Online Strategies for Selecting a Long Increasing Subsequence of a Random Sample

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Ulam's problem

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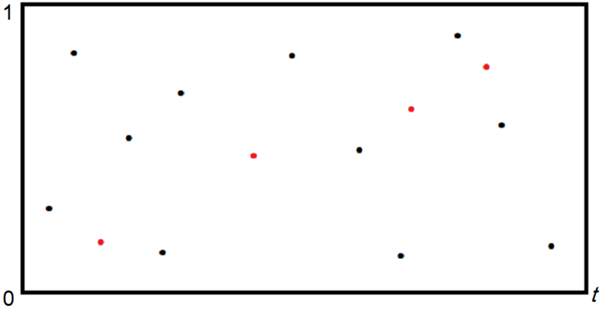
Ulam 1961: *What is the length of the longest increasing subsequence of a random permutation of n integers?*

Instead of permutation, one can consider a sequence of i.i.d. random marks $X_1, \ldots, X_n$ sampled from the uniform-$[0, 1]$ (or any other continuous distribution).

Hammersley 1972 (poissonisation): the marks arrive by a Poisson process on $[0, t]$, so the sample size is random with Poisson(t)-distribution. A increasing subsequence $(x_1, s_2), \ldots, (x_k, s_k)$ of marks/arrival times is a chain in two dimensions: $x_1 < \cdots < x_k, \;\; s_1 < \cdots < s_k$ where each (x_i, s_i) is an atom of the planar Poisson process in $[0, t] \times [0, 1]$



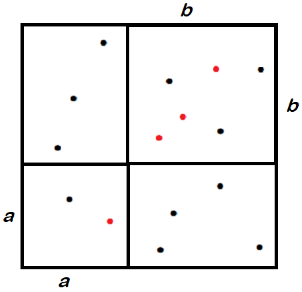
Hammersley: since only the area of rectangle matters, the maximum length $M(t)$ satisfies

$$M((a+b)^2) \geq M(a^2) + M(b^2),$$

hence by subadditivity

$$M(t) \sim c\sqrt{t},\ t \to \infty$$

(in probability and in the mean).



Logan and Shepp 1977, Vershik and Kerov 1977:

$$\mathbb{E}M(t) \sim 2\sqrt{t}, \quad t \to \infty.$$

Baik, Deift and Johansson 1999:

$$\frac{M(t) - 2\sqrt{t}}{t^{1/6}} \xrightarrow{d} \text{Tracy–Widom distribution.}$$

D. Romik 2014: The Surprising Mathematics of Longest Increasing Subsequences.

The online selection problem

Samuels and Steele 1981: The marks are revealed to the observer one-by-one as they arrive by Poisson process on $[0, t]$. Each time a mark is observed, it can be selected or rejected, with decision becoming immediately final. The sequence of selected marks must increase.

The problem is to maximise the expected length of increasing sequence selected by a nonanticipating online strategy.

By subadditivity, the maximum expected length satisfies

$$v(t) \sim c\sqrt{t}, \qquad t \to \infty$$

with some $c > 0$.

The exact solution is possible for small t:

$$v(t) = \int_0^t \frac{1 - e^{-s}}{s} ds, \qquad t < 1.345\cdots,$$

when the optimal strategy is *greedy*, that is selects every consecutive *record* (i.e. a mark bigger than all seen so far). But for large t this is only $\sim \log t$ and too far from optimality.

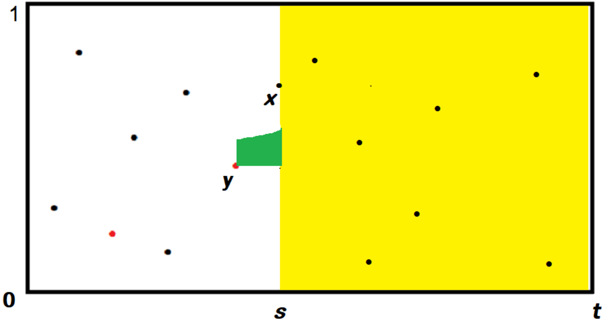
It is sufficient to consider strategies of the kind: if (x, s) is observed and the maximum so far selected mark is y, then x is chosen iff

$$0 < x - y \le \psi(s, t, y)$$

where the function ψ determines the size of the moving acceptance window. In the case

$$\psi(s, t, y) = (1 - y)\varphi((t - s)(1 - y))$$

(for some φ) the strategy is called *self-similar*.



The leading asymptotics

Samuels and Steele 1981:

$$v(t) \sim \sqrt{2t}, \quad t \to \infty,$$

achieved by the strategy with *constant* acceptance window

$$0 < x - y < \sqrt{\frac{2}{t}},$$

where $(x, s) \in [0, 1] \times [0, t]$ is the current arrival, and y the last mark selected before time s.
Under this strategy, the selected sequence grows about linearly from $(0, 0)$ to $(1, t)$.

The optimality equation

The maximal expected length satisfies the dynamic programming equation

$$v'(t) = \int_0^1 \{v(t(1-x)) + 1 - v(t)\}_+ \, dx, \qquad v(0) = 0.$$

Under the optimal strategy (x, s) is accepted iff

$$0 < \frac{x - y}{1 - x} < \varphi^*((t - s)(1 - x))$$

where y is the last selection and $\varphi^*(t)$ is the solution to

$$v(t(1-x)) + 1 - v(t) = 0$$

(for $t > v^{\leftarrow}(1) = 1.345\ldots$).

The tightest known bounds

$$\sqrt{2t} - \log(1+\sqrt{2t}) + c_0 < v(t) < \sqrt{2t}$$

The upper bound: Baryshnikov and G 2000 by comparing with the following bin-packing problem

$$\text{the expected number of choices} \to \max$$

subject to the constraint that

$$\text{the } \textit{expectation of the sum} \text{ of selected marks} \le 1.$$

The lower bound: Bruss and Delbaen 2001, using concavity of $v(t)$ and the optimality equation.

The asymptotic expansion

Let $L_\varphi(t)$ be the length of selected subsequence under the strategy with control function φ, in particular $v(t) = \mathbb{E} L_{\varphi^*}(t)$.

Theorem. *The expected length under the optimal strategy is*

$$v(t) \sim \sqrt{2t} - \frac{1}{12}\log t + c^* + \frac{\sqrt{2}}{144\sqrt{t}} + O(t^{-1})$$

and the variance is

$$\mathrm{Var}[L_{\varphi^*}(t)] = \frac{\sqrt{2t}}{3} + \frac{1}{72}\log t + c_1 + O(t^{-1/2}\log t).$$

The optimal strategy is self-similar with

$$\varphi^*(t) \sim \sqrt{\frac{2}{t}} - \frac{1}{3t} + O(t^{-3/2}).$$

Constants c^*, c_1 are unknown.

Theorem *For every self-similar selection strategy with*

$$\varphi(t) = \sqrt{\frac{2}{t}} + O(t^{-1})$$

the expected length of increasing subsequence is within $O(1)$ *from the optimum, and the CLT holds*

$$\sqrt{3}\,\frac{L_\varphi(t) - \sqrt{2t}}{(2t)^{1/4}} \stackrel{d}{\to} \mathcal{N}(0,1).$$

Bruss and Delbaen 2004, Arlotto et al 2015 (for fixed-n sample) proved the CLT for the optimal strategy using concavity of the value function and martingale methods.
Our approach relies on a renewal approximation to the 'remaining area process'.

Fluctuations of the shape of selected increasing sequence

$Y(s)$ the last mark selected by the optimal strategy by time $s \in [0, t]$.

Theorem For $t \to \infty$

$$(c\, t^{1/4}(Y(\tau t) - \tau))_{\tau \in [0,1]} \Rightarrow \text{Brownian bridge}$$

in the Skorohod topology on $D[0,1]$, where $c = 3^{1/2} 2^{-3/4}$.

A more complex functional Gaussian limit with *random* centering in found in Bruss and Delbaen 2004.

A change of variables

With $z = \sqrt{t}$ as the size parameter and a change of variables, the equation for expected length under self-similar strategy becomes

$$u'(z) = 4 \int_0^1 (u(z-y) + 1 - u(z))_+ (1 - y/z) dy.$$

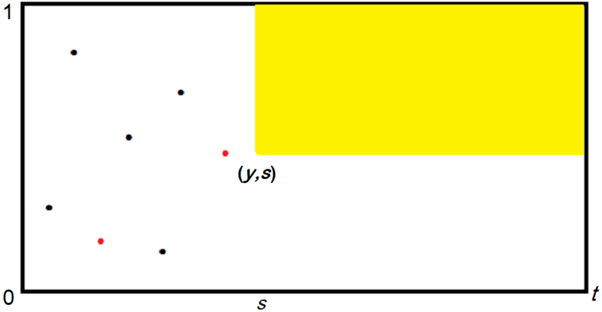
This is a special case of the renewal-type equation

$$u'_{r,\theta}(z) = 4 \int_0^{\theta(z)} (u_{r,\theta}(z-y) + r(z) - u_{r,\theta}(z))(1 - y/z) dy$$

with given reward function $r(z)$ and control function $0 < \theta(z) \leq z$ related to a self-similar strategy via

$$\varphi(z^2) = 1 - \left(1 - \frac{\theta(z)}{z}\right)^2.$$

the last so far selection/time $(y, s) \to z = \sqrt{(t-s)(1-y)}$



Asymptotic comparison method

The operator

$$\mathcal{I}_{\theta,r}g(z) := 4\int_0^{\theta(z)} (g(z-y) + r(z) - g(z))_+ (1 - y/z)dy$$

has shift and monotonicity properties that imply

Lemma *If for large enough z,*

(a) $g'(z) > \mathcal{I}_{\theta,r}g(z)$ *then* $\limsup_{z\to\infty}(u_{\theta,r}(z) - g(z)) < \infty$,

(b) $g'(z) < \mathcal{I}_{\theta,r}g(z)$ *then* $\liminf_{z\to\infty}(u_{\theta,r}(z) - g(z)) > -\infty$.

Example For $g(z) = \alpha z$, in the optimality equation, (a) holds for $\alpha > \sqrt{2}$, and (b) holds for $\alpha < \sqrt{2}$, whence $u(z) \sim \sqrt{2}\, z$.
Iterating twice ,

$$u(z) \sim \sqrt{2}\, z - \frac{1}{6}\log z + O(1), \quad z \to \infty.$$

But the method does not capture the $O(1)$-remainder.

Piecewise deterministic Markov process

For given control function $0 < \theta(z) \le z$, a PDMP process Z on $[0, \infty)$ is defined by

(i) decreases with unit speed until absorption at 0,

(ii) jumps at probability rate $4\lambda(z)$, where

$$\lambda(z) := \theta(z) - \frac{\theta^2(z)}{2z},$$

(iii) if jumps, then from z to $z - y$, with y having density $(1 - y/z)/\lambda(z)$ for $y \in [0, \theta(z)]$.

The number of jumps $N_\theta(z)$ of Z starting from $z = \sqrt{t}$ is equal to $L_\varphi(t)$, the length of increasing subsequence under a self-similar strategy.

Let $U(z_0, dz)$, be the occupation measure on $[0, z_0]$, for the sequence of jump points of Z starting from z_0, and controlled by the optimal $\theta^*(z)$. The density is

$$U(z_0, dz) = 4\lambda(z)p(z_0, z)dz,$$

where $p(z_0, z)$ is the probability that z is a drift point.
Lemma *There exists a pointwise limit* $p(z) := \lim_{z_0\to\infty} p(z_0, z)$*, such that* $\lim_{z\to\infty} p(z) = 1/2$ *and for some* $a, b > 0$

$$|p(z_0, z) - p(z)| < ae^{-b(z_0 - z)}, \quad 0 < z < z_0.$$

The proof is by coupling: two independent Z-processes starting with z_1 and z_2 (where $z_1 < z_2$) with high probability visit the same drift point close to z_1.

The 'mean reward' for Z starting with $z > 0$ has representation

$$u_{\theta^*,r}(z) = \int_0^z r(y)U(z, dy).$$

Corollary *For integrable $r(z)$,*

$$u_{\theta^*,r}(z) \to \int_0^\infty r(y)\lambda(y)p(y)dy, \quad z \to \infty.$$

If $r(z) = O(z^{-\beta})$ with $\beta > 1$ then the convergence rate is $O(z^{-\beta+1})$.

This allows us to obtain the asymptotic expansions of the moments of $N_\theta(t)$ and of the length of selected sequence $L_\varphi(t)$ under self-similar strategies. In particular, $w(z) = (\mathbb{E}N_{\theta^*}(z))^2$ satisfies

$$w'(z) = 4\int_0^{\theta^*(z)} (w(z-y) - w(z) + (1 + 2u(z-y))(1 - y/z)dy,$$

$$w(0) = 0.$$

A renewal approximation to Z

The range of Z (starting with z) is an alternating sequence of drift intervals and gaps skipped by jumps. Let D_z be the size of generic drift interval and J_z that of jump. From

$$\theta^*(z) = \frac{1}{\sqrt{2}} + \frac{1}{12z} + O(z^{-2})$$

follows that for $z \to \infty$ that $4\lambda(z) \to 2\sqrt{2}$ and

$$D_z \stackrel{d}{\to} \frac{E}{2\sqrt{2}}, \quad J_z \stackrel{d}{\to} \frac{U}{\sqrt{2}},$$

where E and U are independent Exponential(1) and Uniform-$[0,1]$ random variables. At distance from 0, the generic jump of Z is approximable by decreasing renewal proces with cycle-size

$$D_z + J_{z-D_z} \stackrel{d}{\to} \frac{E}{2\sqrt{2}} + \frac{U}{\sqrt{2}} =: H$$

CLT by stochastic comparison

Cutsem and Ycart 1994, Haas and Miermont 2011, Alsmeyer and Marynych 2016: limit theorems for absorption times (or jump-counts) for decreasing Markov chains on $\mathbb{N}$.

Adapting the stochastic comparison method of Cutsem and Ycart, we squeeze

$$(1 + c/\underline{z})^{-1} H <_{\rm st} D_z + J_{z - D_z} <_{\rm st} (1 - c/\underline{z})^{-1} H$$

for $z > \underline{z}$, where $\underline{z} = \omega\sqrt{z}$ and ω large parameter. Accordingly, the number of jumps of Z within $[\underline{z}, z]$ is squeezed between two renewal processes which satisfy the CLT.

It is important that the cycle-size of Z is within $O(z^{-1})$ from the limit, by slower convergence rate $O(z^{-1/2+\epsilon})$ the normal approximation may fail.

Online increasing subsequences in higher dimensions

$(x_1, s_1), \ldots, (x_k, s_k) \in [0,1]^d \times [0,t]$ is a chain if the sequence increases in every component.

The longest (offline) chain: $M(t) \sim ct^{1/(d+1)}$, but the constant is unknown (estimates in Bollobas and Winkler 1988).
Online chains in d dimensions. Our methods extend to the online increasing subsequence problem with marks sampled from uniform distribution in $[0,1]^d$. The principal asymptotics is

$$v(t) \sim \frac{(d+1)}{\{(d+1)!\}^{1/(d+1)}} \; t^{1/(d+1)},$$

other results (asymptotic expansions, CLT) generalise as well.