Morse Index of minimal tori in S^4

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Willmore functional and Willmore surfaces

- For a closed surface $f : M \to S^n$, the Willmore energy is defined by
$$W(f) := \int_M (|\vec{H}|^2 + 1)\mathrm{d}M.$$
- Willmore conjecture (1965): If $M^2 = T^2$, then $W(f) \geqslant 2\pi^2$, "$=$" $\Leftrightarrow$ iff f is conformally congruent to the Clifford torus.

Theorem

(Marques & Neves, 2012) If $genus(M^2) \geqslant 1$ and $n = 3$, then $W(f) \geqslant 2\pi^2$, with equality iff f is conformally congruent to the Clifford torus (Minimal in S^3).

Marques & Neves's proof of Willmore conjecture in S^3

The result due to Urbano plays an important role in Marques & Neves's proof:

Theorem

(Urbano, 1990) If f is a minimal surface in S^3 with $genus(M^2) \geqslant 1$, then $Index(f) \geqslant 5$, with equality iff f is conformally congruent to the Clifford torus.

Minimal surfaces in S^n and first eigenvalue of Laplacian

For a surface $f : M \to S^n$, we have

$$\Delta_M f = -2f + 2\vec{H}.$$

Three equivalent characterizations of f being a minimal surface:

- f is a critical surface w.r.t. the area functional $A(f) := \int_M dM$.
- $\vec{H} \equiv 0$.
- The coordinate functions $f_j, j = 1, \cdots, n+1$, are eigenfunctions of Δ_M with eigenvalue $\lambda = 2$.

Minimal surfaces in S^n and first eigenvalue of Laplacian

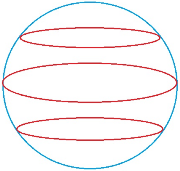
- Let M be a closed Riemann surface with a metric. The eigenvalues of Δ_M are discrete and are tending to $+\infty$:
  $$Spec(\Delta_M) = \{0, \lambda_1, \cdots, \} \text{ and } 0 < \lambda_1 < \lambda_2 < \cdots.$$
  λ_1 the first eigenvalue of Δ_M.

- f is minimal $\Rightarrow 2 \in Spec(\Delta_M) \Rightarrow \lambda_1 \leqslant 2$.

- f is called immersed by its first eigenfunctions if $\{f_j\}$ are eigenfunctions of λ_1, i.e., $\lambda_1 = 2$.

Area functional, Jacobi operator of minimal surfaces

- f minimal $\Leftrightarrow$ For any (normal) variation f_t of f (with $V = \frac{d}{dt}|_{t=0} f_t$), $\frac{d}{dt}|_{t=0} Area(f_t) = -2\int_M \langle \vec{H}, V\rangle dM = 0$.
- f is unstable! For example, any non-isometric conformal transformation decreases the area (in fact corresponding to eigenvalue $\alpha = -2$).



Morse index of minimal surfaces

- The second variation of y:

$$Area(f_t)''|_{t=0} = -\int \langle \mathcal{L}(V), V \rangle dM,$$

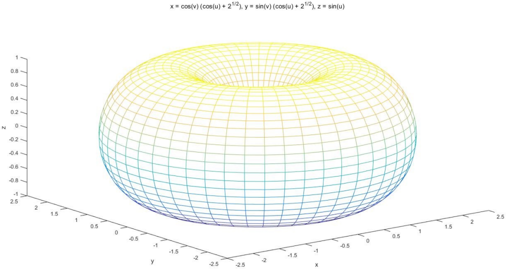
$$\mathcal{L}(V) := \Delta_M^{\perp} V + 2V + \tilde{\mathcal{A}}(V).$$

Here $\langle \tilde{\mathcal{A}}(V), U \rangle = \langle \mathcal{A}(V), \mathcal{A}(U) \rangle$. In Codim-1 case, $\tilde{\mathcal{A}} = |II|^2$.

- $\mathcal{L}$ is self–adjoint: $\int_M \langle \mathcal{L}(V), W \rangle dM = \int_M \langle W, \mathcal{L}(V) \rangle dM$.
- Spectrum of $\mathcal{L}$: $\mathcal{L}(V) + \alpha V = 0$, $\alpha_1 \leqslant \alpha_2 \leqslant \cdots$, $\alpha \to +\infty$. $Spec_L = \{\alpha_1, \cdots, \alpha_n, \cdots\}$.
- Morse Index of f:

  $Index_n(f) :=$The number of negative eigenvalues α_j (with multiplicity) of $\mathcal{L}$ (i.e. the unstable directions).

Clifford torus

- $S^3 \subset \mathbb{C}^2 = \mathbb{R}^4$. $T^2 \subset U(2)$ actions on S^3. The orbits of T^2 with maximal area—Clifford torus.
- 

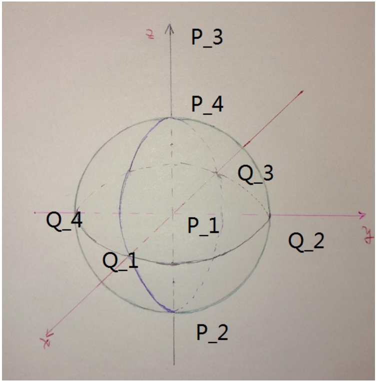
- Codim-1, $V = gN$. So $\mathcal{L}$ actions on g:

$$\mathcal{L}g = \Delta_M g + 2g + |\tilde{\mathcal{A}}|^2 g = \Delta_M g + 4g.$$

- $g = 1 \leftrightarrow \alpha = -4$ (Ros deformation);
- $g = \sin(\sqrt{2}u), \cos(\sqrt{2}u), \sin(\sqrt{2}v), \cos(\sqrt{2}v) \leftrightarrow \alpha = -2$ (non-isometric conformal transformations of S^3).
- $\cdots$, $\leftrightarrow \alpha = 0$, isometric transformations of S^3.
- $Index(T^2) = 5$ in S^3.
- $Index(T^2) = 1 + (n+1) = n + 2$ in S^n.
- Weiner (1978), Guo-Li-Wang(2001): The Clifford torus is Willmore stable in S^3.

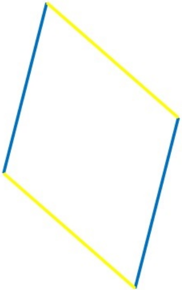
Clifford torus–2

- $y = \frac{1}{\sqrt{2}}(\cos u, \sin u, \cos v, \sin v)$,
- $Ay =$
$\left(\cos\frac{u+v}{2}\cos\frac{u-v}{2}, \sin\frac{u+v}{2}\sin\frac{u-v}{2}, \cos\frac{u+v}{2}\sin\frac{u-v}{2}, \sin\frac{u+v}{2}\cos\frac{u-v}{2}\right)$.

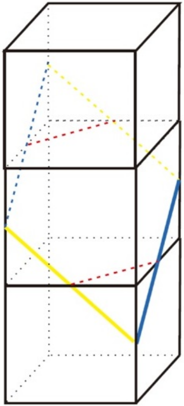


Examples of minimal tori: Bryant-Itoh-Montiel-Ros torus

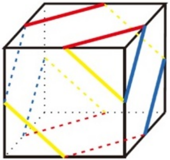
- Clifford 3-torus in $S^5 \subset \mathbb{C}^3 = \mathbb{R}^6$: Consider $T^3 \subset U(3)$ actions on $\mathbb{C}^3$. The maximal volume T^3–Clifford 3-torus.
- $\exp(s,t,r) : \mathfrak{t}^3 \to T^3$. T^3 orbits on $\mathbb{C}^3$: $(ae^{is}, be^{it}, ce^{ir})$ with $a^2+b^2+c^2=1$. Clifford 3-torus: $a=b=c=\frac{1}{\sqrt{3}}$. It is minimal in S^5.
- Consider $SU(3) \subset U(3)$. $T^2 \subset SU(3)$. The one in T^3 orthogonal to a geodesic of S^5 is an equilateral minimal T^2 in S^5! The Bryant-Itoh-Montiel-Ros torus.
- $Volume(T^3) = \left(\frac{2\pi}{\sqrt{3}}\right)^3 = \frac{8\pi^3}{3\sqrt{3}}$.
  $Area(T^2) = \frac{Volume(T^3)}{2\pi/3} = \frac{4\pi^2}{\sqrt{3}} (\approx 7.3\pi < 8\pi)$.
- $Index(T^2) = 8$ in S^5 and $= n+3$ in S^n, $n > 5$.



The BIMR torus



The BIMR torus in 3-copies of the Clifford 3-torus



The BIMR torus in the Clifford 3-torus

Index of minimal tori in S^3

Theorem

(Urbano,1990) Let $f : M^2 \to S^3$ be a closed minimal surface with genus $\geqslant 1$. Then its Morse index is great or equal to 5. Moreover, equality holds if and only if it is the Clifford torus.

Lemma

(J. Simons, 1968; O. Perdomo, 2002; Notes of Schoen; Kusner-W, 2018) Let $f : M^2 \to S^n$ be an oriented minimal surface. Let $\mathcal{E}_{-2}$ be the eigenspace of $\mathcal{L}$ of f, with $\alpha = -2$. Then

- $dim\mathcal{E}_{-2} \geqslant n+1$, *if f is not totally geodesic;*
- $dim\mathcal{E}_{-2} \geqslant n-2$, *if f is totally geodesic.*

Index of minimal tori in S^4

- Set $f: M \to S^n$, $H = 0$ & $|f_z|^2 = \frac{1}{2}e^{2\rho}$.

$$\begin{cases} f_{zz} = 2\rho_z f_z + \Omega, \\ f_{z\bar{z}} = -\dfrac{1}{2}e^{2\rho} f, \\ \psi_z = \nabla^{\perp}_z \psi - 2e^{-2\rho}\langle\psi, \Omega\rangle f_{\bar{z}}. \end{cases}$$

- $\Omega \mathrm{d}z^2$ is a global section defined on M. The Hopf differential $\mathcal{H} := \langle\Omega, \Omega\rangle \mathrm{d}z^4$.
- The Gauss, Codazzi and Ricci equations are as follows:

$$\begin{cases} -K + 1 = 4e^{-4\rho}|\Omega|^2 = \dfrac{S}{2}, \\ \nabla^{\perp}_{\bar{z}}\Omega = 0, \\ R_{z\bar{z}}\psi = \nabla^{\perp}_{\bar{z}}\nabla^{\perp}_{z}\psi - \nabla^{\perp}_{z}\nabla^{\perp}_{\bar{z}}\psi = 2e^{-2\rho}(\langle\psi, \Omega\rangle\bar{\Omega} - \langle\psi, \bar{\Omega}\rangle\Omega). \end{cases}$$

Theorem

(Kusner-W, 2018). Let $f : T^2 \to S^4$ be a minimal torus. Then its Morse index is great or equal to 6. Moreover, equality holds if and only if it is the Clifford torus.

Theorem

(Kusner-W, 2018).

- *Let $f : T^2 \to S^4$ be a full, minimal torus with non-zero Hopf differential. Then $Index(f) \geqslant 7$.*
- *Let $f : T^2 \to S^4$ be a full, minimal torus with vanishing Hopf differential. Then $Index(f) \geqslant 12$.*
- *Let $f : T^2 \to S^n$, $n \geqslant 5$, be a full, minimal torus with vanishing Hopf differential. Then $Index(f) \geqslant n + 3$.*

Index of minimal tori in S^4

- Set $f: M \to S^n$, $H = 0$ & $|f_z|^2 = \frac{1}{2}e^{2\rho}$.

$$\begin{cases} f_{zz} = 2\rho_z f_z + \Omega, \\ f_{z\bar{z}} = -\dfrac{1}{2}e^{2\rho} f, \\ \psi_z = \nabla^\perp_z \psi - 2e^{-2\rho}\langle \psi, \Omega\rangle f_{\bar{z}}. \end{cases}$$

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Index of minimal tori in S^4

- The Laplacian $\Delta^{\perp}$ on normal bundle:

$$\Delta^{\perp}\psi = 2e^{-2\rho}\left(\nabla^{\perp}_{\bar{z}}\nabla^{\perp}_{z}\psi + \nabla^{\perp}_{z}\nabla^{\perp}_{\bar{z}}\psi\right).$$

- Application of the Simons equality for the second fundamental form (as test sections).

$$\Delta^{\perp}\Omega = 4e^{-4\rho}(\langle\Omega,\Omega\rangle\bar{\Omega} - \langle\Omega,\bar{\Omega}\rangle\Omega).$$

Index of minimal tori in S^4

- Application of the Simons equality for the second fundamental form (as test sections).

$$\Delta^{\perp}\Omega = 4e^{-4\rho}(\langle\Omega,\Omega\rangle\bar{\Omega} - \langle\Omega,\bar{\Omega}\rangle\Omega).$$

- For tori with $\langle\Omega,\Omega\rangle = 1$, we have

$$\Delta^{\perp}\Omega_1 = -8e^{-4\rho}|\Omega_2|^2\Omega_1, \qquad \Delta^{\perp}\Omega_2 = -8e^{-4\rho}|\Omega_1|^2\Omega_2.$$

$$\tilde{\mathcal{A}}(\Omega_j) = 8e^{-4\rho}|\Omega_j|^2\Omega_j,\ j = 1,2.$$

- So

$$\mathcal{L}(\Omega_1) = 2\left(1 + 4e^{-4\rho}(|\Omega_1|^2 - |\Omega_2|^2)\right)\Omega_1,$$
$$\mathcal{L}(\Omega_2) = 2\left(1 - 4e^{-4\rho}(|\Omega_1|^2 - |\Omega_2|^2)\right)\Omega_2.$$

Corollary

Let $f : T^2 \to S^n$ be a homogeneous minimal torus with $\mathcal{H} \neq 0$. Then it is Willmore stable if and only if it is congruent to the Clifford torus in some great $S^3 \subset S^n$.

Corollary

Let $f : T^2 \to S^n$ be a homogeneous minimal torus different from Clifford torus. Then it has index $\geqslant n + 3$.

- Define J on the normal bundle

$$J\psi_3 = \psi_4, J\psi_4 = -\psi_3. \tag{2.1}$$

  Here $\psi_3 = \Omega_1/|\Omega_1|$ and $\psi_4 \perp \psi_3$.

- So $\nabla^{\perp} J = J\nabla^{\perp}$ and

$$\mathcal{L}(J\Omega_1) = 2J\Omega_1,\ \mathcal{L}(J\Omega_2) = 2J\Omega_2.$$

- $dimSpan_{\mathbb{R}}\{J\Omega_1, J\Omega_2, e_j^{\perp}, j = 1, \cdots, 5\} \geqslant 6.$

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Thank you for your attention!