Properties of generalized hooking networks

AofA, CIRM

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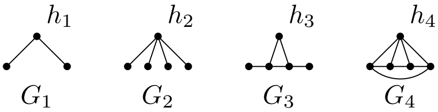
Overview

- $\mathcal{S} = \{G_1, G_2, \ldots, G_m\}$ set of graphs called *blocks*
- Each G_i has a vertex h_i called the *hook* of G_i

Blocks:



Hooking networks:

- Choose a block G_i and let $\mathcal{G}_0$ be a copy of G_i

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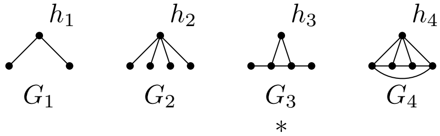

Hooking networks:

- Choose a block G_i and let $\mathcal{G}_0$ be a copy of G_i
- Vertex H corresponding to hook of G_0 called *master hook of the network*

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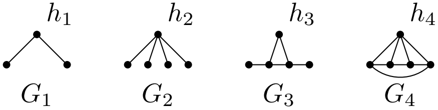

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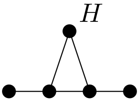

$\mathcal{G}_0$

- Choose v from $\mathcal{G}_{n-1}$, called a *latch*
- Choose a block G_i,
- Attach G_i to $\mathcal{G}_{n-1}$ by joining the hook h_i and the latch v

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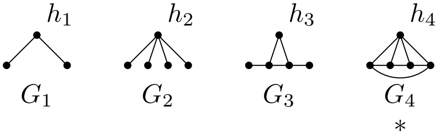

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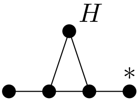

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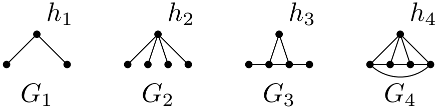

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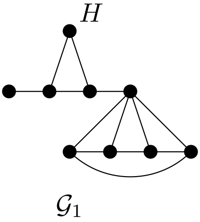

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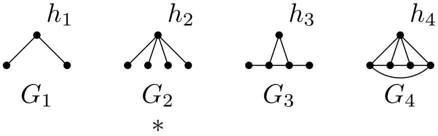

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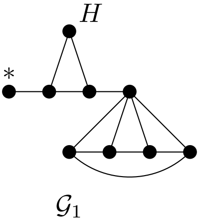

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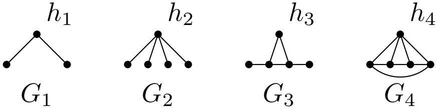

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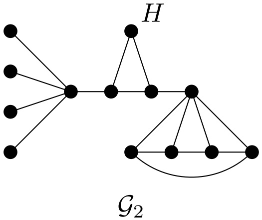

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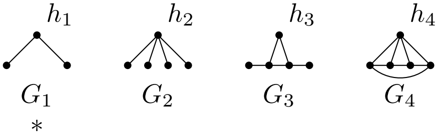

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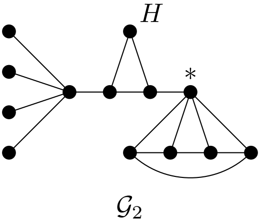

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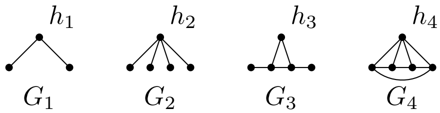

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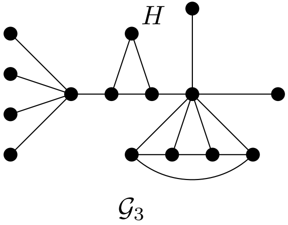
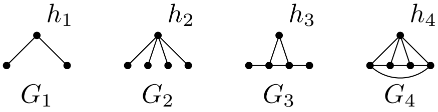


- $k \in \mathbb{Z}^+$ is called an *admissible degree* iff $\exists n$ s.t. $\mathbb{P}(> 1 \text{ vertex with degree } k \text{ in } \mathcal{G}_n) > 0$

Blocks:



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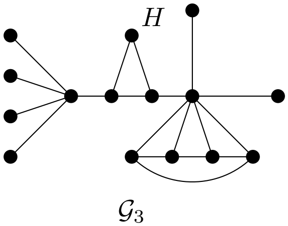
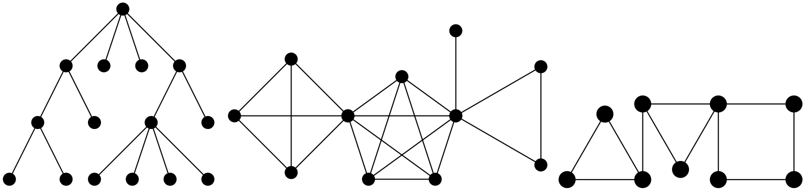


Examples of hooking networks

Trees **Blocks graphs** **Cactus graphs**



Choosing latches and blocks

- Assign probability p_i to each G_i ($\sum p_i = 1$)
- Fix χ, ρ
- Choose latch $v \in \mathcal{G}_{n-1}$ with probability

$$\frac{\deg(v)\chi + \rho}{\sum_{u \in \mathcal{G}_{n-1}} (\deg(u)\chi + \rho)}$$

Main Result: Degrees

- $k_1 < k_2 < \cdots < k_r \leq d$, admissible degrees less than or equal to d
- $X_{n,i} :=$ # vertices with degree k_i in $\mathcal{G}_n$

Theorem (D., Holmgren, 2019+)

Let $\mathcal{X}_n = (X_{n,1}, X_{n,2}, \ldots, X_{n,r})$ *and*

$$\mu_n := \mathbb{E}\mathcal{X}_n = (\mathbb{E}X_{n,1}, \ldots, \mathbb{E}X_{n,r}).$$

Then

$$n^{-1/2}(\mathcal{X}_n - \mu_n) \xrightarrow{d} \mathcal{N}(0, \Sigma)$$

for some covariance matrix Σ.

Previous results

- **Mahmoud and Smythe, 1992:** degrees 0,1,2 in random recursive trees
- **Mahmoud, Smythe, Szymański, 1993:** degrees 0,1,2 in plane-oriented recursive trees
- **Janson, 2005:** degrees in random recursive and plane-oriented recursive trees
- **Gopaladesikan, Mahmoud, and Ward, 2014:** leaves in blocks trees
- **Holmgren, Janson, Šileikis, 2017:** degrees in general preferential attachment trees
- **Mahmoud :** degrees in self-similar hooking networks

Pólya urns

Set up:

- q types (colours) $1, 2, \ldots, q$ of balls
- Assign to type i:
  - $\xi_i = (\xi_{1,i}, \ldots, \xi_{i,q})$, with $\xi_{i,j} \geq 0$ for $i \neq j$ and $\xi_{i,i} \geq -1$
  - an activity a_i
- $X_{n,i} \geq 0$: # of balls of type i in the urn at time n
- Let $X_n = (X_{n,1}, \ldots, X_{n,q})$
- $\mathbb{P}(\text{Type } i \text{ is chosen at time } n) =$

$$\frac{a_i X_{n-1,i}}{\sum_{j=1}^{q} a_j X_{n-1,j}}$$

- Replace a ball i with $\Delta X_{n,j}$ balls of type j for each $j = 1, \ldots, q$, where $\Delta X_n = (\Delta X_{n,1}, \ldots, \Delta X_{n,q}) \sim \xi_i$

Pólya urns

Set up:

- The *intensity matrix* of the Pólya urn is

$$A := (a_j \mathbb{E}\xi_{j,i})_{i,j=1}^{q}$$

- A has a real eigenvalue λ_1 s.t. $\forall \lambda \neq \lambda_1,\ \mathrm{Re}\lambda < \lambda_1$
- Assumptions: A is irreducible, and $\mathbb{E}(\xi_{i,j}^2) < \infty$ for all $i, j = 1, 2, \dots, q$.
- Let $a = (a_1, \dots, a_q)$ and let v_1 be the right eigenvector of A corresponding to the eigenvalue λ_1 normalized so that $a \cdot v_1 = 1$

Pólya urns

Theorem (Janson, 2004)

Let $(X_n)_{n=0}^{\infty}$ be a Pólya urn with intensity matrix A. Assume that $Re\lambda < \lambda_1/2$ for all eigenvalues $\lambda \neq \lambda_1$. Define $\mu := \lambda_1 v_1$. Then as $n \to \infty$,

$$n^{-1/2}(X_n - n\mu) \xrightarrow{d} \mathcal{N}(0, \Sigma)$$

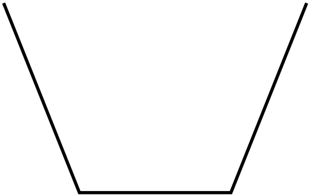
for some covariance matrix Σ.

Pólya urns corresponding to hooking networks:

Recall: $k_1 < \cdots < k_r \leq d$, admissible degrees less than or equal to d.

- balls of type $1, 2, \ldots, r$, and balls of *special type* $*$
- vertex of degree $k_i \Leftrightarrow$ ball of type i
- master hook of network $H \Leftrightarrow \deg(H)\chi + \rho$ balls of special type $*$
- vertex v of degree $> d \Leftrightarrow \deg(v)\chi + \rho$ balls of special type $*$
- activity of ball i is $a_i = k_i\chi + \rho$
- activity of special type $*$ is $a_* = 1$

Example, $\chi = 1, \rho = 0$:



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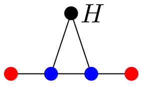

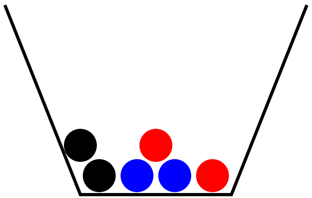

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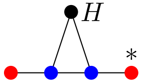

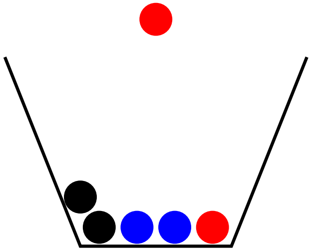

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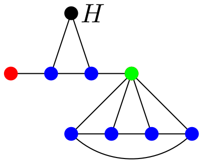

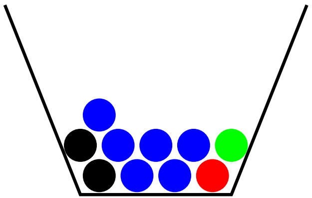

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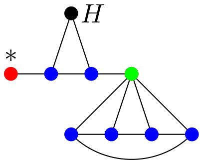

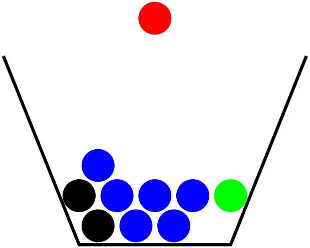

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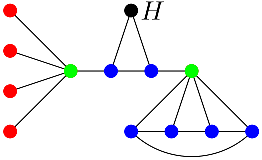

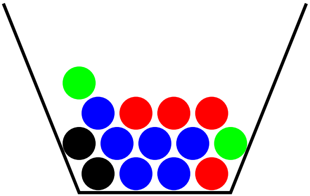

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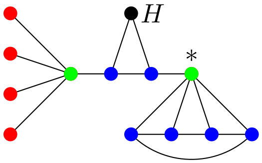

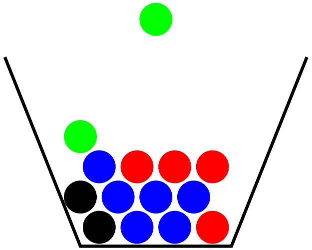

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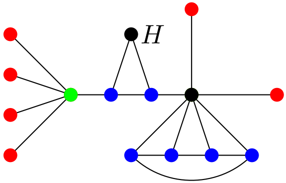

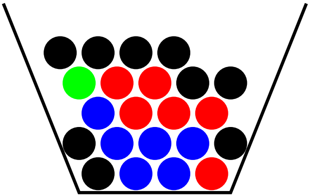

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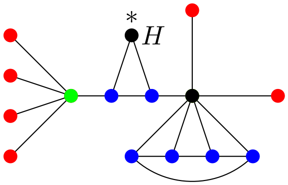

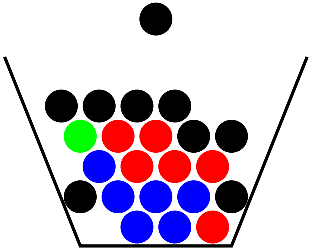

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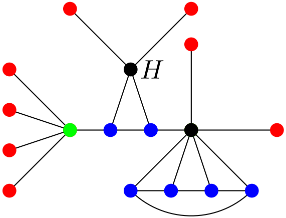
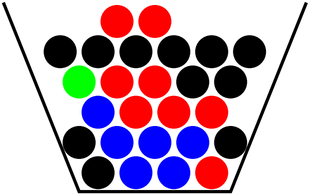
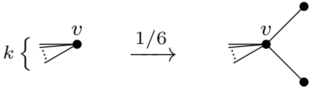
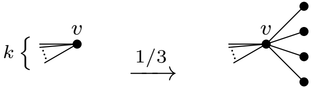






Type k — 2x Type 1, Type $k+2$



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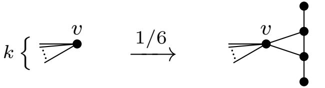
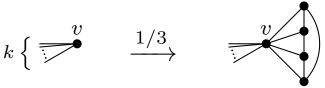


Type k — 2x Type 1, 2x Type 3, Type $k+2$



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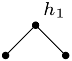
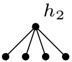
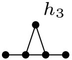


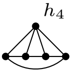
$G_1,\ p_1 = 1/6$



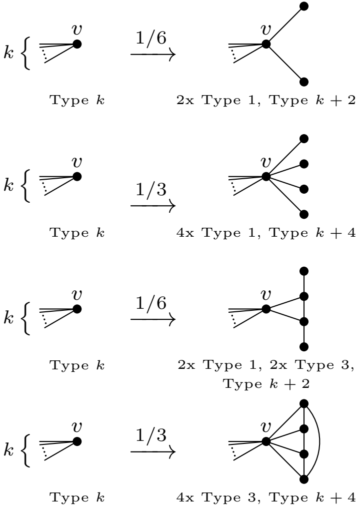
$G_2,\ p_2 = 1/3$



$G_3,\ p_3 = 1/6$

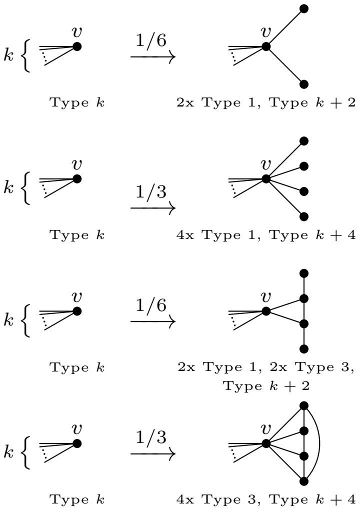


$G_4,\ p_4 = 1/3$



row/column 1 → vertices degree 1
row/column 2 → vertices degree 3
row/column 3 → vertices degree 5
row/column 4 → master hook
& vertices degree >5

$$
\begin{aligned}
\mathbb{E}\xi_1 &= \frac{1}{6}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + \frac{1}{3}\begin{pmatrix}3\\0\\1\\0\end{pmatrix} \\
&+ \frac{1}{6}\begin{pmatrix}1\\3\\0\\0\end{pmatrix} + \frac{1}{3}\begin{pmatrix}-1\\4\\1\\0\end{pmatrix} \\
&= \frac{1}{6}\begin{pmatrix}6\\12\\4\\0\end{pmatrix}
\end{aligned}
$$



row/column 1 → vertices degree 1
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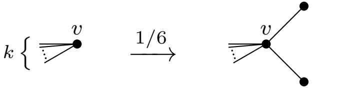
$$\begin{aligned}\mathbb{E}\xi_3 &= \frac{1}{6}\begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 4 \\ -1 \\ 0 \\ 7\chi+\rho \end{pmatrix} \\ &+ \frac{1}{6}\begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 \\ 3 \\ 0 \\ 7\chi+\rho \end{pmatrix} \\ &= \frac{1}{6}\begin{pmatrix} 12 \\ 4 \\ 2 \\ 28\chi+4\rho \end{pmatrix}\end{aligned}$$



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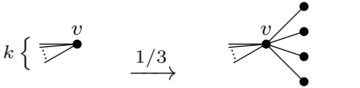
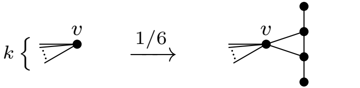
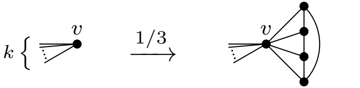


Type k — 4x Type 1, Type $k+4$



Type k — 2x Type 1, 2x Type 3, Type $k+2$



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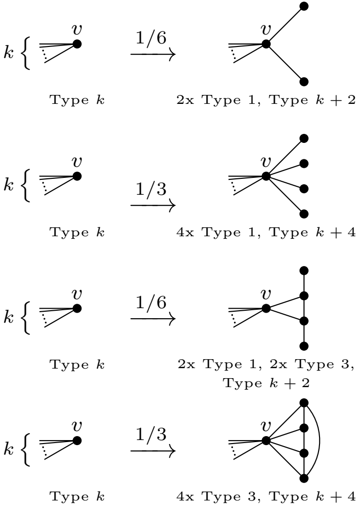
$$
\begin{aligned}
\mathbb{E}\xi_5 &= \frac{1}{6}\begin{pmatrix} 2 \\ 0 \\ -1 \\ 7\chi+\rho \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 4 \\ 0 \\ -1 \\ 9\chi+\rho \end{pmatrix} \\
&+ \frac{1}{6}\begin{pmatrix} 2 \\ 2 \\ -1 \\ 7\chi+\rho \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 \\ 4 \\ -1 \\ 9\chi+\rho \end{pmatrix} \\
&= \frac{1}{6}\begin{pmatrix} 12 \\ 10 \\ -6 \\ 50\chi+6\rho \end{pmatrix}
\end{aligned}
$$



row/column 1 → vertices degree 1
row/column 2 → vertices degree 3
row/column 3 → vertices degree 5
row/column 4 → master hook
& vertices degree >5

$$\mathbb{E}\xi_* = \frac{1}{6}\begin{pmatrix} 2 \\ 0 \\ 0 \\ 2\chi \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 4 \\ 0 \\ 0 \\ 4\chi \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 2 \\ 2 \\ 0 \\ 2\chi \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 \\ 4 \\ 0 \\ 4\chi \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 12 \\ 10 \\ 0 \\ 20\chi \end{pmatrix}$$

Intensity matrix A =

$$\frac{1}{6}\begin{pmatrix} 6(\chi+\rho) & 12(3\chi+\rho) & 12(5\chi+\rho) & 12 \\ 12(\chi+\rho) & 4(3\chi+\rho) & 10(5\chi+\rho) & 10 \\ 4(\chi+\rho) & 2(3\chi+\rho) & -6(5\chi+\rho) & 0 \\ 0 & (28\chi+4\rho)(3\chi+\rho) & (50\chi+6\rho)(5\chi+\rho) & 20\chi \end{pmatrix}$$

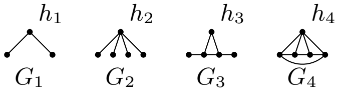
Eigenvalues:

$$\frac{31}{3}\chi+\frac{11}{3}\rho, -\chi-\rho, -3\chi-\rho, -5\chi-\rho.$$

Let $\mathcal{S} = \{G_1, G_2, \ldots, G_m\}$ be a set of blocks. Each G_i with vertex set V_i and hook h_i. For all positive integer j, define:

$$f(j) := \sum_{G_i \in \mathcal{S}} p_i |\{v \in V_i \setminus \{h_i\} : \deg(v) = j\}|$$

$$g(j) := \sum_{\substack{G_i \in \mathcal{C} \\ \deg(h_i) = j}} p_i$$



f(1) = 2, f(3) = 5/3
g(2) = 1/3, g(4) = 2/3

Intensity matrix $A = [a_{ij}]$ given by

$$a_{ij} = \begin{cases} (k_j\chi + \rho)f(k_i) & i < j \le r \\ f(k_i) & i < j = r+1 \\ (k_j\chi + \rho)(f(k_i) - 1) & j = i \le r \\ (k_j\chi + \rho)(f(k_i) + g(k_i - k_j)) & j < i \le r \\ (k_j\chi + \rho)\sum_{k>d}(k\chi + \rho)(f(k) + g(k - k_j)) & j < i = r+1 \\ \sum_{k>d}(k\chi + \rho)f(k) + \sum_{k\ge 1} k\chi g(k) & j = i = r+1, \end{cases}$$

Eigenvalues:

$$\lambda_1 = \sum_{k\ge 1}((k\chi + \rho)f(k) + k\chi g(k))$$

$$\lambda_2 = -k_1\chi - \rho$$

$$\vdots$$

$$\lambda_{r+1} = -k_r\chi - \rho$$

Let $v_{11} = f(k_1)/(\lambda_1 + k_1\chi + \rho)$, define recursively for $i = 2, \ldots, r$:

$$v_{1i} = \frac{1}{\lambda_1 + k_i\chi + \rho}\left(f(k_i) + \sum_{j=1}^{i-1}(k_j\chi + \rho)g(k_i - k_j)v_{1j}\right),$$

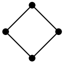
and let $v_{1r+1} = 1 - \sum_{j=1}^{r}(k_j\chi + \rho)v_{1j}$. Define

$$v_1 = (v_{11}, v_{12}, \ldots, v_{1r}, v_{1r+1}).$$

Then

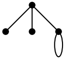
$$n^{-1/2}(X_n - n\lambda_1 v_1) \xrightarrow{d} \mathcal{N}(0, \Sigma).$$

h_1



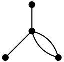
$G_1, \;\; p_1 = \frac{1}{5}$

h_2



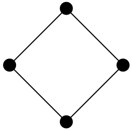
$G_2, \;\; p_2 = \frac{1}{5}$

h_3



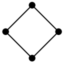
$G_3, \;\; p_3 = \frac{3}{5}$

H



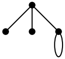
$\mathcal{G}_0$

h_1



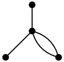
$G_1, \;\; p_1 = \frac{1}{5}$

h_2



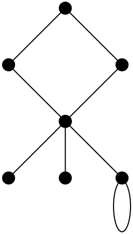
$G_2, \;\; p_2 = \frac{1}{5}$

h_3

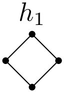


$G_3, \;\; p_3 = \frac{3}{5}$

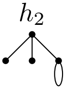
H



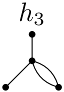
$\mathcal{G}_1$



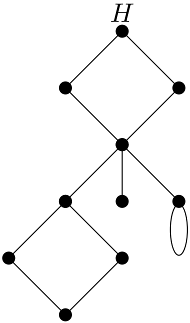
$G_1, \;\; p_1 = \frac{1}{5}$



$G_2, \;\; p_2 = \frac{1}{5}$

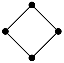


$G_3, \;\; p_3 = \frac{3}{5}$



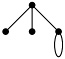
$\mathcal{G}_2$

h_1



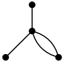
$G_1, \ p_1 = \frac{1}{5}$

h_2



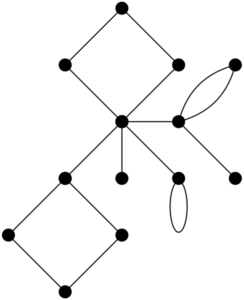
$G_2, \ p_2 = \frac{1}{5}$

h_3

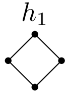


$G_3, \ p_3 = \frac{3}{5}$

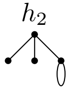
H



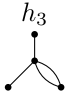
$\mathcal{G}_3$



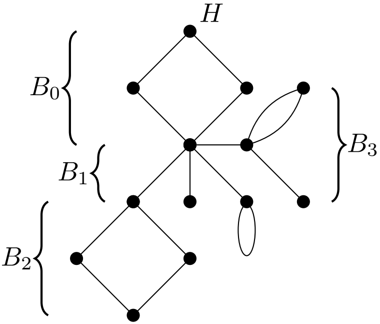
$G_1, \;\; p_1 = \frac{1}{5}$



$G_2, \;\; p_2 = \frac{1}{5}$

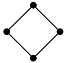


$G_3, \;\; p_3 = \frac{3}{5}$



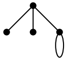
$\mathcal{G}_3$

h_1



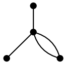
$G_1, \quad p_1 = \frac{1}{5}$

h_2



$G_2, \quad p_2 = \frac{1}{5}$

h_3



$G_3, \quad p_3 = \frac{3}{5}$

- $s_i = \sum_{v \in V(G_i)} (\deg(v)\chi + \rho) - \rho$
- We want $s = s_1 = s_2 = \cdots = s_m$
- Happens when $\chi = 0$ and $|V(G_1)| = \cdots = |V(G_m)|$
- Happens when $\rho = 0$ and $|E(G_1)| = \cdots = |E(G_m)|$
- $s = s_1 = s_2 = s_3 = 8\chi + 3\rho$
- Δ_n : depth of random vertex within block B_n after hooking
- Δ_0 : adjusted for B_0

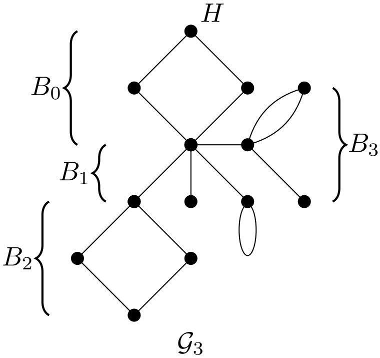
- Define $\tau_n = sn + \rho = \sum_{v \in \mathcal{G}_n} (\deg(v)\chi + \rho)$
- X_n : depth of latch chosen to construct $\mathcal{G}_n$.
- $\mathbb{P}(X_n = X_0 + \Delta_0) = (s+\rho)/\tau_{n-1}$
- $\mathbb{P}(X_n = X_i + \Delta_i) = s/\tau_{n-1}$
- $\psi_{X_n}(u) = \frac{s+\rho}{\tau_{n-1}} \mathbb{E}\left[e^{(X_0+\Delta_0)u}\right]$

$$+ \frac{s}{\tau_{n-1}} \sum_{i=1}^{n-1} \mathbb{E}\left[e^{(X_i+\Delta_i)u}\right]$$

$$= \psi_{\Delta_0}(u) \prod_{i=1}^{n-1} \frac{\rho+si+s\psi_{\Delta}(u)}{\rho+s(i+1)}$$

- $\mathbb{E}[X_n] \sim \mathbb{E}[\Delta] \ln n$

$$\frac{X_n - \mathbb{E}[X_n]}{\sqrt{\ln n}} \xrightarrow{d} \mathcal{N}\left(0, \mathbb{V}ar[\Delta] + \mathbb{E}^2[\Delta]\right)$$

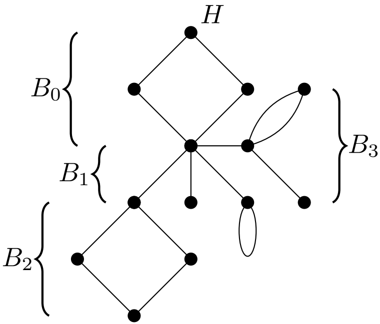


- Suppose further $v_1 = \cdots = v_m$
- $\widetilde{D_n}$: depth of vertex chosen uniformly at random in $\mathcal{G}_n$.
- $\psi_{\widetilde{D}_n}\left(\frac{u}{\sqrt{\ln n}}\right)$

  $\sim \frac{1}{n}\sum_{i=1}^{n}\psi_{X_i}\left(\frac{u}{\sqrt{\ln n}}\right)$

  $\sim \psi_{X_{n+1}}\left(\frac{u}{\sqrt{\ln n}}\right)$
- $\mathbb{E}[\widetilde{D}_n] \sim \mathbb{E}[\Delta]\ln n$

  $\frac{\widetilde{D}_n - \mathbb{E}[\widetilde{D}_n]}{\sqrt{\ln n}} \xrightarrow{d} \mathcal{N}\left(0, \mathbb{V}ar[\Delta] + \mathbb{E}^2[\Delta]\right)$



$\mathcal{G}_3$

Thank you!