Local limits of high genus triangulations

Baptiste Louf (IRIF Paris Diderot)

joint work with Thomas Budzinski

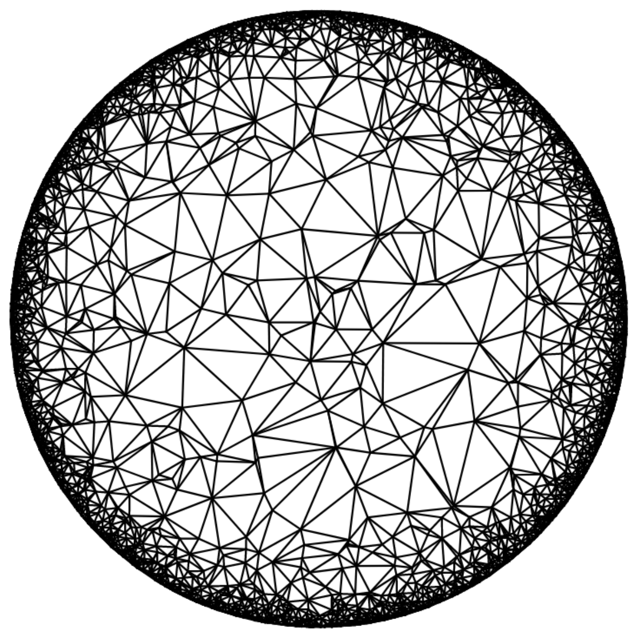
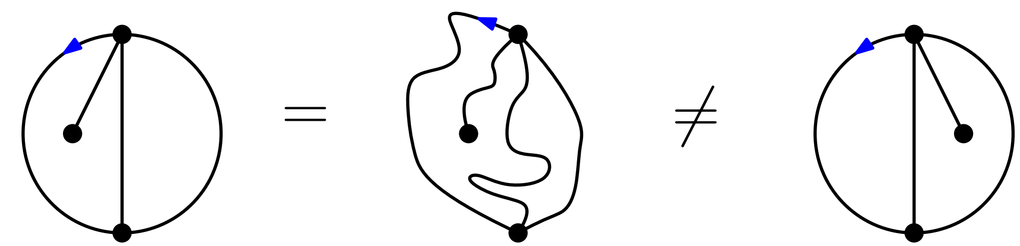


image : N. Curien

Work supported by the grant ERC – Stg 716083 – “CombiTop”

Definition : maps and triangulations

Map = embedding up to homeomorphism of a connected multigraph (loops and multiple edges allowed) in a compact connected orientable surface.
Rooted = an oriented edge is distinguished



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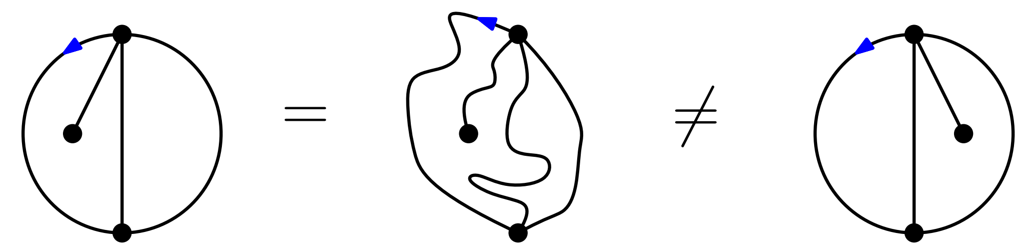
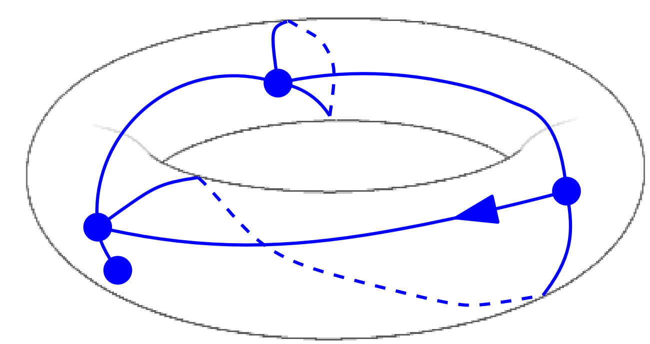


Genus g of the map = genus of the surface = # of handles



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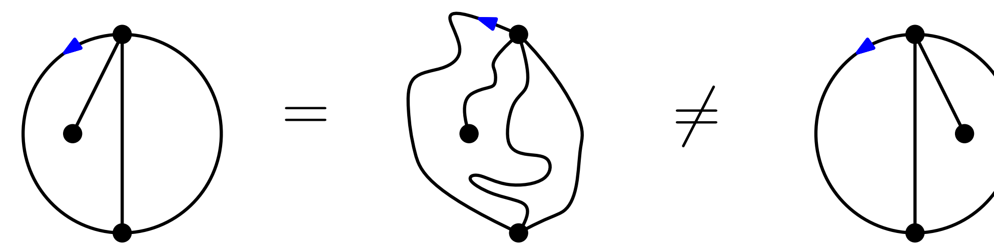

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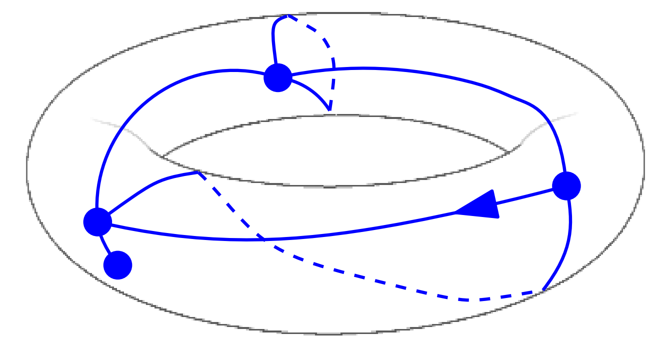
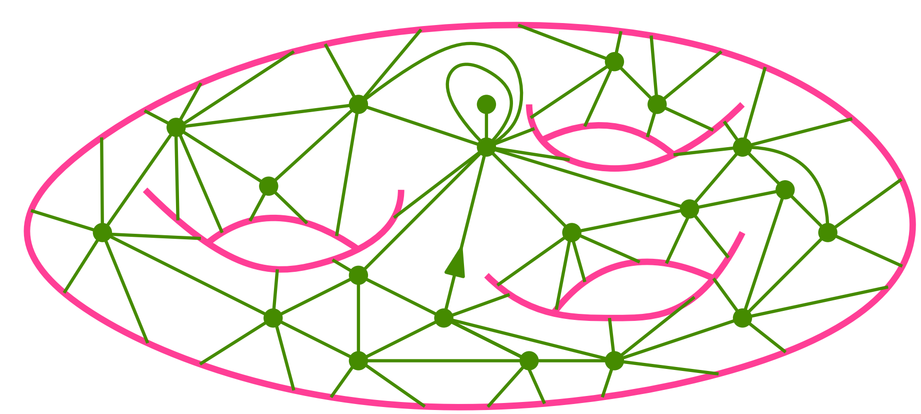


Triangulation $=$ all faces of degree 3



What does a triangulation look like around the root ?

How similar are two triangulations locally ?

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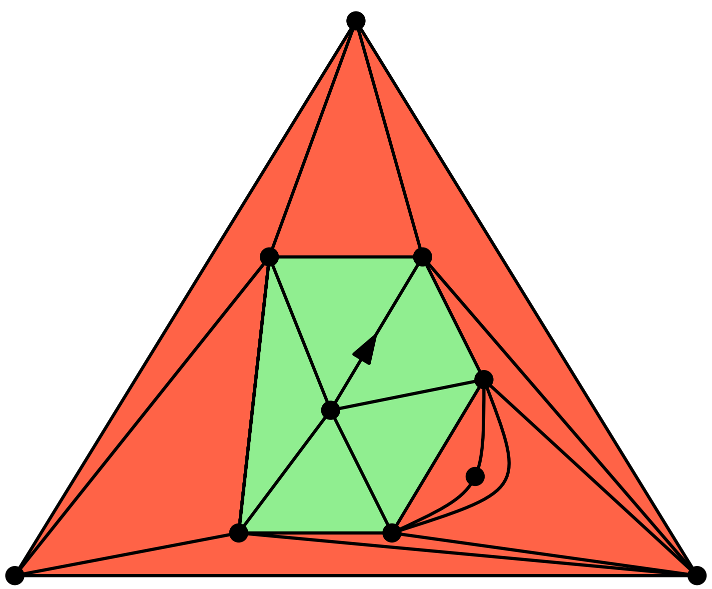
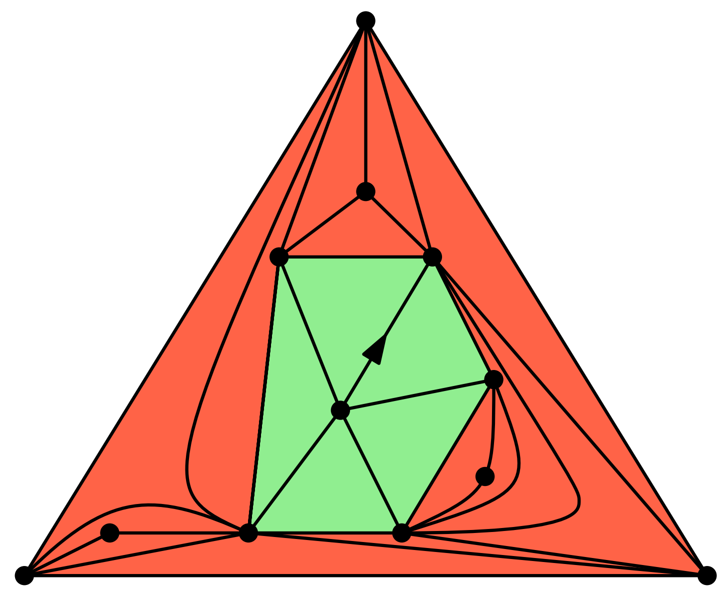
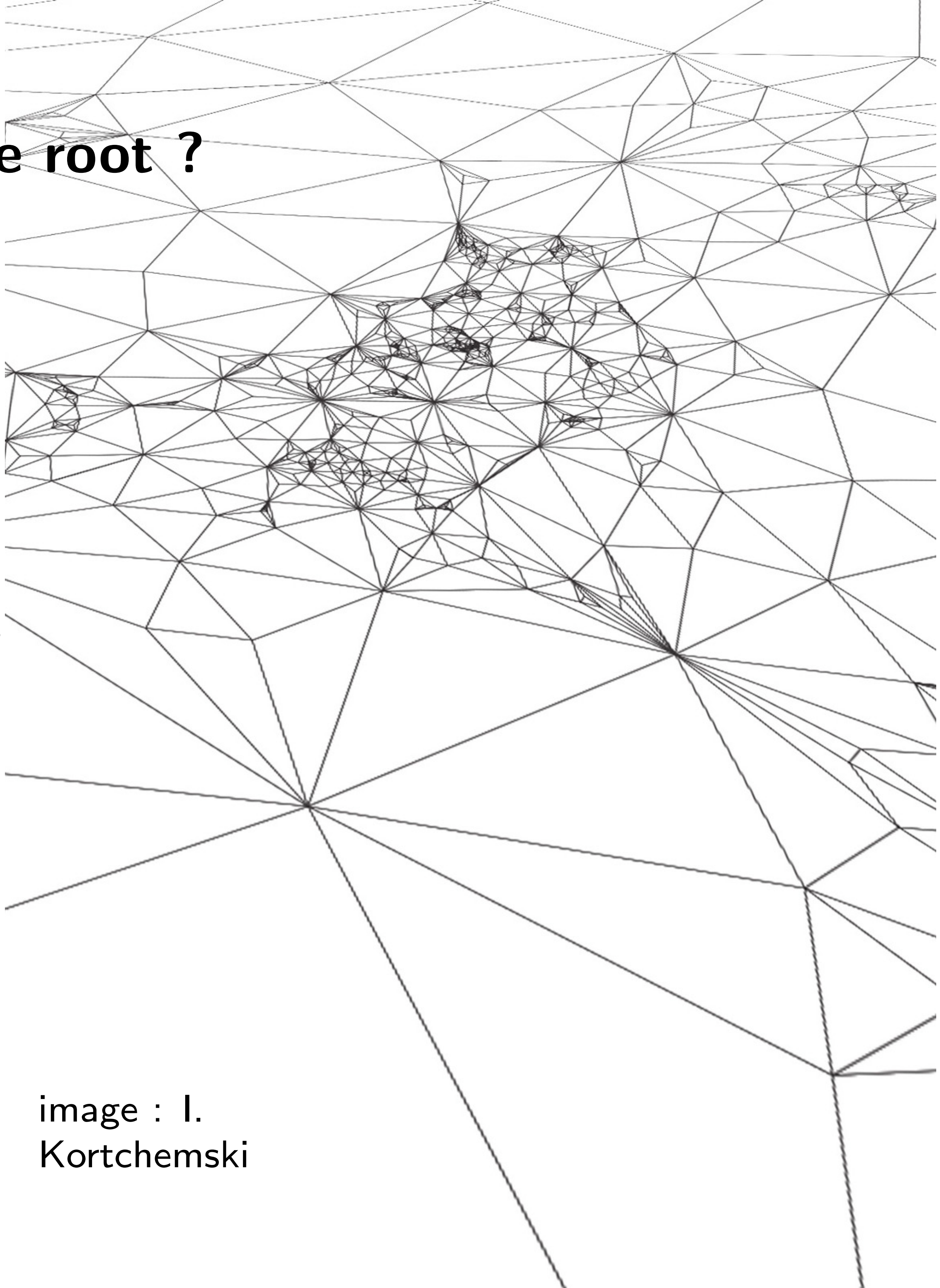
The local distance :
$d_{loc}(T, T') = (1 + \sup\{r | B_r(T) = B_r(T')\})^{-1}$





What does a (large, random) triangulation look like around the root ?



The limit (in law) w.r.t d_{loc} is called the local limit (= convergence (in law) of the balls of radius r).

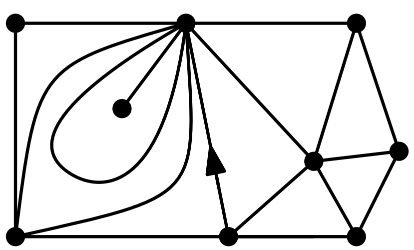
Question : let (T_n) be a sequence of random triangulations, whose size $\to \infty$, is there a local limit ? What does it look like ?

[Angel, Schramm '02] : uniform planar triangulations converge to an infinite triangulation called the Uniform Infinite Planar Triangulation (UIPT).

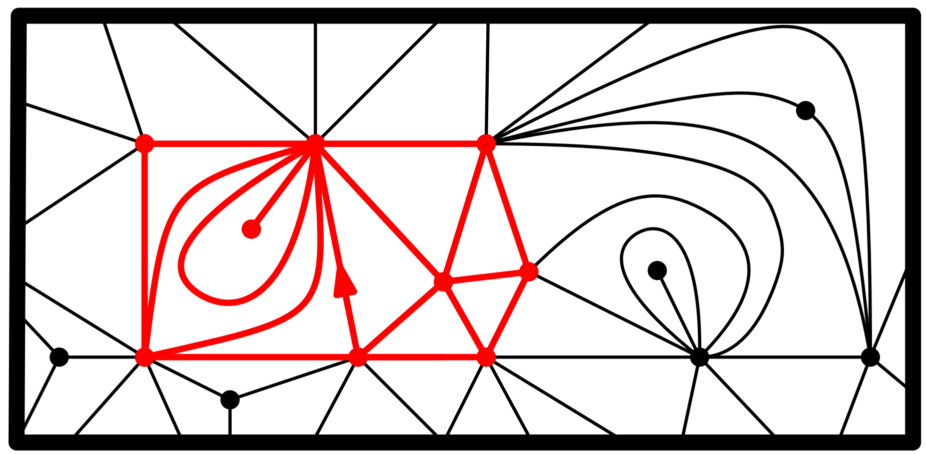
image : I. Kortchemski

Properties of the UIPT

Spatial Markov property : $\mathbb{P}(t \subset \mathbb{T}) = C_p \lambda_c^{|v|}$



$\subset$



$p = 7,\ |v| = 9$

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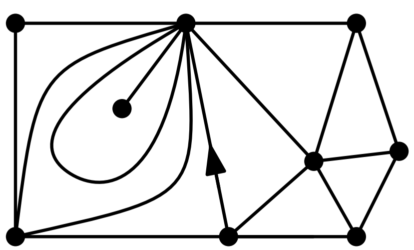

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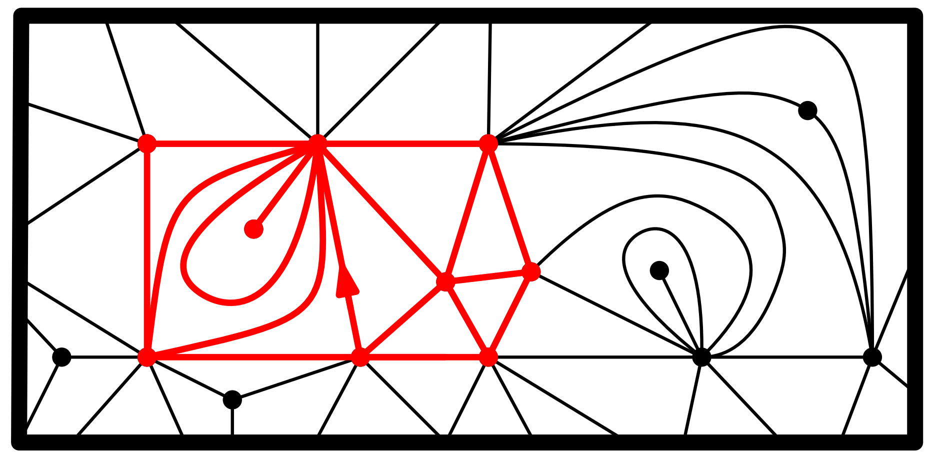

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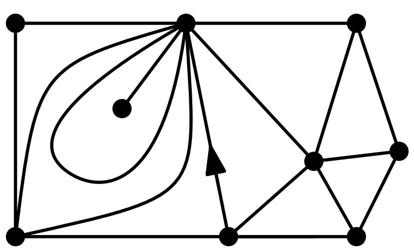

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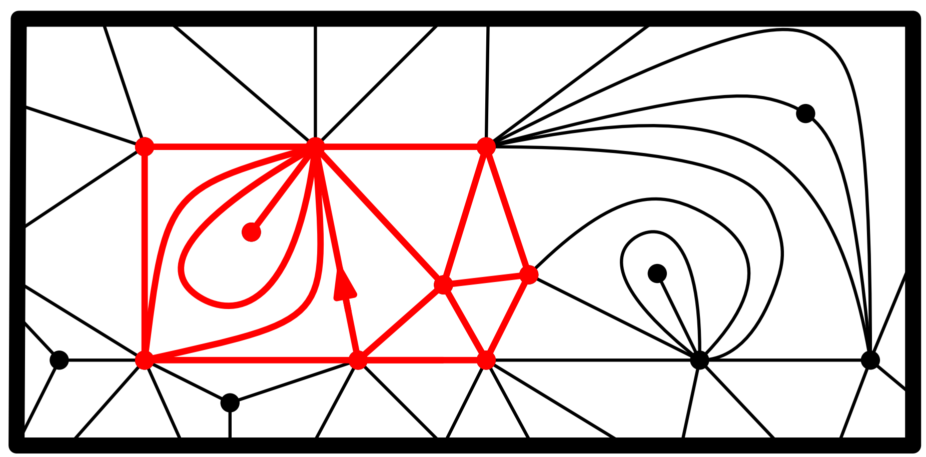
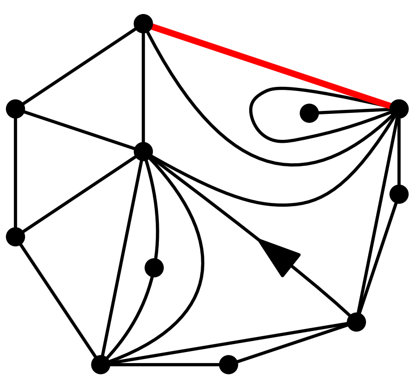


λ_c = rcv of the series of planar triangulations !

Peeling process : Discover $\mathbb{T}$ step by step, unveil triangles.



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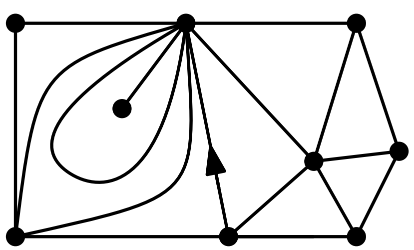

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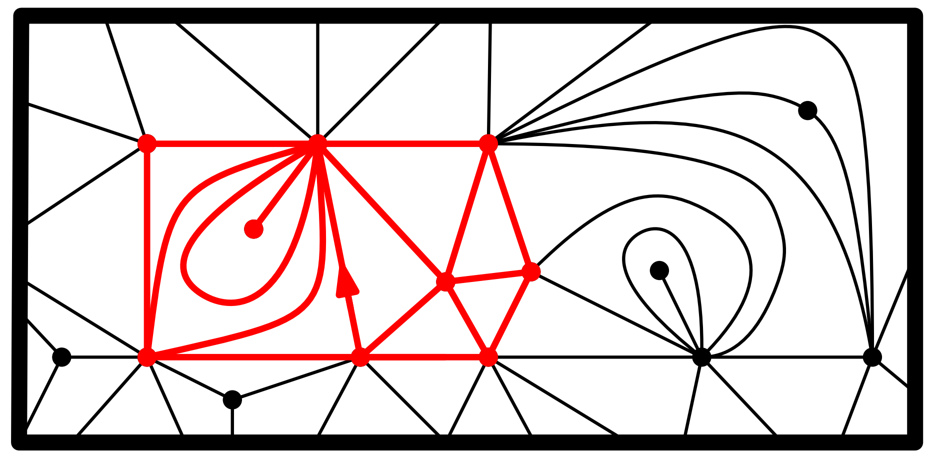

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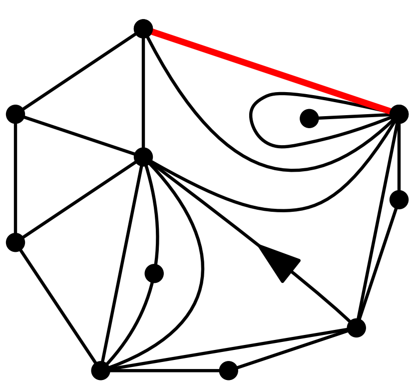

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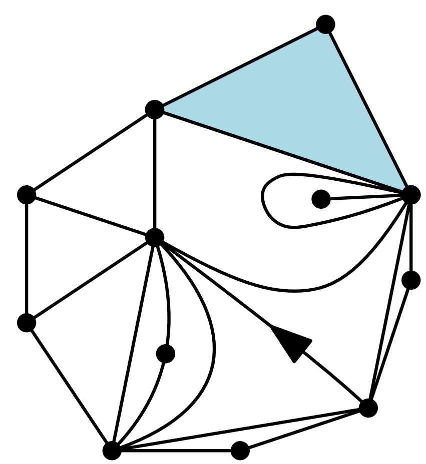

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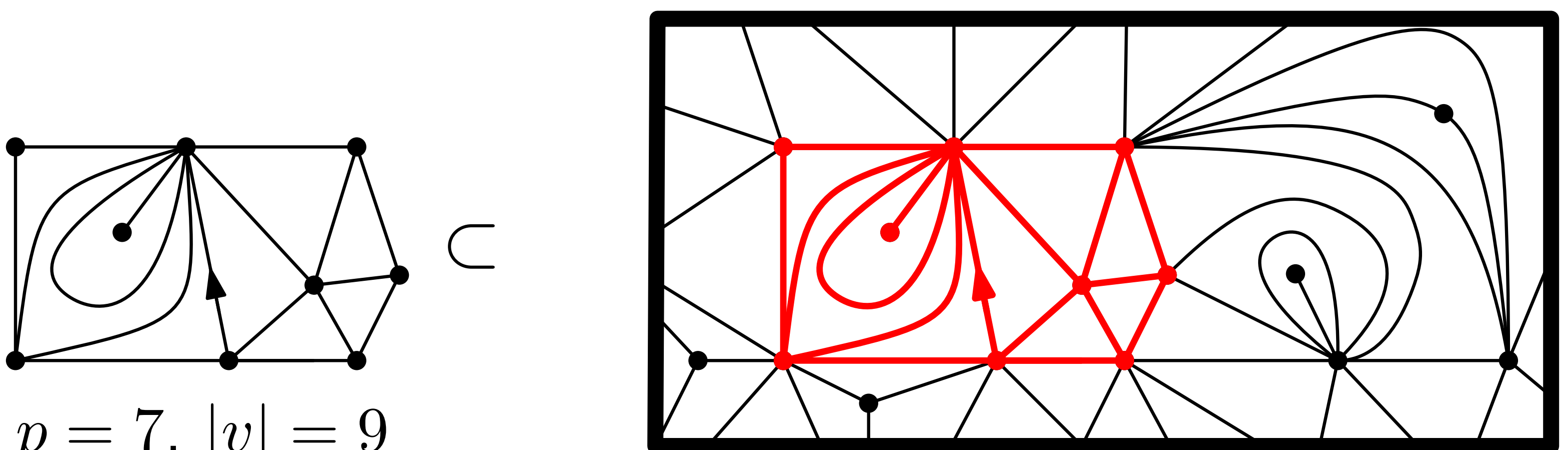

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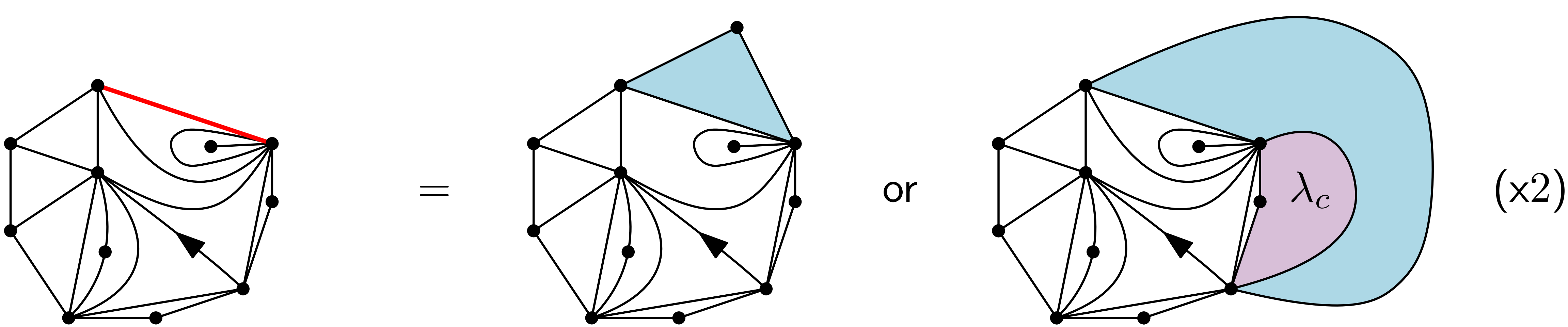


The PSHIT

Introduced by Curien in 2012

Defined in the same way as the UIPT, by with $\lambda \in]0, \lambda_c]$.

For $\lambda < \lambda_c$, has an hyperbolic flavour : the "average degree" of a vertex is higher than 6 (the value in a regular planar triangulation), the balls have exponential growth, ...

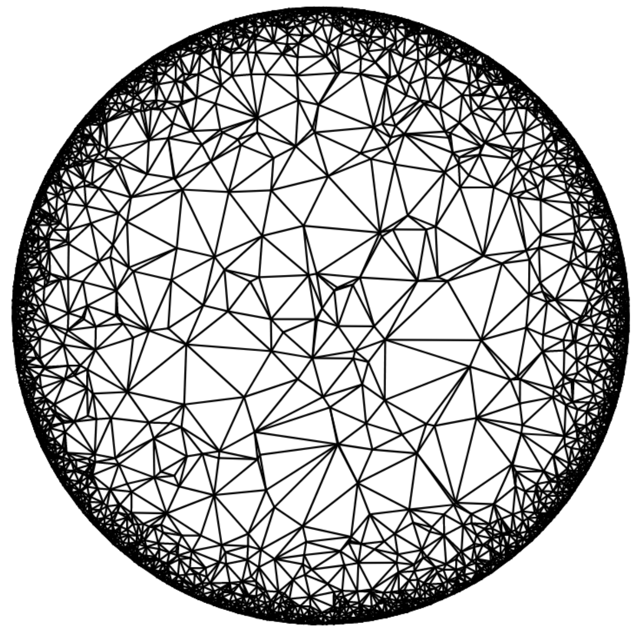


image : N. Curien

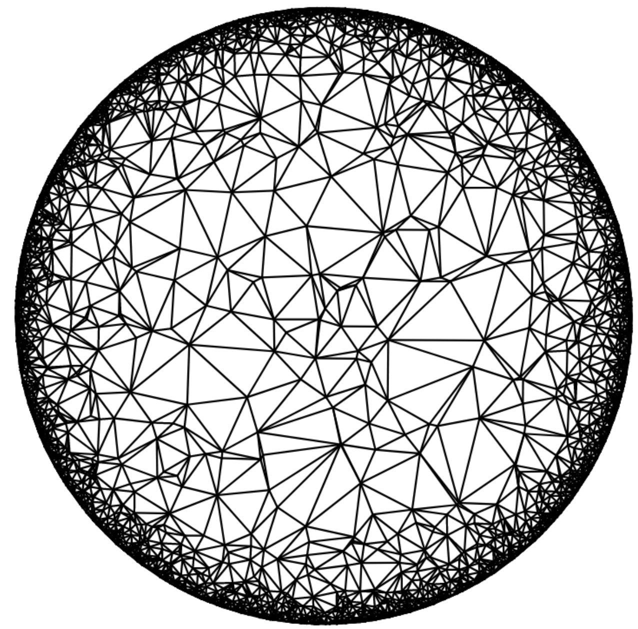
Question : Can the PSHITs be interpreted as local limits ?

A conjecture

Let $\frac{g_n}{n} \to \theta$ with $\theta \in [0, \frac{1}{2}[$.
Let (T_n) be a sequence of random triangulations, such that T_n is drawn uniformly among all triangulations of genus g_n with $2n$ triangles.

Conjecture [Benjamini, Curien '12] : (T_n) has a local limit, and it is a PSHIT of parameter λ, with λ a function of θ.

For g_n constant, the limit is the UIPT (well known,but never written anywhere).



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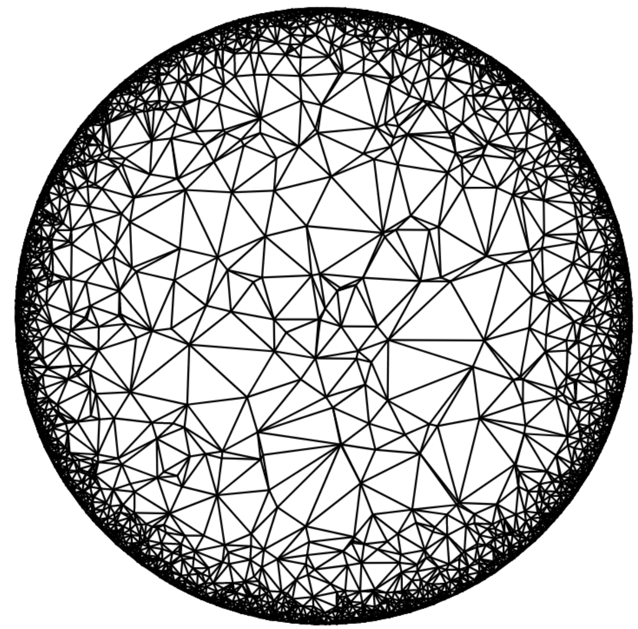

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A similar result [Angel, Chapuy, Curien, Ray '13] : the local limit of one-faced maps of high genus is an infinite hyperbolic tree

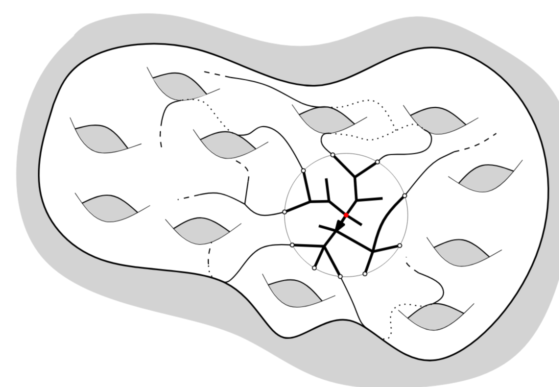
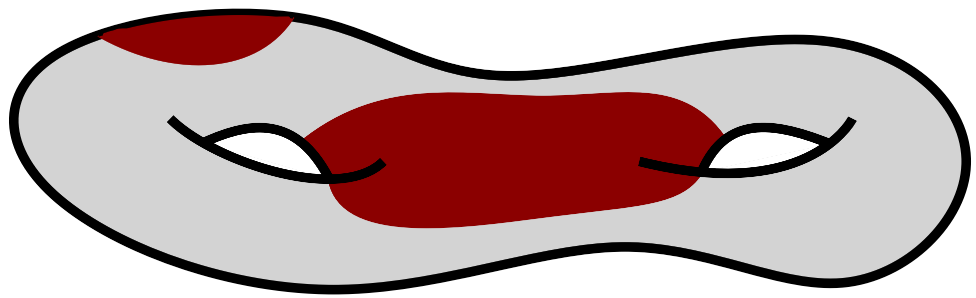


image : ACCR

The intuition behind the conjecture

For fixed genus . . .



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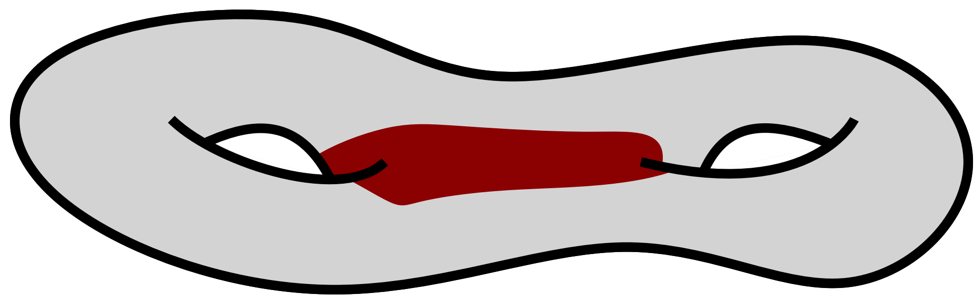

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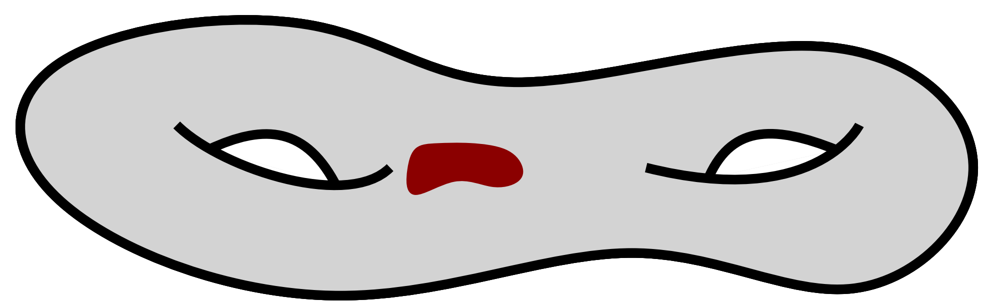

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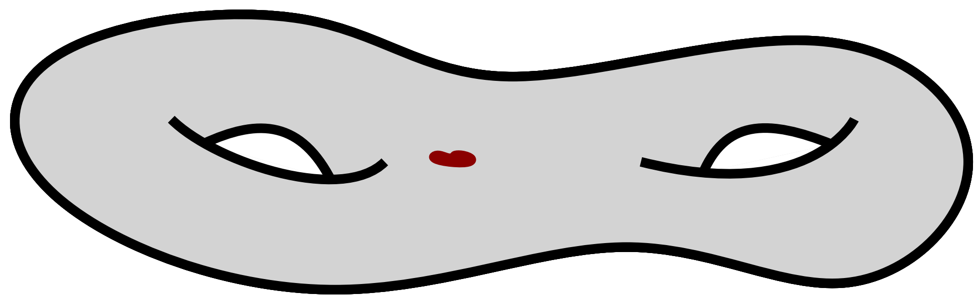

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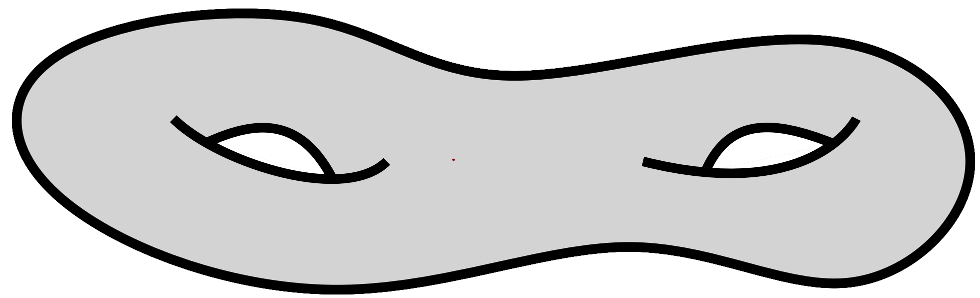

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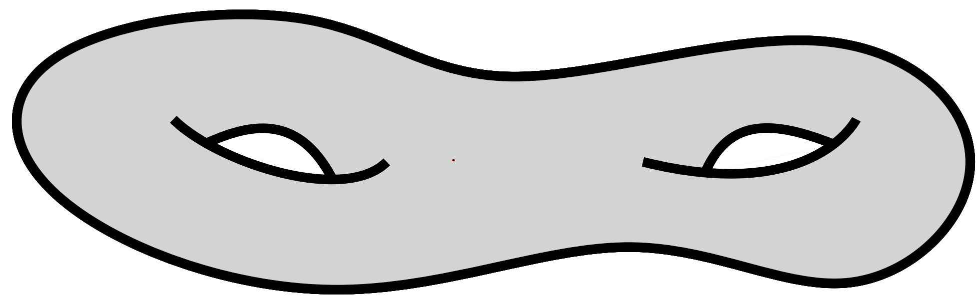

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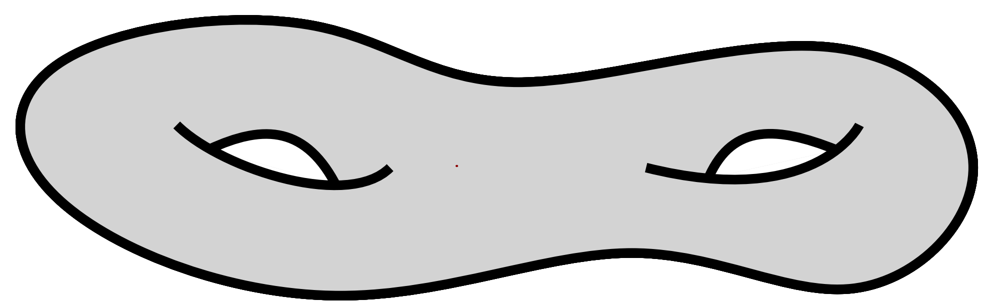

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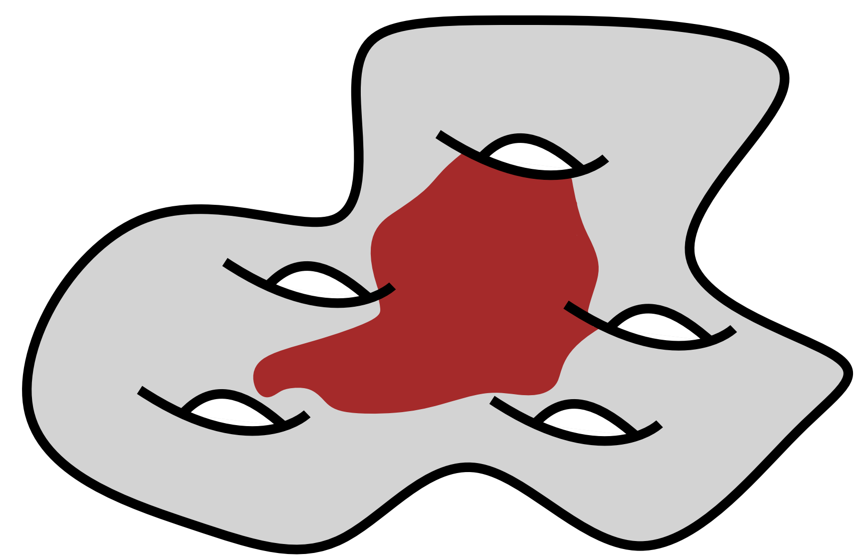
When the genus increases linearly with the size, in the end we don't see the genus but we still "feel the curvature"



Our result

Theorem [Budzinski, L. '18+] : the conjecture of Benjamini and Curien is true.

Let's get to (a part of) the proof !

First idea :

Obtain precise asymptotics for $\tau(n,g)$ (the number of maps of genus g with $2n$ triangles) as $\frac{g}{n} \to \theta$

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$\Big\downarrow$

TOO HARD

Outline of the proof :

1) Tightness ($+$ planarity and one-endedness)
$\rightarrow$ every subsequence has a converging subsubsequence

2) Every possible limit is a PSHIT with random parameter Λ

3) Λ is deterministic and depends only on θ

Tightness : how do we prove it ?

Two main ingredients :

• The bounded ratio lemma : for $\frac{g}{n} < \frac{1}{2} - \varepsilon$, there is a constant C_ε s.t. :

$$\frac{\tau(n,g)}{\tau(n-1,g)} < C_\varepsilon$$

• Planarity and one-endedness

The bounded ratio lemma

For $\frac{g}{n} < \frac{1}{2} - \varepsilon$, $\frac{\tau(n,g)}{\tau(n-1,g)} < C_\varepsilon$

There are $n + 2 - 2g \geq 2\varepsilon n$ vertices, and the average degree is $\frac{6n}{n+2-2g} \leq \frac{3}{\varepsilon}$
$\rightarrow$ there are $\geq \varepsilon n$ vertices of degree $\leq \frac{6}{\varepsilon}$

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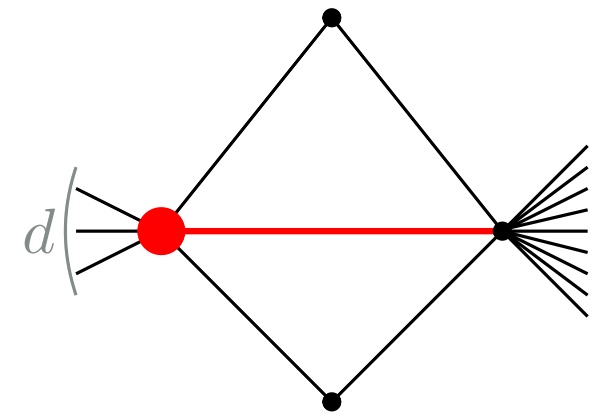

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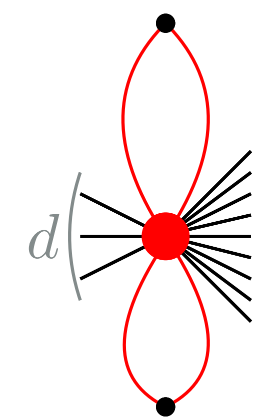

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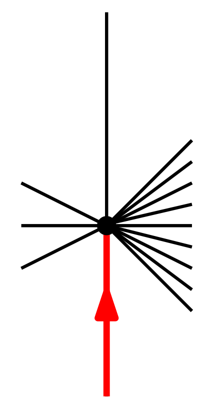
Remember $\deg(v)$ and an oriented edge ...



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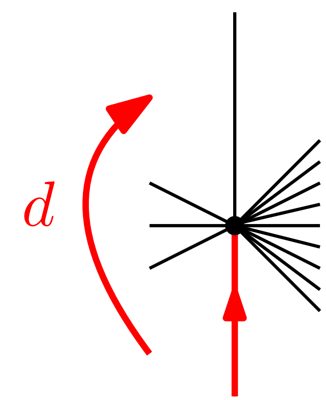

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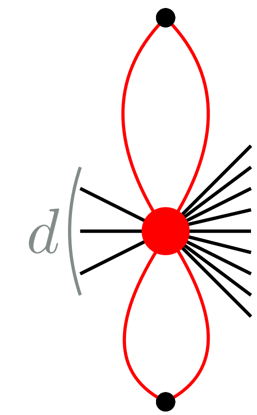

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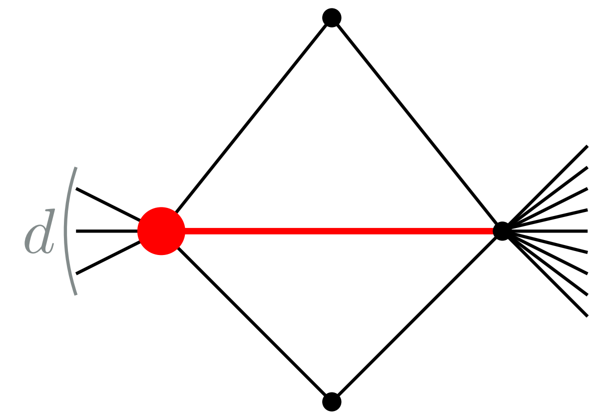

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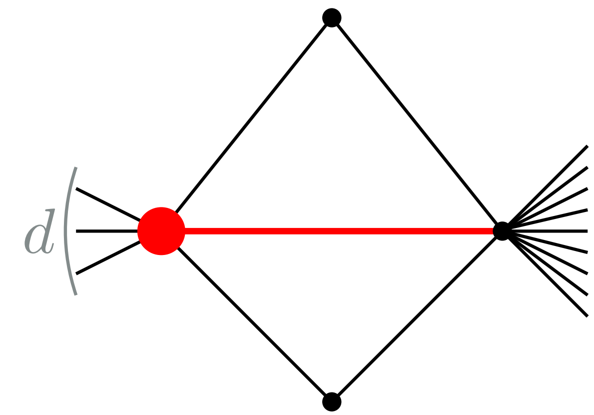

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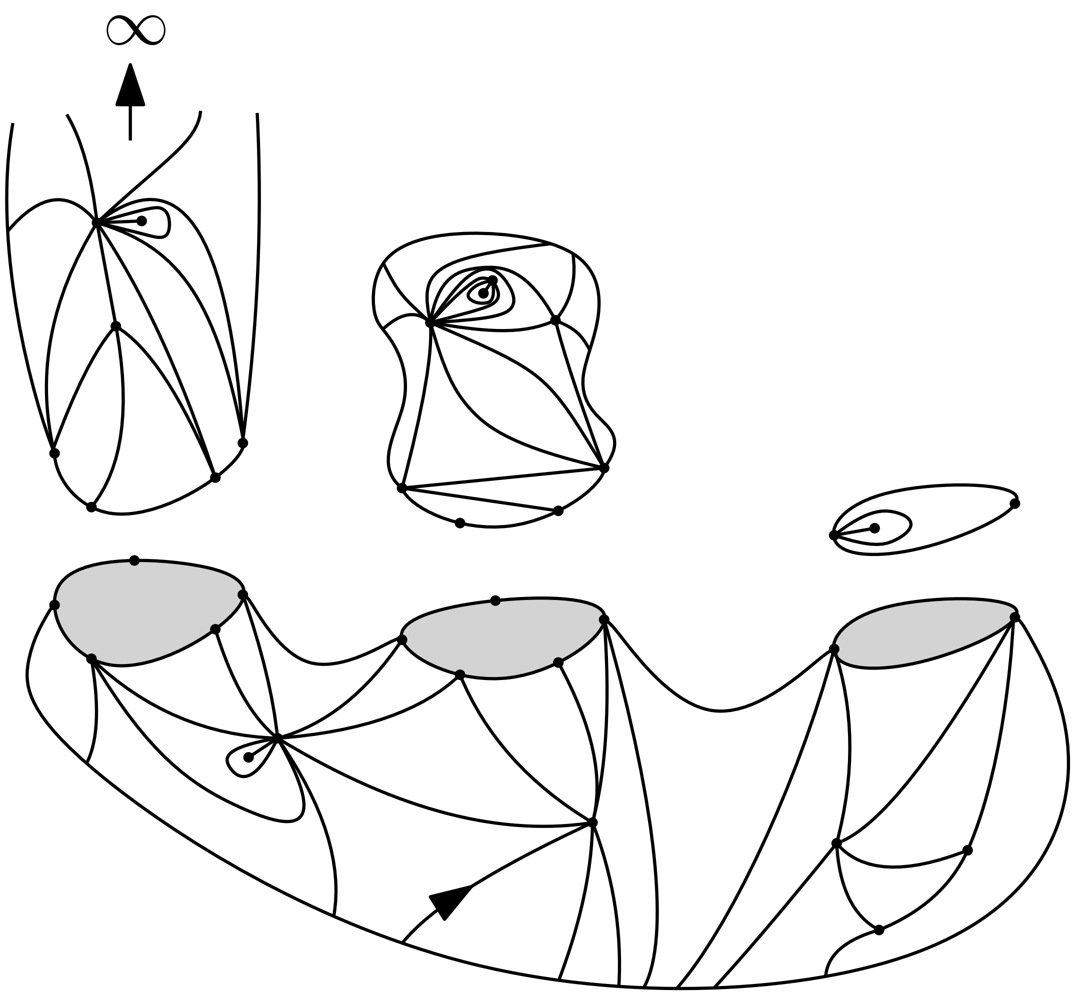
Thus $\varepsilon n \tau(n, g) \leq \frac{6}{\varepsilon} \cdot 6n\tau(n-1, g)$

One-endedness and planarity



Proven by using the Goulden–Jackson formula (and the bounded ratio lemma) :

$$\boxed{\begin{aligned}(n+1)\tau(n,g) = {} & 4n(3n-2)(3n-4)\tau(n-2,g-1) + 4(3n-1)\tau(n-1,g) \\ & +4\sum_{i+j=n-2}\sum_{g_1+g_2=g}(3i+2)(3j+2)\tau(i,g_1)\tau(j,g_2)\end{aligned}}$$

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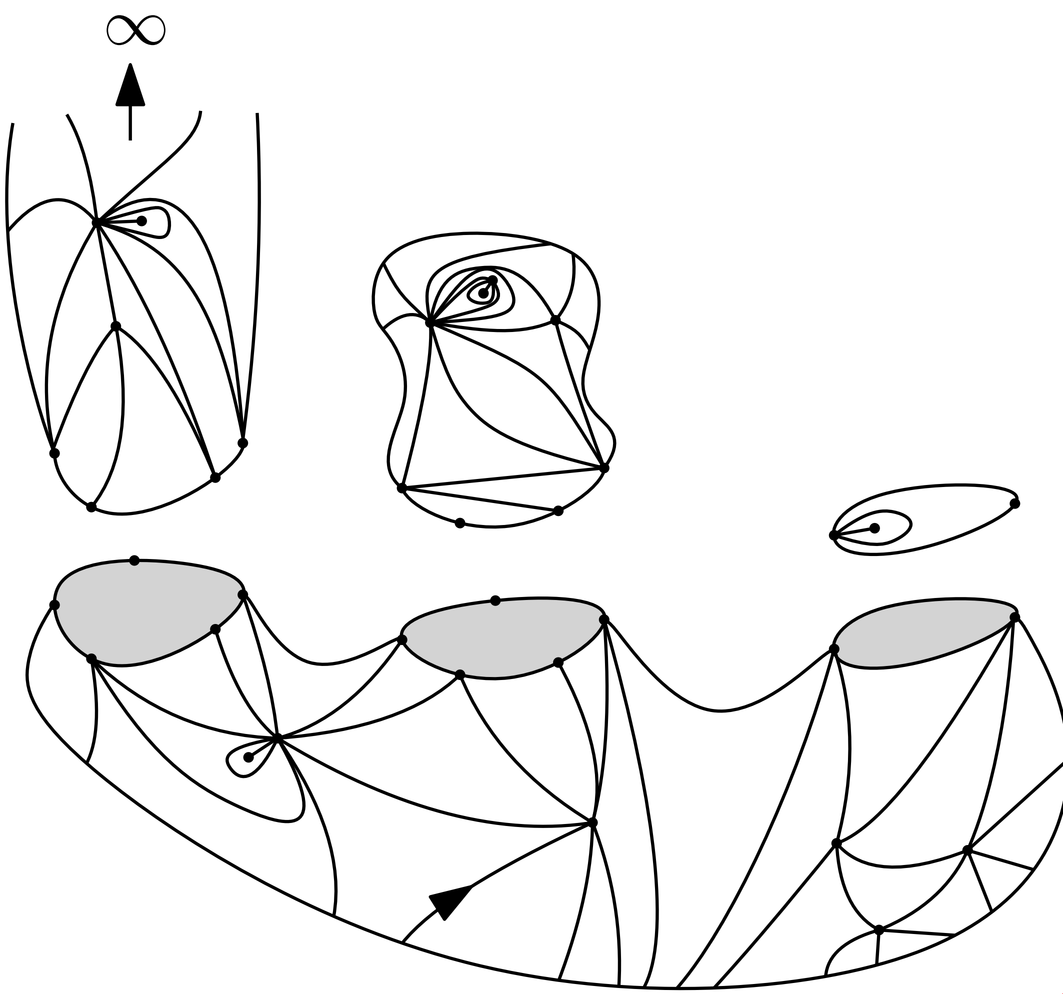

Proven using algebraic properties of maps and the KP hierarchy

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(Tutte equation is not enough)

to be continued ...

Bonus : asymptotics

$$\frac{\tau(n, g_n)}{\tau(n-1, g_n)} \to c(\theta)$$

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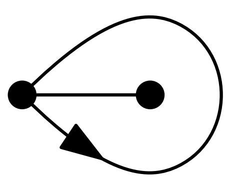
Proof :

$\mathbb{P}($  $\in \mathbb{T})$

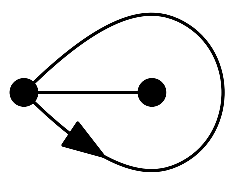
$= C_1 \lambda^2$ in PSHITs

$= \frac{\tau(n-1, g_n)}{\tau(n, g_n)}$ in finite maps

Bonus : asymptotics

$$\frac{\tau(n, g_n)}{\tau(n-1, g_n)} \to c(\theta)$$

Proof :

$\mathbb{P}($  $\in \mathbb{T})$

$= C_1\lambda^2$ in PSHITs

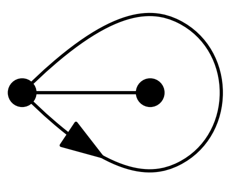
$= \frac{\tau(n-1,g_n)}{\tau(n,g_n)}$ in finite maps

(exercise !)

Bonus : asymptotics

$$\frac{\tau(n, g_n)}{\tau(n-1, g_n)} \to c(\theta)$$

Proof :

$\mathbb{P}($  $\in \mathbb{T})$

$= C_1 \lambda^2$ in PSHITs

$= \frac{\tau(n-1, g_n)}{\tau(n, g_n)}$ in finite maps

(exercise !)

$$\tau(n, g_n) = n^{2g_n} \exp(nf(\theta) + o(n))$$

What's next ?

Boltzmann maps (arbitrary face degrees)

$\rightarrow$ requires a formula similar to Goulden–Jackson
but for bipartite maps with prescribed degrees
([L. 18+])

More info on high genus triangulations (diameter, . . .)

Maps decorated with "matter" ?

What happens when $\frac{g}{n} \rightarrow \frac{1}{2}$?

Thank you !