Construction of constant mean curvature surfaces from minimal surfaces

Martin Traizet

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Can we do gluing constructions with DPW method ?

1) *n*-noids (genus zero, *n* unduloid ends) with small necks (Kapouleas 1990, particular case)

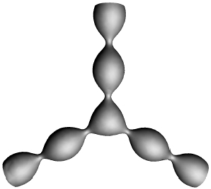

Image by N. Schmitt

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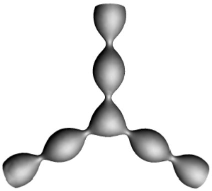

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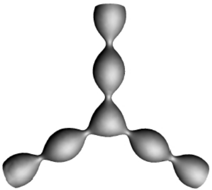

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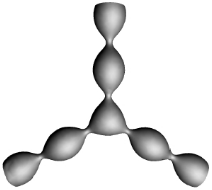

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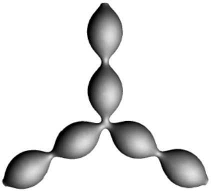
- Kapouleas general construction (higher genus) can be done with DPW by opening nodes but is way more complicated (T. 2018)

2) *n*-noids without central sphere



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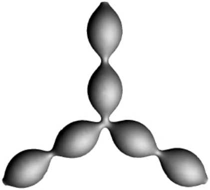

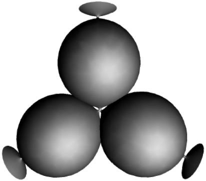

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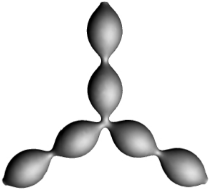

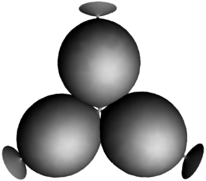

$\longrightarrow$ blowup

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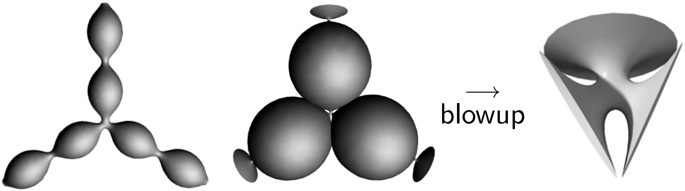

Construction of CMC *n*-noids by gluing half Delaunays to a minimal *n*-noid (Mazzeo Pacard 2001)

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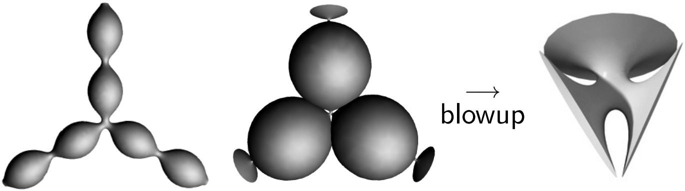

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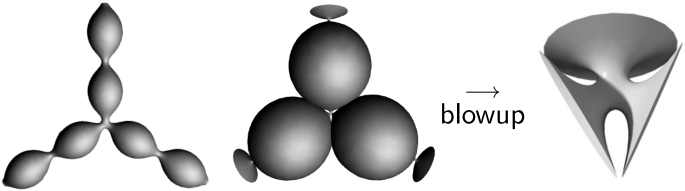

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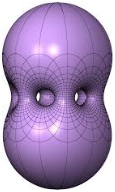
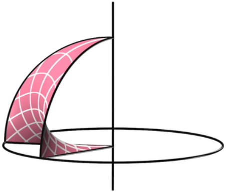
Question : how to produce a DPW potential from the Weierstrass data of a minimal surface ?

3) High-genus Lawson minimal surfaces in $\mathbb{S}^3$ (with S. & L. Heller)





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genus$=k$

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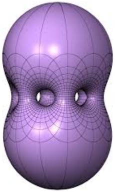

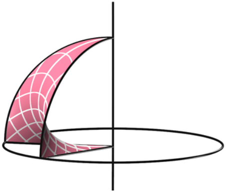

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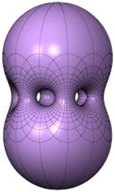

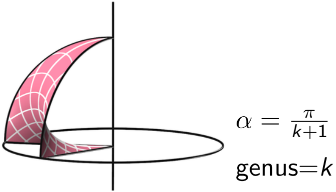

- Lawson surfaces of large genus k can be constructed by DPW
- Corollary : $\text{Area}(\text{Lawson}_k) = 8\pi\left(1 - \frac{\log 2}{k} + O(\frac{1}{k^2})\right)$

2. The DPW method

Identify $\mathbb{R}^3$ with $\mathfrak{su}(2)$ by

$$(x_1, x_2, x_3) \in \mathbb{R}^3 \longleftrightarrow -i \begin{pmatrix} -x_3 & x_1 + i x_2 \\ x_1 - i x_2 & x_3 \end{pmatrix} \in \mathfrak{su}(2)$$

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Loop groups

- $\Lambda SL(2, \mathbb{C}) = \{\phi : \mathbb{S}^1 \overset{C^\infty}{\to} SL(2, \mathbb{C})\}$
- $\Lambda SU(2) = \{F : \mathbb{S}^1 \to SU(2)\}$ (unitary loops)
- $\Lambda_+^{\mathbb{R}} SL(2, \mathbb{C}) = \{B \in \Lambda SL(2, \mathbb{C}) \mid B$ extends holo. to $\mathbb{D}$, $B(0) = \begin{pmatrix} \rho & \mu \\ 0 & \rho^{-1} \end{pmatrix}, \rho > 0\}$ (positive loops)

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Theorem ("Iwasawa" decomposition)

Any $\Phi \in \Lambda SL(2,\mathbb{C})$ can be decomposed uniquely as $\Phi = FB$ with $F \in \Lambda SU(2)$ and $B \in \Lambda_+^{\mathbb{R}} SL(2,\mathbb{C})$

DPW potential : matrix-valued 1-form

$$\xi(z,\lambda) = \begin{pmatrix} \alpha(z,\lambda) & \lambda^{-1}\beta(z,\lambda) \\ \gamma(z,\lambda) & -\alpha(z,\lambda) \end{pmatrix}$$

α, β, γ holomorphic with respect to $z \in \Sigma$ and $\lambda \in \overline{\mathbb{D}}$

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- Gauss map : $N = -i\,F\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}F^{-1}|_{\lambda=1}$
- differential $df = 2i\rho^2 F\begin{pmatrix} 0 & \beta^0 \\ \overline{\beta^0} & 0 \end{pmatrix}F^{-1}|_{\lambda=1}$

  where $\rho = B_{11}|_{\lambda=0}$, $\beta^0 = \beta|_{\lambda=0}$.

Examples

- $\xi = \begin{pmatrix} 0 & \lambda^{-1}dz \\ 0 & 0 \end{pmatrix}$ on $\mathbb{C} \cup \{\infty\}$ with $\Phi(0) = I_2$ gives $\mathbb{S}^2$.

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The Monodromy Problem (or period problem)

If Σ is not simply connected, Φ and f are only well-defined on the universal cover $\widetilde{\Sigma}$ of Σ.

For $\gamma \in \pi_1(\Sigma, z_0)$, let $\mathcal{M}(\Phi, \gamma)$ be the monodromy of Φ.

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For $\gamma \in \pi_1(\Sigma, z_0)$, let $\mathcal{M}(\Phi, \gamma)$ be the monodromy of Φ.

Fact : assume that $\forall \gamma \in \pi_1(\Sigma, z_0)$:

$\mathcal{M}(\Phi, \gamma) \in \Lambda SU(2)$ intrinsic closing condition

$$\left.\begin{array}{l} \mathcal{M}(\Phi,\gamma)|_{\lambda=1} = \pm I_2 \\ \frac{\partial}{\partial \lambda}\mathcal{M}(\Phi,\gamma)|_{\lambda=1} = 0 \end{array}\right\} \text{extrinsic closing condition for } \mathbb{R}^3$$

Then f descends to a well defined immersion on Σ.

3. An (easy) blow-up result

ξ_t smooth family of DPW potentials on Σ

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Proposition

Assume Φ_t solves Monodromy Problem and Φ_0 is constant :

$$\Phi_0(z,\lambda) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}$$

Then $\lim_{t\to 0} \frac{1}{t} f_t$ is a (branched) minimal immersion with Weierstrass data

$$g = \frac{-a}{c} \qquad \omega = 4c^2\beta' \qquad \textit{with } \beta' = \frac{\partial \beta_t}{\partial t}|_{t=0,\lambda=0}$$

i.e. parametrized by Weierstrass Representation Formula

$$z \mapsto Re\int_{z_0}^{z} \left(\tfrac{1}{2}(1-g^2)\omega, \tfrac{i}{2}(1+g^2)\omega, g\omega\right)$$

Proof : Iwasawa decompose Φ_0 :

$$F_0 = \frac{1}{\sqrt{|a|^2 + |c|^2}} \left(\begin{smallmatrix} a & -\overline{c} \\ c & \overline{a} \end{smallmatrix}\right) \qquad B_0 = \frac{1}{\sqrt{|a|^2 + |c^2}} \left(\begin{smallmatrix} |a|^2+|c|^2 & \overline{a}b+\overline{c}d \\ 0 & 1 \end{smallmatrix}\right)$$

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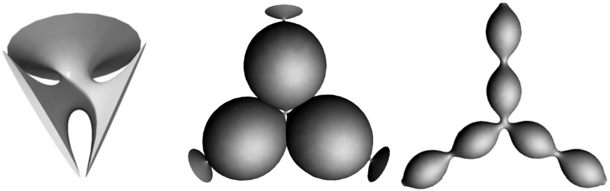
$$\begin{aligned}
\frac{\partial df_t}{\partial t} &= 2i\rho_0^2 F_0 \begin{pmatrix} 0 & \beta' \\ \overline{\beta'} & 0 \end{pmatrix} F_0^{-1} \\
&= 2i \begin{pmatrix} -ac\beta' - \overline{ac\beta'} & a^2\beta' - \overline{c^2\beta'} \\ \overline{a^2\beta'} - c^2\beta' & ac\beta' + \overline{ac\beta'} \end{pmatrix} \\
&= \mathrm{Re}\left(2(c^2-a^2)\beta', 2i(c^2+a^2)\beta', -4ac\beta'\right) \\
&= \mathrm{Re}\left(\tfrac{1}{2}(1-g^2)\omega, \tfrac{i}{2}(1+g^2)\omega, g\omega\right)
\end{aligned}$$

4. Construction of CMC n-noids from minimal n-noids

Theorem

Let $M_0 \subset \mathbb{R}^3$ be a non-degenerate, Alexandrov-embedded minimal surface with genus zero and n catenoidal ends. There exists a smooth family $(M_t)_{0<t<\varepsilon}$ of genus zero, Alexandrov-embedded CMC-1 surfaces in $\mathbb{R}^3$ with n unduloid ends and $\lim_{t\to 0} \frac{1}{t} M_t = M_0$.



1) Choice of the DPW potential

(g, ω) Weierstrass data of M_0. Consider the potential

$$\xi_t = \begin{pmatrix} 0 & \frac{1}{4} t(\lambda - 1)^2 \lambda^{-1} \omega \\ dg & 0 \end{pmatrix}$$

$$\Phi_t(z_0, \lambda) = \begin{pmatrix} g(z_0) & 1 \\ -1 & 0 \end{pmatrix}$$

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Blow-up proposition applies :

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$$\beta_t|_{\lambda=0} = \frac{t}{4}\omega$$

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Can we solve Monodromy Problem ? Why do we get unduloid ends ?

Parameters : need to perturb (g, ω) to solve Monodromy Problem

$$g(z) = \frac{A(z)}{B(z)} \qquad \omega = \frac{B(z)^2\, dz}{\prod_{i=1}^{n}(z - p_i)^2}$$

$$A(z) = \sum_{i=0}^{n-1} a_i z^i \qquad B(z) = \sum_{i=0}^{n-1} b_i z^i$$

$$x = (a_0, \cdots, a_{n-1}, b_0, \cdots, b_{n-1}, p_1, \cdots, p_{n-3})$$

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Functional spaces for functions $u : \mathbb{S}^1 \to \mathbb{C}$:

$$\mathcal{W} = \{u = \sum_{i\in\mathbb{Z}} u_i \lambda^i : \|u\| = \sum |u_i| R^{|i|} < \infty\} \quad (R > 1 \text{ fixed})$$

$$\mathcal{W}^+ = \{u = \sum_{i\in\mathbb{N}} u_i \lambda^i : \|u\| < \infty\}$$

Take $x \in (\mathcal{W}^+)^{3n-3}$ in a neighborhood of the (constant) value x_0 corresponding to M_0.

2) The Monodromy Problem

$\gamma_1, \cdots, \gamma_{n-1}$ generators of $\pi_1(\Sigma, z_0)$
Observe : if $t(\lambda - 1)^2 = 0$ then $\xi_t = \begin{pmatrix} 0 & 0 \\ dg & 0 \end{pmatrix}$ so $\mathcal{M}(\Phi_t, \gamma_i) = I_2$.

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$$M_i(0) \in \Lambda\mathfrak{su}(2) \Leftrightarrow \begin{cases} P_{i,1} \in i\mathbb{R} \\ P_{i,2} = \overline{P_{i,0}} \end{cases} \Leftrightarrow \text{Period Problem for } M_0$$

So Renormalized Monodromy Problem is solved at $(t, x) = (0, x_0)$.

Implicit Function Argument

Notation : $u^*(\lambda) = \overline{u(1/\overline{\lambda})} = \sum_{i \in \mathbb{Z}} \overline{u_{-i}} \lambda^i$. Define

$$\begin{cases} \mathcal{F}_i = M_{i,11} + M^*_{i,11} \\ \mathcal{G}_i = M_{i,12} + M^*_{i,21} \end{cases} \qquad \text{so } M_i \in \Lambda\mathfrak{su}(2) \Leftrightarrow \mathcal{F}_i = \mathcal{G}_i = 0$$

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Notation : $u = u^+ + u^0 + u^- \in \mathcal{W}^{>0} \oplus \mathcal{W}^0 \oplus \mathcal{W}^{<0}$.
Claim : if M_0 is non-degenerate, the differential of

$$(\mathcal{F}_i^+, \mathcal{G}_i^+, \mathcal{G}_i^{-*}, \mathrm{Re}\mathcal{F}_i^0, \mathcal{G}_i^0)_{1\leq i\leq n-3}$$

with respect to x at $(t,x) = (0, x_0)$ is surjective from $(\mathcal{W}^+)^{3n-3}$ to $((\mathcal{W}^{>0})^3 \times \mathbb{R} \times \mathbb{C})^{n-1}$.

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By Implicit Function Theorem, for t small enough, there exists a unique $x(t)$ in a neighborhood of x_0 such that Renormalized Monodromy Problem is solved.

3) Use gauges to understand the singularities

ξ_t has poles at $p_1, \cdots, p_n$ and the poles of g (all now depending on t and λ)

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A gauge on Σ is $G(z,\lambda) \in SL(2,\mathbb{C})$ holomorphic with respect to $z \in \Sigma$, $\lambda \in \mathbb{D}$, such that $G(z,0)$ is upper-triangular.

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Claim : The poles of g are apparent singularities

Proof : take the gauge

$$G = \begin{pmatrix} g^{-1} & -1 \\ 0 & g \end{pmatrix}$$

$$\widehat{\xi}_t = \begin{pmatrix} 0 & \frac{1}{4}t(\lambda-1)^2\lambda^{-1}g^2\omega \\ g^{-2}dg & 0 \end{pmatrix}$$

is holomorphic near poles of g, so f_t extends analytically at poles of g.

Claim : ξ_t has moving singularities at $p_i(t,\lambda)$ but these can be gauged to fixed singularities at $p_i(t,0)$ and f_t is defined on $\mathbb{C} \cup \{\infty\} \setminus \{p_1(t,0), \cdots, p_n(t,0)\}$.

Proof : define $\psi_\lambda(z) = z + p_i(t,\lambda) - p_i(t,0)$ and take

$$G(z,\lambda) = \Phi_t(z,\lambda)^{-1}\Phi_t(\psi_\lambda(z),\lambda)$$

Use Gronwall Inequality and Riemann Extension Theorem to prove that G extends holomorphically at $\lambda = 0$ so is a gauge. So Φ_t and $\psi_\lambda^*\Phi_t$ define the same immersion. But $\psi_\lambda^*\Phi_t$ has a fixed singularity at $p_i(t,0)$.

Claim : f_t has a Delaunay end at p_i

Proof : take local coordinate $w = g(z) - g(p_i)$. Let $r, s \in \mathbb{R}$ be the solution of

$$\begin{cases} rs = \frac{1}{4} t\alpha_i & \text{with } \alpha_i = \operatorname{Res}_{p_i} w\,\omega \\ r + s = \frac{1}{2} \\ r < s \end{cases}$$

$$G = \begin{pmatrix} \frac{\sqrt{r+s\lambda}}{\sqrt{w}} & \frac{-1}{2\sqrt{r+s\lambda}\sqrt{w}} \\ 0 & \frac{\sqrt{w}}{\sqrt{r+s\lambda}} \end{pmatrix}$$

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Raujouan (2018) : $\exists$ uniform $\varepsilon > 0$ such that $f_t(D(p_i, \varepsilon))$ is embedded.