gamma_1}{\gamma}}^{\frac{\gamma_2}{\gamma}} \frac{e^{-\frac{x^2}{2s}}}{\sqrt{2\pi s}}\, ds \qquad \forall\, x \in \mathbb{R}$$

$f_T(x) \quad \rightsquigarrow \quad$ heat equation in dimension 1

It is known that

$$\frac{Y_n - \beta n}{\sqrt{\gamma n}} \xrightarrow{d} T$$

# Local limit law of T type

[GLV19]

**Theorem**

Under the same hps $\begin{cases} \lambda_1 = \lambda_2, \beta_1 = \beta_2, \gamma_1 \neq \gamma_2 \\ M_1, M_2 \text{ primitive}, A_j \neq 0 \neq B_j,\ j = 1, 2 \end{cases}$

if $(A_1, B_1), (A_2, B_2)$ aperiodic and $\gamma = \frac{\gamma_1 + \gamma_2}{2}$ then

$$\left| \sqrt{\gamma n} \Pr(Y_n = k) \ - \ f_T \left( \frac{k - \beta n}{\sqrt{\gamma n}} \right) \right| \ = \ O\left( \frac{(\log n)^2 \, \tau_n}{\sqrt{n}} \right)$$

uniformly for $k \in \{0, 1, \ldots, n\}$
where $\{\tau_n\} \subset \mathbb{R} : \tau_n \to +\infty$, $\tau_n = o(\log \log n)$ (arbitrarily slow)

# Equipotent bicomponent models with equal $\beta$'s and $\gamma$'s

**Theorem**

Assume the bicomponent model,
let $\lambda_1 = \lambda_2$, $\beta_1 = \beta_2$, $\gamma_1 = \gamma_2$
$M_1$, $M_2$ primitive, $A_j \neq 0 \neq B_j$ for $j = 1, 2$.
If $(A_1, B_1)$, $(A_2, B_2)$ are aperiodic then

$$\left| \sqrt{n} \Pr(Y_n = k) - \frac{e^{-\frac{(k-\beta n)^2}{2\gamma n}}}{\sqrt{2\pi\gamma}} \right| = O\left(n^{-1/2}\right)$$

uniformly for every $k \in \{0, 1, \ldots, n\}$.

# Summary of results in [GLV19]

| | Primitive Models | Bicomponent Models | | | |
|---|---|---|---|---|---|
| | | | $\lambda_1 = \lambda_2$ | | |
| | | $\lambda_1 \neq \lambda_2$ | $\beta_1 \neq \beta_2$ | $\beta_1 = \beta_2$ <br> $\gamma_1 \neq \gamma_2$ | $\beta_1 = \beta_2$ <br> $\gamma_1 = \gamma_2$ |
| Local limit distribution | $N_{0,1}$ [BCGL03] | $N_{0,1}$ | $U_{\beta_1,\beta_2}$ [GLV18] | $T$ | $N_{0,1}$ |
| Convergence rate | $O(n^{-1/2})$ | $O(n^{-1/2})$ | $O\left(\frac{\tau_n \log^{3/2} n}{\sqrt{n}}\right)$ | $O\left(\frac{\tau_n \log^2 n}{\sqrt{n}}\right)$ | $O(n^{-1/2})$ |

# Conclusions

- The results strengthen under aperiodicity conditions the convergence in distribution obtained in [dGL04]
- The convergence rate is $O(n^{-1/2})$ for all Gaussian limits
- The convergence rate is arbitrarily "slower" than $O(n^{-1/2})$ in the other cases
- For bicomponent models without communication with aperiodicity conditions

  $$A_0 + B_0 = 0,\ r = s + t \text{ for } s, t \in \mathbb{R}_+^{\text{rat}}\langle\langle a, b\rangle\rangle$$

  we get [GLV19b] (with similar convergence rate):
  - Gaussian local limit for $Y_n$ in dominant cases $\lambda_1 \neq \lambda_2$
  - shifted Bernoullian local limit if $\lambda_1 = \lambda_2$, $\beta_1 \neq \beta_2$
  - local limit towards a convex combination of 2 Gaussian laws if $\lambda_1 = \lambda_2$, $\beta_1 = \beta_2$ and $\gamma_1 \neq \gamma_2$
  - Gaussian local limit for $Y_n$ if $\lambda_1 = \lambda_2$, $\beta_1 = \beta_2$, $\gamma_1 = \gamma_2$

Thank you !

## Theorem [Nicodeme-Salvy-Flajolet 02, BCGL03]

For every regular $R \subseteq \Sigma^*$ (pattern) and every Markovian source $(\pi, P)$ over $\Sigma$ there exists $r \in \mathbb{R}_+^{\text{rat}}\langle\langle a, b\rangle\rangle$ s.t.

$$\Pr(O_n(\pi, P, R) = k) = \Pr(Y_n(r) = k) \qquad \forall\, k = 0, ..., n$$

[Proof sketch]
Consider a det. f.s. automaton $(Q, p, \delta, F)$ recognizing $\Sigma^* R$.
Define a new set of states $Q' = \{p\} \cup \{(q, \sigma) \mid q \in Q, \sigma \in \Sigma\}$.
Define a linear representation $(\xi, A, B, \eta)$ over $Q'$ such that:

- $\xi = \chi_p$, $\eta = \chi_{Q'}$
- $A_{p,(q,\sigma)} = \begin{cases} \pi_\sigma & \text{if } q = \delta(p, \sigma) \wedge q \in F \\ 0 & \text{otherwise} \end{cases}$,

  $A_{(q,\sigma),(q',\tau)} = \begin{cases} P_(\sigma, \tau) & \text{if } q' = \delta(q, \sigma) \wedge q' \in F \\ 0 & \text{otherwise} \end{cases}$, $\forall\, \sigma, \tau \in \Sigma, q, q' \in Q$
- $B_{p,(q,\sigma)} = \begin{cases} \pi_\sigma & \text{if } q = \delta(p, \sigma) \wedge q \notin F \\ 0 & \text{otherwise} \end{cases}$,

  $B_{(q,\sigma),(q',\tau)} = \begin{cases} P_(\sigma, \tau) & \text{if } q' = \delta(q, \sigma) \wedge q' \notin F \\ 0 & \text{otherwise} \end{cases}$, $\forall\, \sigma, \tau \in \Sigma,\ q, q' \in Q$

## (Continue)

Let $\mu : \{a, b\}^* \to \mathbb{R}_+^{Q' \times Q'}$ be the morphism generated by $A = \mu(a)$, $B = \mu(b)$.
Then, for every $n \in \mathbb{N}$ and every $k \in \{0, 1 \ldots, n\}$, we have

$$\Pr(Y_n = k) = \frac{\sum_{x \in \{a,b\}^n, |x|_a = k} \xi' \mu(x) \eta}{\xi'(A+B)^n \eta} = \frac{\sum_{w \in \Sigma^n, |w|_R = k} \Pr(w)}{1} = \Pr(O_n = k)$$

Note that $A + B$ is a stochastic matrix
and the bivariate generating function

$$\sum_{n=0}^{+\infty} \sum_{k=0}^{n} \Pr(O_n = k) x^k y^n \;=\; \xi' \left(I - (Ax + B)y\right)^{-1} \eta$$

is rational.

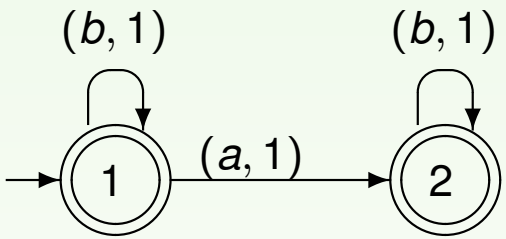
## Theorem [BCGL03]

For some $r \in \mathbb{R}_+^{\mathrm{rat}}\langle\langle a, b\rangle\rangle$ the generating fucntion

$$f_r(x,y) = \sum_{n=0}^{+\infty}\sum_{k=0}^{n} \Pr(Y_n = k)x^k y^n$$

is not rational.

[Proof sketch]
Consider the series $r$ defined by the weighted automaton




$$r(x) = \begin{cases} 1 & \text{if } |x|_a \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Then $\Pr(Y_n = k) = \begin{cases} \frac{1}{n+1} & \text{if } k = 0 \\ \frac{n}{n+1} & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$, and

$$\sum_{n=0}^{+\infty}\sum_{k=0}^{n} \Pr(Y_n = k)x^k y^n = (x-1)\frac{\log(1-z)}{y} + \frac{x}{1-y}$$