Satisfiability of Regular Occupation Problems

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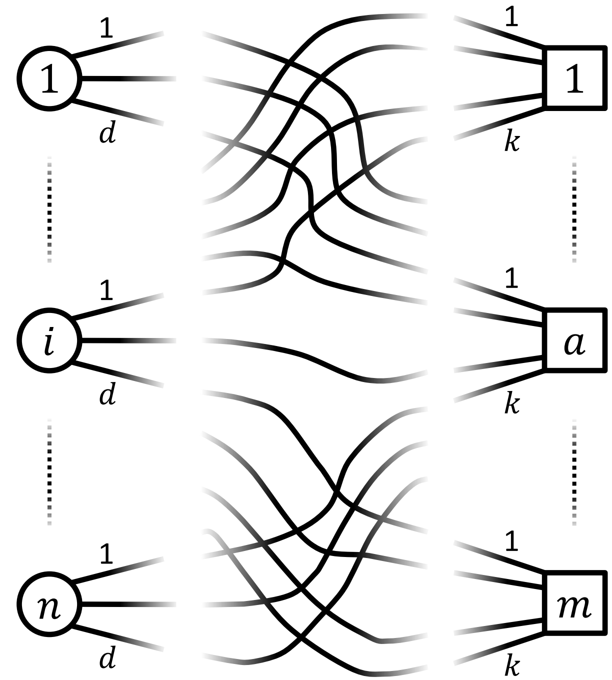
Contents

- Random Occupation Problems
- First and Second Moment Method
- Contraction Coefficient

The d-regular r-in-k Occupation Problem

- **Parameters**
  - Number $n > 0$ of variables
  - Number $m > 0$ of constraints
  - Variable degree $d > 1$
  - Constraint degree $k > 1$
  - Occupation number $r > 0$

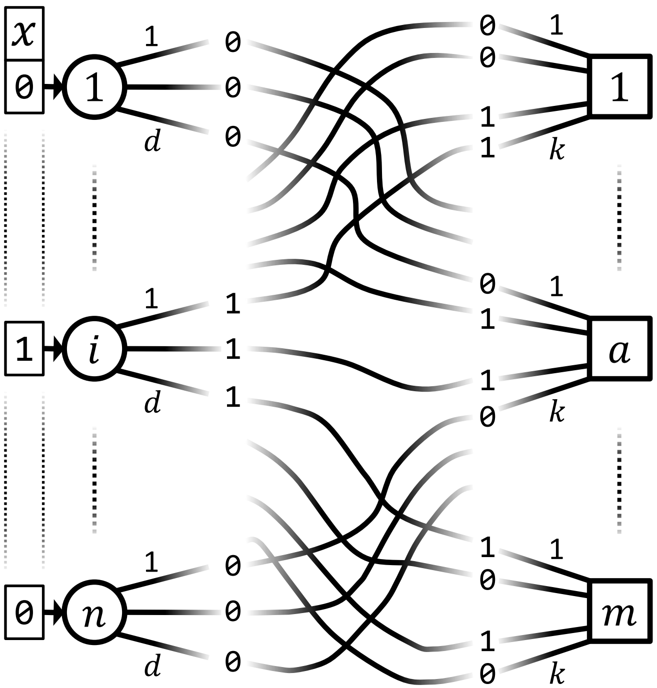


#Edges	dn	$=$	km

- **Instance of the d-regular r-in-k occupation problem**
  A (d, k)-biregular graph $g = ([n], [m], E)$ with $m = dn/k$

The d-regular r-in-k Occupation Problem

- **Solutions of g**
  - Assignments $x \in \{0,1\}^n$
  - x solution of g if each constraint sees r 1´s on the k edges
  - → Restrict to $rn \in k\mathbb{Z}$



#Ones	$^{mm}/_{kd}$	rm	$=$	rm

- **Decision Problem**

  Does there exist a solution x of g?

The d-regular r-in-k Occupation Problem

- **Asymptotical Average Case**
  - Fix r, k and d
  - For fixed n draw random instance G uniformly from all (d,k)-biregular graphs $([n],[m],E)$
  - Is G satisfiable with high probability in the large n limit?

- **History of Random Constraint Satisfaction Problems**
  - Probabilistic method for random graph models developed by Erdős and Rényi in the 1960s
  - Introduction of non-rigorous replica/cavity method and solution space clustering picture by physicists in the 1980s [Mézard, Montanari '09]
  - Active research to date, major recent successes include k-SAT satisfiability threshold for large k [Ding, Sly, Sun '15]

The d-regular r-in-k Occupation Problem

- **Satisfiability Thresholds**
  - Fix k and r
  - We conjecture a sharp satisfiability threshold d^* with respect to d

$$\lim_{n\to\infty} \mathbb{P}[G \text{ satisfiable}] = 1 \text{ for } d < d^*$$
$$\lim_{n\to\infty} \mathbb{P}[G \text{ satisfiable}] = 0 \text{ for } d > d^*$$

  - Conjectured location of threshold

$$d^* = d^*(r,k) = \frac{kH\left({}^{r}/_{k}\right)}{kH\left({}^{r}/_{k}\right) - \ln\binom{k}{r}}$$

Binary entropy $H(p) = -p\ln p - (1-p)\ln(1-p)$

The d-regular r-in-k Occupation Problem

- **Cooper et al (1996)**

  Confirmed sharp satisfiability threshold $d^*(r,k)$ for $r = 1$ and $k \geq 3$

- **Mora (2007)**

  Introduction of Occupation Problems

- **Mézard, Zdeborová (2008) and Krzakala, Zdeborová (2011)**

  Discussion of Occupation Problems with first and second moment

The d-regular r-in-k Occupation Problem

- **Moore (2016)**

  Confirmed sharp satisfiability threshold $d^*(r,k)$ for $r = 1$ and $k \geq 3$

- **Panagiotou, P. (2019)**
  - Confirmed sharp satisfiability threshold $d^*(r,k)$ for $r = 2$ and $k = 4$
  - Confirmed $d^*(r,k)$ for $r = 2$ and $k \geq 4$ modulo an analytical optimization problem

- **Applications**
  - The d-regular positive r-in-k SAT
  - Existence of r-factors in k-regular d-uniform hypergraphs
    → Perfect matchings for $r = 1$

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First and Second Moment Method

- **Outline**
  - **Configuration Model** [Bollobás ’80]
    - Draw bijection $G: [dn] \to [km]$ uniformly at random
    - Translate sharp satisfiability threshold to random graph using contiguity
  - **First Moment Method**
    - G not satisfiable with high probability for $d > d^*$
    - Markov´s inequality for number Z of solutions of G

$$\lim_{n\to\infty} \mathbb{P}[Z > 0] \le \lim_{n\to\infty} \mathbb{E}[Z] = 0$$

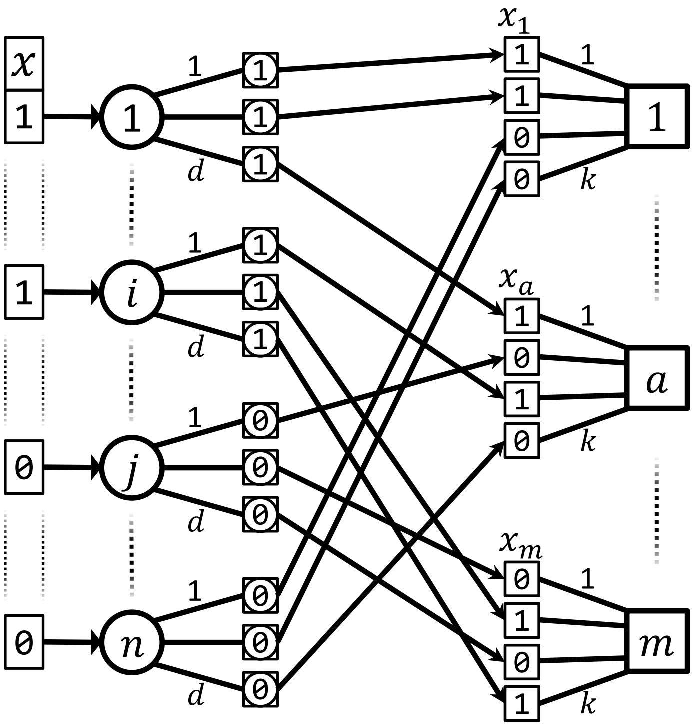
    - Discussion of $\mathbb{E}[Z]$ straightforward compared to $\mathbb{E}[Z^2]$
  - **Second Moment Method**
    - G satisfiable with positive probability for $d < d^*$
    - Paley–Zygmund inequality for Z

$$\lim_{n\to\infty} \mathbb{P}[Z > 0] \ge \lim_{n\to\infty} \frac{\mathbb{E}[Z]^2}{\mathbb{E}[Z^2]} > 0$$

    - Obtain $Z > 0$ with high probability via small subgraph conditioning [Robinson, Wormald ’94, Janson ’95, Molloy et al ‘97]

First and Second Moment Method

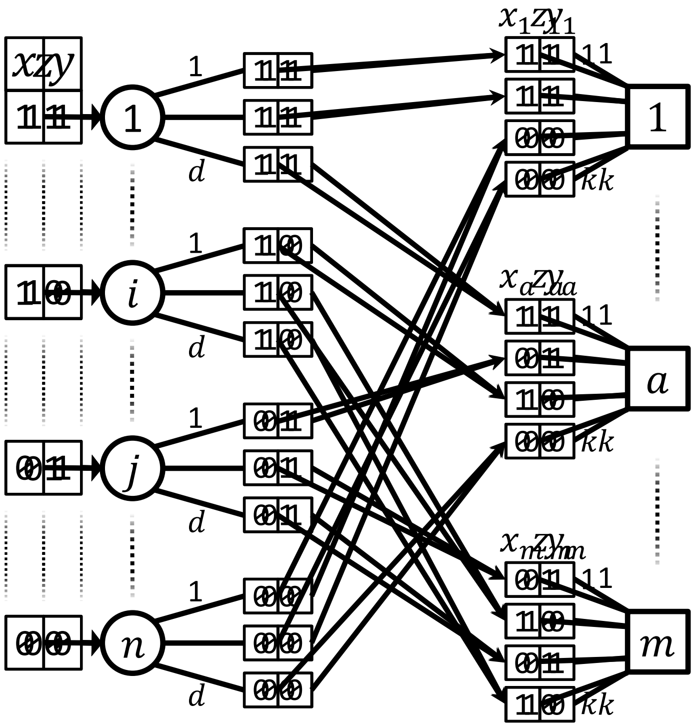
- **Instances of the Problem**
  - Random bijection $G\colon [dn] \to [km]$
    $\mathbb{P}[G = g] = \frac{1}{(dn)!}$
  - Graphical representation with dn v-edges and km c-edges



- **Solutions**
  - Assignments $x \in \{0,1\}^n$ to variables
    → Constraint assignments $x_a \in \{0,1\}^k$
  - Assignment x is a solution of g if the number of 1´s of x_a equals r for all $a \in [m]$
    → $x_a \in \binom{[k]}{r}$ for all $a \in [m]$

First and Second Moment Method

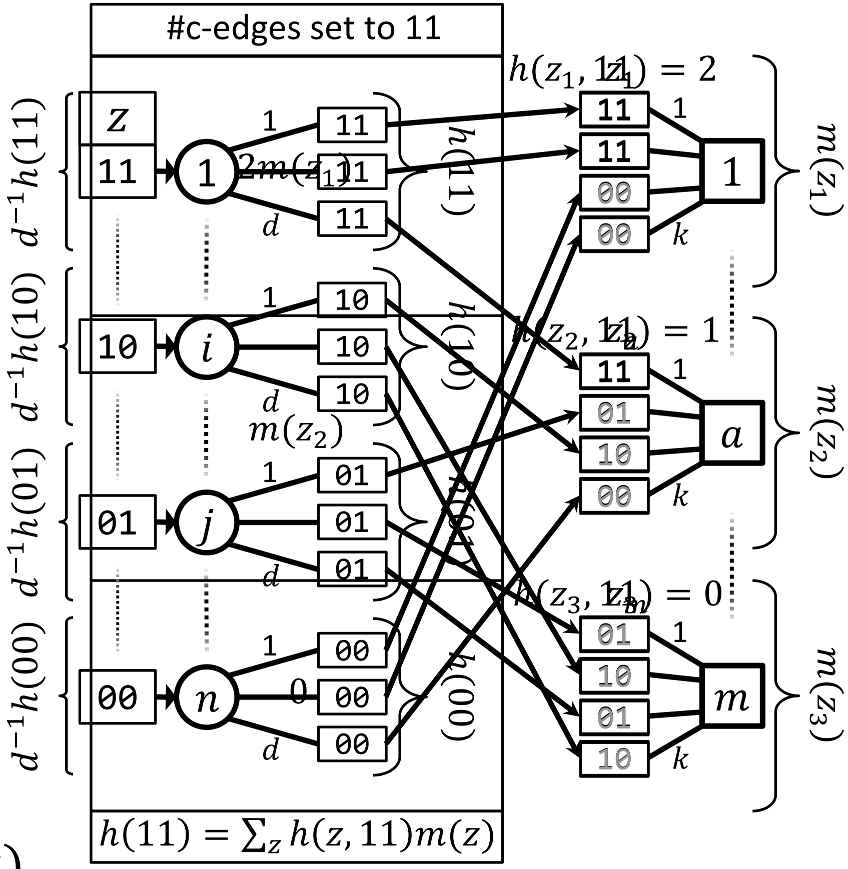
- **Pairs of Solutions**
  - Two solutions $x, y \in \{0,1\}^n$
    $\rightarrow x_a, y_a \in \binom{[k]}{r}$ for all $a \in [m]$
  - Solution pair $z = (x, y) \in (\{0,1\}^n)^2$
  - Equivalently $z \in (\{0,1\}^2)^n$
    $\rightarrow$ Assignment z to n variables with values in $\{0,1\}^2$
    $\rightarrow$ Constraint assignments
    $z_a \in \binom{[k]}{r}^2$ for all $a \in [m]$



- **Second Moment**
  - Basic algebra yields $\mathbb{E}[Z^2] = \frac{1}{(dn)!} |\{(g, z) : z \text{ solution pair of } g\}|$
  - Problem: There is no closed form for the number of pairs (g, z)

First and Second Moment Method

- **Second Moment Combinatorics**
  - Fix number $m(z)$ of constraints that see $z \in \binom{[k]}{r}^2 \rightarrow \sum_z m(z) = m$
  - For $x \in \{0,1\}^2$ we know the number $h(z,x)$ of x´s in z
  - Number $h(x)$ of c-edges set to x is $h(x) = \sum_z h(z,x)m(z)$ $\rightarrow \sum_x h(x) = km$
  - C-Edge assignment permutation of v-edge assignment $\rightarrow$ Number of v-edges set to x is $h(x)$



  - V-Edge Assignment obtained from d copies of variable assignment $\rightarrow$ Number of variables set to x is $d^{-1}h(x)$
  - Fixes number of constraint/variable partitions and bijections

First and Second Moment Method

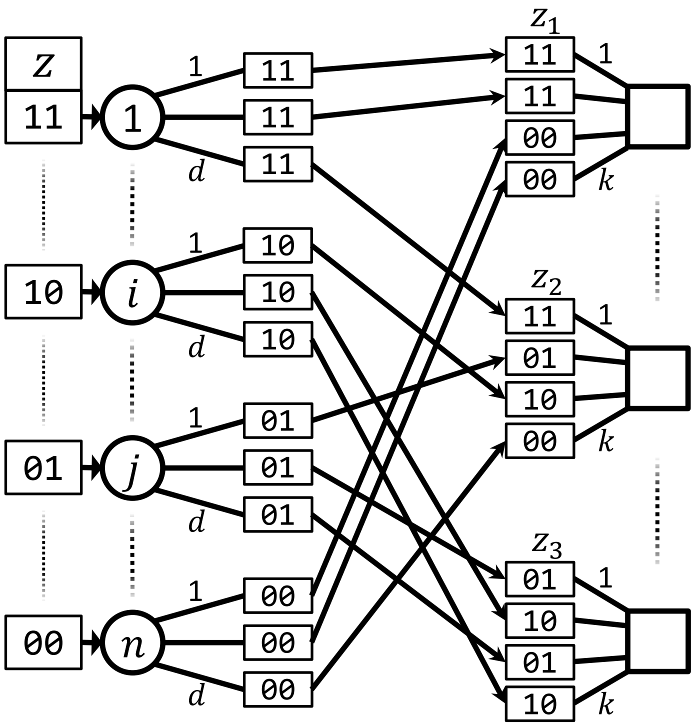
- **Constraint/Edge Distributions**
  - Constraint distribution κ on $\binom{[k]}{r}^2$

    $\kappa(z) = \frac{m(z)}{m} = \frac{1}{m}\#\text{constraints with } z$
  - Edge distribution κ_{e} on $\{0,1\}^2$

    $\kappa_{\mathrm{e}}(x) = \frac{h(x)}{km} = \frac{1}{km}\#\text{c}-\text{edges set to } x$
  - With $h(x) = \sum_z h(z,x)m(z)$

    $\kappa_{\mathrm{e}}(x) = \sum_z \frac{h(z,x)}{k}\kappa(z)$
  - Matrix notation

    $\kappa_{\mathrm{e}} = W\kappa$ with $W_{xz} = \frac{h(z,x)}{k} = \frac{1}{k}\#x\text{'s in } z$
  - W column stochastic transition probability matrix



First and Second Moment Method

- **Combinatorics for fixed $\boldsymbol{\kappa}$**
  - Let $\mathbb{E}[Z^2] = \sum_\kappa E(\kappa)$ with contributions $E(\kappa)$ to $\mathbb{E}[Z^2]$ for fixed κ
  - Compute $E(\kappa)$ combinatorically (similarly to $\mathbb{E}[Z]$)

- **Asymptotics for fixed $\boldsymbol{\kappa}$**
  - Fix (limiting) constraint distribution κ
  - Stirling´s formula $n! \sim \sqrt{2\pi n}(n/e)^n$ gives
  $$\frac{E(\kappa)}{\mathbb{E}[Z]^2} \sim \frac{\sqrt{d}}{\sqrt{\prod_z \kappa(z)}\sqrt{2\pi m}^r} \exp(-n\phi_2(\kappa))$$
  - Need $\phi_2(\kappa) \geq 0$ for all κ and $d < d^*$, otherwise $\frac{\mathbb{E}[Z^2]}{\mathbb{E}[Z]^2} \to \infty$ and Paley–Zygmund inequality trivial

First and Second Moment Method

- **Objective Function**
  - $\phi_2(\kappa) = {}^{d}/_{k}\, D(\kappa \parallel \kappa^*) - (d-1) D(\kappa_{\mathrm{e}} \parallel \kappa_{\mathrm{e}}^*)$
  - Relative entropy $D(p \parallel p^*) = \sum_x p(x) \ln\left(\frac{p(x)}{p^*(x)}\right)$
  - Reference distribution κ^* uniform on $\binom{[k]}{r}^2$
- **Results for $r = 2$**
  - **If** $\phi_2(\kappa) > 0$ for $\kappa \neq \kappa^*$, then $\frac{\mathbb{E}[Z^2]}{\mathbb{E}[Z]^2} \sim \sqrt{\frac{k-1}{k-d}}$
    Proof based on Laplace´s method for sums
  - **If** $\phi_2(\kappa) > 0$ for $\kappa \neq \kappa^*$, then $Z > 0$ whp
    Proof based on small subgraph conditioning
  - Showed for $k = 4$ that $\phi_2(\kappa) > 0$ for $\kappa \neq \kappa^*$
    Proof based on connection to contraction coefficient

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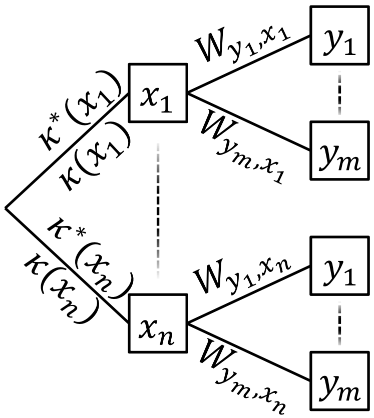
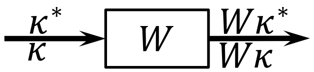
Contraction Coefficient

- **Contraction Coefficients**
  - Pair of random variables X^* and Y^*
  - $X^* \sim \kappa^*$ reference (input) distribution
  - Channel W with $W_{yx} = \mathbb{P}[Y^* = y | X^* = x]$
  - $Y^* \sim W\kappa^*$ reference (output) distribution
  - Contraction Coefficient

    $s^*(X^*, Y^*) = s^*(W, \kappa^*) = \sup_{\kappa \neq \kappa^*} \frac{D(W\kappa \| W\kappa^*)}{D(\kappa \| \kappa^*)}$





- **Applications**
  - $s^*(X^*, Y^*)$ is correlation coefficient for $X^*$and Y^*
    $\rightarrow 0 \leq s^*(X^*, Y^*) \leq 1$
  - Strengthens data processing inequality
    $\rightarrow D(W\kappa \parallel W\kappa^*) \leq s^*(W, \kappa^*) D(\kappa \parallel \kappa^*)$

Contraction Coefficient

- **Occupation Problem**
  - $X^* \in \binom{[k]}{r}^2$ uniformly random constraint assignment
  - $Y^* \in \{0,1\}^2$ corresponding edge distribution
  - $W_{yx} = \mathbb{P}[Y^* = y | X^* = x] = \frac{1}{k} \# y$´s in x
  - $s^*(X^*, Y^*) =$ maximum distance of Y to Y^* compared to distance of X to X^* under a variation of X keeping $Y|X \sim Y^*|X^*$ fixed

Proposition (Contraction Coefficient)

For any r, k the following statements are equivalent

- $\phi_2 \geq 0$ and $\phi_2(\kappa) = 0$ iff $\kappa = \kappa^*$ for all $d < d^*$
- $s^*(X^*, Y^*) = \frac{H\left({}^{r}/_{k}\right)}{\ln\binom{k}{r}}$

Contraction Coefficient

- **Computing the Contraction Coefficient**
  - Easy bound $s^* \geq \frac{D(W\kappa \| W\kappa^*)}{D(\kappa \| \kappa^*)} = \frac{H(r/_k)}{\ln\binom{k}{r}}$ for $\kappa(x,y) = \binom{k}{r}^{-1} \mathbb{1}\{x = y\}$
  - Image of W is one-dimensional
  - Explicit Minimization of $D(\kappa \parallel \kappa^*)$ for fixed $W\kappa = \rho$ possible
  - Maximization over remaining degree of freedom
    only solved for $r = 2$ and $k = 4$ using basic analysis and numerics
- **Agenda**
  - Identify and exploit information-theoretic results on contraction coefficients to show conjecture for arbitrary r and k
  - We know that $s^* = \frac{H(r/_k)}{\ln\binom{k}{r}}$ if and only if $\inf_\kappa \frac{H\left(\frac{1}{k}\sum_h \kappa_h\right)}{H(\kappa)} = \frac{H(r/_k)}{\ln\binom{k}{r}}$

    → Implications for subadditivity of entropy, i.e. $\inf_\kappa \frac{\sum_h \frac{1}{k} H(\kappa_h)}{H(\kappa)}$?

Thank you!