Topological Aspects of Random Graphs

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Guiding questions

(1) What is a typical genus of Erdős-Rényi random graph?

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* genus of a graph G is the minimum number of handles that must be attached to a sphere in order to be able to embed G without any crossing edges

(2) How does a topological constraint, such as being planar or imposing an upper bound on the genus, influence the component structure of a random graph?

Throughout the talk

- Let $G(n, m)$ denote a uniform random graph:

  a graph taken u.a.r. from the set $\mathcal{G}(n, m)$ of all graphs on vertex set $[\,n\,] := \{1, \ldots, n\}$ with $m = m(n)$ edges

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- All asymptotics are as $n \to \infty$

- whp = with probability tending to one as $n \to \infty$

Emergence of the giant

Let L_1 denote the largest component in $G(n, m)$.

Theorem [Erdős-Rényi 1959–60]

- If $\frac{2m}{n} < 1$, then whp $|L_1| = O(\log n)$
- If $\frac{2m}{n} > 1$, then whp $|L_1| = \Theta(n)$

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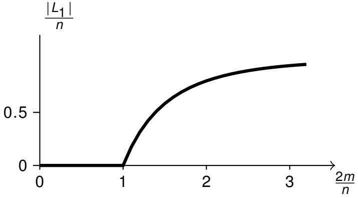

Let $d := \frac{2m}{n} > 1$. Then whp

$$|L_1| = (1 + o(1))\ \rho n$$

where $1 - \rho = \exp(-d\rho)$

(via coupling of BFS with Galton Watson process with $\mathrm{Po}(d)$)

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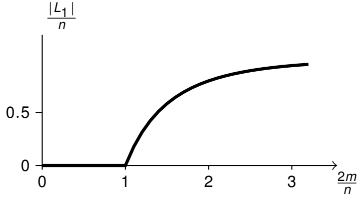
Let L_1 denote the largest component in $G(n, m)$.

Theorem [Erdős-Rényi 1959–60]

- If $\frac{2m}{n} < 1$, then whp $|L_1| = O(\log n)$ and L_1 is planar.
- If $\frac{2m}{n} > 1$, then whp $|L_1| = \Theta(n)$ and L_1 is not planar.



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Random planar graphs

Let $P(n, m)$ denote a random planar graph:

> a graph taken uniformly at random from the set $\mathcal{P}(n, m)$ of all graphs on vertex set $[\,n\,]$ with $m = m(n)$ edges that are embeddable on the sphere without crossing edges

Random planar graph $P(n, m)$

Let L_1 denote the largest component in $P(n, m)$.

Theorem [K.–Łuczak 2012]

- If $\frac{2m}{n} < 1$, then whp $|L_1| = O(\log n)$.
- If $\frac{2m}{n} \to d \in (1, 2)$, then whp
$$|L_1| = (1 + o(1))\,(d - 1)n.$$
- If $\frac{2m}{n} \to d \in [2, 3]$, then whp
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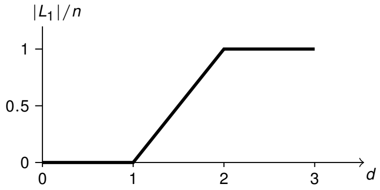
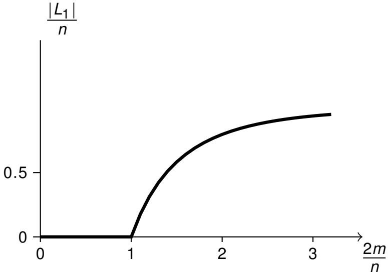
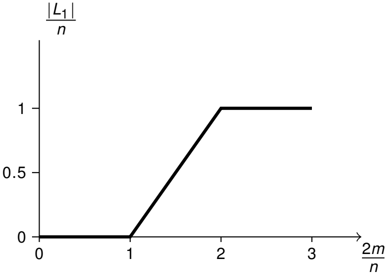


Critical phases



Uniform random graph $G(n, m)$



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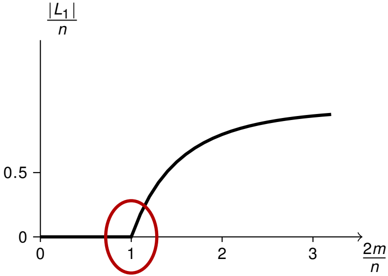

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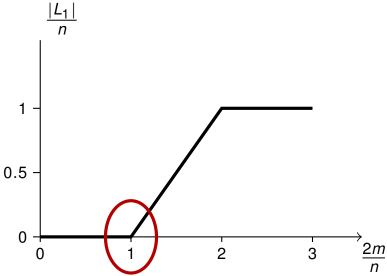

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Weakly supercritical random graphs

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Uniform random graph $G(n, m)$ [BOLLOBÁS 84; ŁUCZAK 90]

whp $$|L_1| = (4 + o(1))\, s$$

Random planar graph $P(n, m)$ [K.–ŁUCZAK 2012]

whp $$|L_1| = (2 + o(1))\, s$$

Random graph on a surface

Random graph $S_g(n, m)$ on an orientable surface:

a graph taken uniformly at random from the set $\mathcal{S}_g(n, m)$ of all graphs on $[n]$ with m edges and with genus $\leq g$

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For which $g = g(n)$, are $S_g(n,m)$ and $G(n,m)$ contiguous (indistinguishable under viewpoint of whp-properties)?

Genus of weakly supercritical $G(n, m)$

Let $m = \frac{n}{2} + s$ for $s > 0$, $n^{2/3} \ll s \ll n$.

Let g denote the genus of $G(n, m)$.

Theorem [Dowden–K.–Krivelevich 2019]

whp
$$g = (1 + o(1)) \frac{8s^3}{3n^2}.$$

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$$g = (1 + o(1))\frac{8s^3}{3n^2}.$$

For every $\varepsilon > 0$,

- if $g \geq (1+\varepsilon)\frac{8s^3}{3n^2}$, $G(n, m)$ and $S_g(n, m)$ are contiguous
  ($\forall$ a property $\mathcal{P}$, whp G satisfies $\mathcal{P}$ iff whp S_g satisfies $\mathcal{P}$);
- if $g \leq (1-\varepsilon)\frac{8s^3}{3n^2}$, $G(n, m)$ and $S_g(n, m)$ are not contiguous.

$\Longrightarrow$ $T = \frac{8s^3}{3n^2}$ is called the contiguity threshold.

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Largest component in weakly supercritical $S_g(n,m)$

Let $m = \frac{n}{2} + s$ for $s > 0$, $n^{2/3} \ll s \ll n$ and let $T = \frac{8s^3}{3n^2}$.

Let L_1 denote the largest component in $S_g(n,m)$.

Theorem [DOWDEN–K.–MOSSHAMMER–SPRÜSSEL 2019+]

whp

- $|L_1| = (4 + o(1))\, s \qquad$ *if* $\quad g \geq (1 + o(1))T$

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Theorem [Dowden–K.–Mosshammer–Sprüssel 2019+]

whp

- $|L_1| = (4 + o(1))\, s$ *if* $g \geq (1 + o(1))T$
- $|L_1| = (f(c) + o(1))\, s$ *if* $g = (c + o(1))T$ *for* $c \in (0,1)$
- $|L_1| = (2 + o(1))\, s$ *if* $g = o(T)$

where $f(c) \to 2$ as $c \to 0$ and $f(c) \to 4$ as $c \to 1$.

Genus of supercritical $G(n, m)$

Let $\frac{2m}{n} \to d > 1$ and g denote the genus of $G(n, m)$.

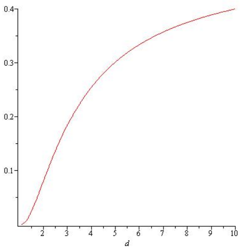
Theorem [Dowden–K.–Krivelevich 2019]

whp

$$g = (1 + o(1))\,\mu(d)\,\frac{dn}{2},$$

where $\mu(d) \to 0$ as $d \to 1$ and $\mu(d) \to \frac{1}{2}$ as $d \to \infty$.

$g/m \sim \mu(d)$



Largest component L_1 in supercritical $S_g(n,m)$

Theorem [DOWDEN–K.–MOSSHAMMER–SPRÜSSEL 2019+]

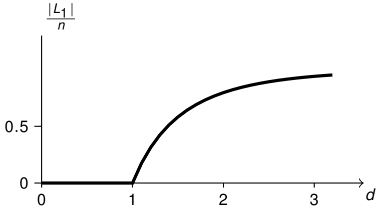
Assume $\frac{2m}{n} \to d > 1$ and $g \gg n$

(so $S_g(n,m)$ and $G(n,m)$ are contiguous).

Then whp

$$|L_1| = (1 + o(1))\,\rho\, n,$$

where $1 - \rho = \exp(-d\,\rho)$.



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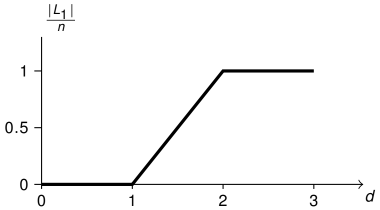
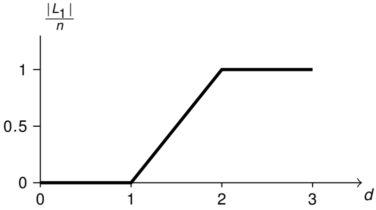
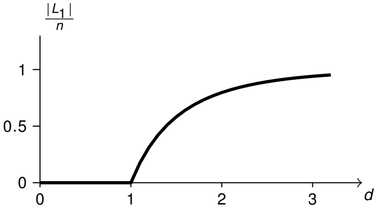


Summary

Largest component L_1 in $S_g(n, m)$ with $\frac{2m}{n} \rightarrow d > 1$.



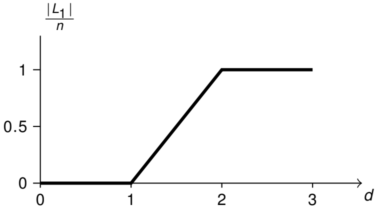
$S_g(n, m)$ for $g \ll n$



$S_g(n, m)$ for $g \gg n$

Summary & open problem

Largest component L_1 in $S_g(n,m)$ with $\frac{2m}{n} \to d > 1$.



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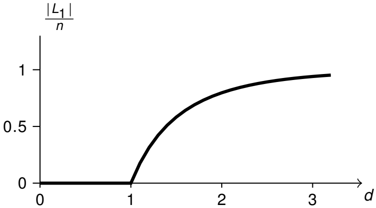

$S_g(n,m)$ for $g \gg n$

$\Longrightarrow$ behaviour of largest component L_1 when $g = \Theta(n)$?