HIGHER-POWER HARMONIC MAPS AND SECTIONS

CHRIS WOOD

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φ is a *r-power harmonic map,* or *r-harmonic map* if $\mathcal{E}_r$ stationary at φ.

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$$\varepsilon_1^v(\sigma) = \|d^v\sigma\|^2 = 2e^v(\sigma),$$

$$\varepsilon_2^v(\sigma) = \|d^v\sigma \wedge d^v\sigma\|^2, \quad \text{etc.}$$

Definition. σ is *r-horizontal* if $\varepsilon_r^v(\sigma) = 0$ (identically).

1-horizontal $\Leftrightarrow$ horizontal;

2-horizontal $\Leftrightarrow$ vertical rank $\leqslant 1$, etc.

Definition. The *r-th vertical energy* of σ is:

$$\mathcal{E}_r^v(\sigma) = \frac{1}{2}\int_M \varepsilon_r^v(\sigma)\,\mathrm{vol}(g).$$

Note: $\mathcal{E}_r^v(\sigma) = 0$ for all σ if $r > p - m$.

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Thus σ *is a* r*-harmonic section if and only if:*

$$\tau_r^v(\sigma) + \mathrm{tr}_\nu(\sigma^*\mathcal{A}) = 0.$$

Corollary. *If* π *has totally geodesic fibres then* σ *is a* r*-harmonic section*

if and only if $\tau_r^v(\sigma) = 0$.

r-harmonic sections vs. (sections that are) r-harmonic maps.

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Lemma. *If π is a Riem. submersion with t. g. fibres then for all $1 \leqslant r \leqslant m$:*

$$\varepsilon_r(\sigma) = \varepsilon_r^v(\sigma) + (m-r+1)\varepsilon_{r-1}^v(\sigma) + \cdots + \binom{m-1}{r-1}\varepsilon_1^v(\sigma) + \frac{1}{2}\binom{m}{r},$$

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The vertical component of $\tau_r(\sigma)$ is:

$$\tau_r(\sigma)^v = \tau^v_r(\sigma) + (m-r+1)\tau^v_{r-1}(\sigma) + \cdots + \binom{m-1}{r-1}\tau^v_1(\sigma).$$

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Theorem. *σ is a r-harmonic map precisely when σ is a twisted r-skyrmion with coupling constants $c_i = \binom{m-i}{r-i}$ and the horizontal component of $\tau_r(\sigma)$ vanishes.*

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ii) *If π is a Riem. submersion with t. g. fibres, and σ is horizontal, then σ is a r-harmonic map for all $1 \leqslant r \leqslant m$.*

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In particular, σ is a r-harmonic section of E for all $r \geqslant \operatorname{rank} E$.

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the equation for *harmonic unit vector fields* (when $E = TM$).

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The λ_i (relative signs) classify $\mathfrak{g}$ algebraically:

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$\mathfrak{e}(1,1)$: $\lambda_1 > 0,\ \lambda_2 = 0,\ \lambda_3 < 0.\quad \mathcal{H}_1 = \mathcal{H}_2 = \mathcal{C}_{13} \cup \mathcal{P}_2,\quad \mathcal{Z}_1 = \mathcal{Z}_2 = \emptyset,$

unless $\lambda_1 = -\lambda_3$: $\quad \mathcal{Z}_2 = \mathcal{C}_{13}.$

$\mathfrak{e}(2)$: $\lambda_1, \lambda_2 > 0,\ \lambda_3 = 0.\quad \mathcal{H}_1 = \mathcal{H}_2 = \mathcal{C}_{12} \cup \mathcal{P}_3,\quad \mathcal{Z}_1 = \mathcal{Z}_2 = \emptyset,$

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or $\lambda_1 = \lambda_2 - \lambda_3$: $\quad \mathcal{H}_2 = \mathcal{C}_{13} \cup \mathcal{P}_2,\quad \mathcal{Z}_2 = \mathcal{C}_{13}.$

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Observation. Left-invariant metrics with $\mathcal{H}_1 \subsetneq \mathcal{H}_2$ supported (only) on $\mathfrak{e}(2)$, $\mathfrak{sl}(2)$ and $\mathfrak{su}(2)$.

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Remark. Examples where $\mathcal{H}_1 \not\subseteq \mathcal{H}_2$ occur when G non-unimodular.

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Remarks.

- $r = 3$ characterises minimal immersions.

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Remarks.

- $r = 3$ characterises minimal immersions.
- Results of Gonzalez-Davila & Vanhecke (2002), and Tsukuda & Vanhecke (2000) recovered when $r = 1, 3$.

Fin!