Minimal Maslov number of *R*-spaces canonically embedded in Einstein-Kähler *C*-spaces

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Plan of this talk

0. What is an $\boldsymbol{R}$-space?
1. How to make $\boldsymbol{R}$-spaces and their canonical embeddings
2. The canonical embedding of $\boldsymbol{R}$-spaces into Kähler $\boldsymbol{C}$-spaces
3. The canonical embedding of $\boldsymbol{R}$-spaces into Einstein-Kähler $\boldsymbol{C}$-spaces
4. Minimal Maslov number of Lagrangian submanifolds
5. Main results
6. Some examples
8. References

What is an *R*-space?

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Masaru Takeuchi, *Cell decompositions and Morse equalities on certain symmetric spaces*, Fac. Sci. Univ. Tokyo, I, **12** (1965), 81–192.

Masaru Takeuchi and Shoshichi Kobayashi, *Minimal imbeddings of **R**-spaces*, J. Differential Geom. **2** (1969), 203–215.

Dirk Ferus, *Immersionen mit paralleler zweiter Fundamentalform: Beispiele und Nicht-Beispiele*. (in German) Manuscripta Math. **12** (1974), 153–162.

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What is an R-space?

$L = \mathbb{R}P^2 \quad \subset \quad \mathbb{C}P^2 = M$

What is an *R*-space?

***R*-space** | Einstein-Kähler ***C*-space**

$$L = \mathbb{R}P^2 \subset \mathbb{C}P^2 = M$$

canonical embedding

What is an R-space?

$$\begin{tikzcd} S^2 = \{\mathbf{x} \in \mathbb{R}^3 \mid \|\mathbf{x}\| = 1\} \subset \mathbb{R}^3 \text{: 2-dim. std. sphere} \arrow[d, "\pi"', "\mathbb{Z}_2"] \\ \mathbb{R}P^2 := \{V \subset \mathbb{R}^3 \mid 1\text{-dim. vect. subsp. of } \mathbb{R}^3\} \end{tikzcd}$$

2-dim. real projective space ("an ***R***-space")

$$\begin{array}{ccl} \mathbb{R}^3 & \subset & \mathbb{C}^3 = \mathbb{R}^3 + \sqrt{-1}\,\mathbb{R}^3 \\ \cup & & \cup \\ V & \longmapsto & V^{\mathbb{C}} = V + \sqrt{-1}\,V \\ & & \Cap \end{array}$$

$\mathbb{C}P^2 = \{\text{complex } 1\text{-dim. vect. subsp. of } \mathbb{C}^3\}$

2-dim. complex projective space ("an Einstein-Kähler ***C***-space")

$$
\begin{array}{ccccccc}
\mathbb{R}^3 & \supset & V & \longrightarrow & V^{\mathbb{C}} & \subset & \mathbb{C}^3 \\
\cup & & & & & & \cup \\
S^2 & & & \subset & & & S^5 \\
\Big\downarrow{\scriptstyle \color{red}{\mathbb{Z}_2}} & & & & & & \Big\downarrow{\scriptstyle \color{red}{\text{Hopf fib.}}} \\
\mathbb{R}P^2 & \ni & V & \longmapsto & V^{\mathbb{C}} & \in & \mathbb{C}P^2 \\
\| & & & & & & \|
\end{array}
$$

$$
\{\mathbf{1}\text{-dim. vect. subsp. of } \mathbb{R}^3\} \quad \subset \quad \{\text{cplx. } \mathbf{1}\text{-dim. vect. subsp. of } \mathbb{C}^3\}
$$

2-dim. real proj. sp.   **2**-dim. complex proj. sp.

$$
\begin{array}{ccccccc}
SO(3) & & & \subset & & & SU(3) \\
\mathbb{R}^3 & \supset & V & \longrightarrow & V^{\mathbb{C}} & \subset & \mathbb{C}^3 \\
\cup & & & & & & \cup \\
S^2 & & & \subset & & & S^5 \\
\Big\downarrow {\color{red}\mathbb{Z}_2} & & & & & & \Big\downarrow {\color{red}\text{Hopf fib.}} \\
\mathbb{R}P^2 & \ni & V & \longmapsto & V^{\mathbb{C}} & \in & \mathbb{C}P^2 \\
\| & & & & & & \| \\
\{1\text{-dim. vect. subsp. of } \mathbb{R}^3\} & & & \subset & & & \{\text{cplx. } 1\text{-dim. vect. subsp. of } \mathbb{C}^3\} \\
{\color{red}2\text{-dim. real proj. sp.}} & & & & & & {\color{red}2\text{-dim. complex proj. sp.}} \\
\wr\| & & & & & & \wr\| \\
SO(3)/S(O(1)\times O(2)) & & & & & & SU(3)/S(U(1)\times U(2))
\end{array}
$$

$$\begin{array}{ccccccc}
\mathfrak{o}(3) & & & \subset & & & \mathfrak{su}(3) \\
SO(3) & & & \subset & & & SU(3) \\
\mathbb{R}^3 & \supset & V & \longrightarrow & V^{\mathbb{C}} & \subset & \mathbb{C}^3 \\
\cup & & & & & & \cup \\
S^2 & & & \subset & & & S^5 \\
\downarrow \mathbb{Z}_2 & & & & & & \downarrow \text{Hopf fib.} \\
\mathbb{R}P^2 & \ni & V & \longmapsto & V^{\mathbb{C}} & \in & \mathbb{C}P^2 \\
\| & & & & & & \| \\
\{1\text{-dim. vect. subsp. of } \mathbb{R}^3\} & & & \subset & & & \{\text{cplx. } 1\text{-dim. vect. subsp. of } \mathbb{C}^3\} \\
2\text{-dim. real proj. sp.} & & & & & & 2\text{-dim. complex proj. sp.} \\
\wr & & & & & & \wr \\
SO(3)/S(O(1) \times O(2)) & & & & & & SU(3)/S(U(1) \times U(2))
\end{array}$$

$$\mathfrak{su}(3) = \mathfrak{o}(3) + \sqrt{-1}S_0^2(\mathbb{R}^3)$$

$$\Psi \qquad H = \sqrt{-1}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{array}{ccc} \mathfrak{o}(3) & \subset & \mathfrak{su}(3) \\ SO(3) & \subset & SU(3) \end{array}$$

$$\begin{array}{ccccccc} \mathbb{R}^3 & \supset & V & \longrightarrow & V^{\mathbb{C}} & \subset & \mathbb{C}^3 \\ \cup & & & & & & \cup \\ S^2 & & & \subset & & & S^5 \\ \downarrow \mathbb{Z}_2 & & & & & & \downarrow \text{Hopf fib.} \\ \mathbb{R}P^2 & \ni & V & \longmapsto & V^{\mathbb{C}} & \in & \mathbb{C}P^2 \\ \| & & & & & & \| \end{array}$$

$\{1$-dim. vect. subsp. of $\mathbb{R}^3\}$ $\subset$ $\{$cplx. 1-dim. vect. subsp. of $\mathbb{C}^3\}$

2-dim. real proj. sp.   2-dim. complex proj. sp.

$$\wr\| \qquad\qquad \wr\|$$

$$SO(3)/S(O(1)\times O(2)) \qquad SU(3)/S(U(1)\times U(2))$$

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$$\mathfrak{su}(3) = \mathfrak{o}(3) + \sqrt{-1}S_0^2(\mathbb{R}^3)$$

$$\ni \quad H = \sqrt{-1}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{array}{ccccccc}
\mathfrak{o}(3) & & & \subset & & & \mathfrak{su}(3) \\
K = SO(3) & & & \subset & & & G = SU(3) \\
\mathbb{R}^3 & \supset & V & \longrightarrow & V^{\mathbb{C}} & \subset & \mathbb{C}^3 \\
\cup & & & & & & \cup \\
S^2 & & & \subset & & & S^5 \\
\downarrow \mathbb{Z}_2 & & & & & & \downarrow \\
\mathbb{R}P^2 & \ni & V & \longmapsto & V^{\mathbb{C}} & \in & \mathbb{C}P^2 \\
\| & & & & & & \| \\
\{1\text{-dim. vect. subsp. of } \mathbb{R}^3\} & & & \subset & & & \{\text{cplx. } 1\text{-dim. vect. subsp. of } \mathbb{C}^3\} \\
2\text{-dim. real proj. sp.} & & & & & & 2\text{-dim. complex proj. sp.} \\
\cong & & & & & & \cong \\
SO(3)/S(O(1)\times O(2)) & & & & & & SU(3)/S(U(1)\times U(2)) \\
\| & & & & & & \| \\
K/K_H & & & & & & G/G_H
\end{array}$$

where $K_H := K \cap G_H$ $\qquad$ $G_H := \{a \in G \mid \mathrm{Ad}(a)H = H\}$

$$\mathfrak{su}(3) = \mathfrak{o}(3) + \sqrt{-1}S^2_0(\mathbb{R}^3)$$

$$\rotatebox{90}{$\in$}$$

$$H = \sqrt{-1}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{array}{ccccccc}
 & \mathfrak{o}(3) & & & \subset & & \mathfrak{su}(3) \\
{\color{red}K =} & SO(3) & & & \subset & & {\color{red}G =}\ SU(3) \\
 & \mathbb{R}^3 & \supset & V & \longrightarrow & V^{\mathbb{C}} \subset & \mathbb{C}^3 \\
 & \cup & & & & & \cup \\
 & S^2 & & & \subset & & S^5 \\
 & \downarrow {\color{red}\mathbb{Z}_2} & & & & & \downarrow \\
 & \mathbb{R}P^2 & \ni & V & \longmapsto & V^{\mathbb{C}} \in & \mathbb{C}P^2 \\
 & \| & & & & & \|
\end{array}$$

$$\{1\text{-dim. vect. subsp. of } \mathbb{R}^3\} \quad \subset \quad \{\text{cplx. } 1\text{-dim. vect. subsp. of } \mathbb{C}^3\}$$

$$\begin{array}{ccc}
{\color{red}2\text{-dim. real proj. sp.}} & & {\color{red}2\text{-dim. complex proj. sp.}} \\
\wr\| & & \wr\| \\
SO(3)/S(O(1)\times O(2)) & & SU(3)/S(U(1)\times U(2)) \\
{\color{red}\wr\|} & & {\color{red}\wr\|} \\
{\color{red}(\mathrm{Ad}K)H \subset \sqrt{-1}S^2_0(\mathbb{R}^3)} & & {\color{red}(\mathrm{Ad}G)H \subset \mathfrak{su}(3)}
\end{array}$$

$$\mathfrak{su}(3) = \mathfrak{o}(3) + \sqrt{-1}\,S^2_0(\mathbb{R}^3) \qquad \theta(X) := \bar{X} \quad (\forall X \in \mathfrak{su}(3))$$

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p} \qquad \mathfrak{su}(3) \ni H = \sqrt{-1}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{array}{ccccccccc}
\mathfrak{k} = \mathfrak{o}(3) & & & & \subset & & & \mathfrak{g} = \mathfrak{su}(3) & \\
K = SO(3) & & & & \subset & & & G = SU(3) & \\
\mathbb{R}^3 & \supset & V & & \longrightarrow & V^{\mathbb{C}} & \subset & \mathbb{C}^3 & \\
\cup & & & & & & & \cup & \\
S^2 & & & & \subset & & & S^5 & \\
\downarrow \mathbb{Z}_2 & & & & & & & \downarrow \text{Hopf fib.} & \\
L = \mathbb{R}P^2 & \ni & V & & \longmapsto & V^{\mathbb{C}} & \in & \mathbb{C}P^2 & = M \\
\parallel & & & & & & & \parallel & \\
\{1\text{-dim. vect. subsp. of } \mathbb{R}^3\} & & & & \subset & & & \{\text{cplx. } 1\text{-dim. vect. subsp. of } \mathbb{C}^3\} & \\
2\text{-dim. real proj. sp.} & & & & & & & 2\text{-dim. complex proj. sp.} & \\
\cong & & & & & & & \cong & \\
SO(3)/S(O(1)\times O(2)) & & & & & & & SU(3)/S(U(1)\times U(2)) & \\
\cong & & & & & & & \cong & \\
(\mathrm{Ad}K)H \subset \mathfrak{p} & & & & & & & (\mathrm{Ad}G)H \subset \mathfrak{g} &
\end{array}$$

(G,K,θ) is a Riemannian symmetric pair!

$$\mathfrak{su}(3) = \mathfrak{o}(3) + \sqrt{-1}S^2_0(\mathbb{R}^3) \qquad \theta(X) := \bar{X} \quad (\forall X \in \mathfrak{su}(3))$$

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

$$\rotatebox{90}{$\in$}$$

$$H = \sqrt{-1}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{array}{ccccccc}
\mathfrak{k} = \mathfrak{o}(3) & & & \subset & & & \mathfrak{g} = \mathfrak{su}(3) \\
K = SO(3) & & & \subset & & & G = SU(3) \\
\mathbb{R}^3 & \supset & V & \longrightarrow & V^{\mathbb{C}} & \subset & \mathbb{C}^3 \\
\cup & & & & & & \cup \\
S^2 & & & \subset & & & S^5 \\
\downarrow \mathbb{Z}_2 & & & & & & \downarrow \text{Hopf fib.} \\
L = \mathbb{R}P^2 & \ni & V & \longmapsto & V^{\mathbb{C}} & \in & \mathbb{C}P^2 = M \\
\| & & & & & & \| \\
\{1\text{-dim. vect. subsp. of } \mathbb{R}^3\} & & & \subset & & & \{\text{cplx. } 1\text{-dim. vect. subsp. of } \mathbb{C}^3\}
\end{array}$$

R-space (left) **Einstein-Kähler C-space** (right)

2-dim. real proj. sp. $\qquad$ 2-dim. complex proj. sp.

$$\cong \qquad\qquad \cong$$

$$SO(3)/S(O(1)\times O(2)) \qquad SU(3)/S(U(1)\times U(2))$$

$$\cong \qquad\qquad \cong$$

$$(\mathrm{Ad}K)H \subset \mathfrak{p} \qquad (\mathrm{Ad}G)H \subset \mathfrak{g}$$

How to make *R*-spaces and their canonical embeddings

(G, K, θ): a compact Riemannian symmetric pair.

$\mathfrak{g} = \mathfrak{k} + \underset{\forall H}{\underset{\Cup}{\mathfrak{p}}}$: canonical decomp. of sym. Lie alg. $(\mathfrak{g}, \mathfrak{k}, \theta)$

R-space $\qquad\qquad\qquad\qquad\qquad\qquad$ Kähler *C*-space

$$\begin{array}{ccccc}
K & & \subset & & G \\
\downarrow & & & & \downarrow \\
L = K/K_H & \ni \; aK_H & \longmapsto & aG_H \; \in & G/G_H = M \\
\wr\| & & & & \wr\| \\
(\mathrm{Ad}K)H & & \subset & & (\mathrm{Ad}G)H \\
\cap & & & & \cap \\
\mathfrak{p} \quad + \quad \mathfrak{k} & & = & & \mathfrak{g}
\end{array}$$

where $\quad K_H := K \cap G_H \qquad\qquad G_H := \{a \in G \mid \mathrm{Ad}(a)H = H\}$

Suppose

$$(G, K, \theta)\text{: compact Riemannian symmetric pair.}$$

Here G: connected compact Lie group.
$\langle\ ,\ \rangle$: $\mathrm{Ad}(G)$- and θ-inv. inner product of $\mathfrak{g}$.

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

: the canonical decomp. of $\mathfrak{g}$ as a symm. Lie alg.
For each $H \in \mathfrak{p}$, the compact homogeneous space

$$K/K_H \cong \mathrm{Ad}(K)H \subset \mathfrak{p}$$

is called an *R-space*, where

$$K_H := \{a \in K \mid \mathrm{Ad}_{\mathfrak{p}}(a)(H) = H\} \text{ (isot. subgp. of } K \text{ at } H\text{).}$$

It has the *standard imbedding* into the Euclidean space $\mathfrak{p}$:

$$\phi_H : K/K_H \ni aK_H \longmapsto \mathrm{Ad}_{\mathfrak{p}}(a)(H) \in \mathfrak{p}.$$

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Set

$$G_H := \{a \in G \mid \mathrm{Ad}_{\mathfrak{p}}(a)(H) = H\} \text{ (isot. subgp. of } G \text{ at } H).$$

The compact homogeneous space G/G_H is called a *generalized flag manifold*. Since a generalized flag manifold admits G-invariant Kähler structures, It is also called a *Kähler C-space*. It has the *standard imbedding* into the Euclidean space $\mathfrak{g}$:

$$\psi_H : G/G_H \ni aG_H \longmapsto \mathrm{Ad}(a)(H) \in \mathfrak{g}.$$

Then it is classically well-known that

$$G_H \text{ is connected}$$

and by definition we have

$$K_H = K \cap G_H.$$

Set

$$\mathfrak{g} = \mathfrak{g}_H + \mathfrak{m}, \quad T_{eG_H}M \cong \mathfrak{m},$$
$$\mathfrak{k} = \mathfrak{k}_H + \mathfrak{l}, \quad T_{eK_H}L \cong \mathfrak{l}.$$

Note that $\mathfrak{k}_H = \mathfrak{k} \cap \mathfrak{g}_H$.

Lemma

$$\theta(G_H) = G_H \text{ and } \theta(\mathfrak{g}_H) = \mathfrak{g}_H$$

$$\begin{aligned}\mathfrak{g} =&(\mathfrak{g}_H \cap \mathfrak{k}) + (\mathfrak{g}_H \cap \mathfrak{m}) + (\mathfrak{m} \cap \mathfrak{k}) + (\mathfrak{m} \cap \mathfrak{p}) \\ =&\mathfrak{k}_H + \mathfrak{l} + (\mathfrak{g}_H \cap \mathfrak{p}) + (\mathfrak{m} \cap \mathfrak{p})\end{aligned}$$

We have $\mathfrak{m} = \mathfrak{m} \cap \mathfrak{k} + \mathfrak{m} \cap \mathfrak{p}$, $\mathfrak{l} = \mathfrak{m} \cap \mathfrak{k}$.

$$(\mathrm{ad}H) : \mathfrak{m} \cap \mathfrak{k} \longrightarrow \mathfrak{m} \cap \mathfrak{p}, \quad (\mathrm{ad}H) : \mathfrak{m} \cap \mathfrak{p} \longrightarrow \mathfrak{m} \cap \mathfrak{k}$$

are injective and thus $\dim \mathfrak{m} \cap \mathfrak{k} = \dim \mathfrak{m} \cap \mathfrak{p}$.

Lemma

$$2 \dim L = \dim M.$$

The G-invariant *symplectic form* ω_H on $M = G/G_H$ is defined by

$$\omega_H(X, Y) := \langle [H, X], Y \rangle \quad (X, Y \in \mathfrak{m}).$$

For each $X, Y \in \mathfrak{l}$,

$$\omega_H(X, Y) = \langle [H, X], Y \rangle = 0.$$

Hence

Proposition

The canonical embedding

$$\iota_H : L = K/K_H \ni aK_H \longmapsto aG_H \in G/G_H = M$$

is a Lagrangian embedding into a symplectic manifold (M, ω_H).

(G, K, θ): a compact Riemannian symmetric pair.

$\mathfrak{g} = \mathfrak{k} + \underset{\forall H}{\underset{\Psi}{\mathfrak{p}}}$: canonical decomp. of sym. Lie alg. $(\mathfrak{g}, \mathfrak{k}, \theta)$

$$\begin{array}{ccccc}
K & & \subset & & G \\
\downarrow & & & & \downarrow \\
\textit{R}\text{-space} & & & & \text{Kähler } \textit{C}\text{-space} \\
L = K/K_H \ni aK_H & & \longmapsto & & aG_H \in (G/G_H, \omega_H) = M \\
\wr\!\parallel & & & & \wr\!\parallel \\
(\mathrm{Ad}K)H & & \subset & & (\mathrm{Ad}G)H \\
\cap & & & & \cap \\
\mathfrak{p} & + & \mathfrak{k} & = & \mathfrak{g}
\end{array}$$

where $\quad K_H := K \cap G_H \qquad G_H := \{a \in G \mid \mathrm{Ad}(a)H = H\}$

The involutive automorphism θ of G induces the involutive diffeomorphism $\hat{\theta}_H$ of $M = G/G_H$ as follows:

$$\hat{\theta}_H : M = G/G_H \ni aG_H \longmapsto \theta(a)G_H \in G/G_H = M.$$

Then we have

$$\hat{\theta}_H^* \omega_H = -\omega_H.$$

Proposition

$\hat{\theta}_H : M = G/G_H \to G/G_H = M$ is an anti-symplectic involutive diffeomorphism with respect to ω_H.

Define

$$\mathrm{Fix}(M, \hat{\theta}_H) := \{p \in M \mid \hat{\theta}_H(p) = p\}$$

: the fixed point subset of $\hat{\theta}_H$ on M

Here note that

$$\iota_H(L) \subset \mathrm{Fix}(M, \hat{\theta}_H) \quad \text{as a connected component.}$$

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The group action of G on $M = G/G_H$ is a Hamiltonian group action wrt. ω_H with the moment map (which is given by the standard imbedding)

$$\mu_G = \psi_H : M = G/G_H \longrightarrow \mathfrak{g} \cong \mathfrak{g}^*.$$

The group action of K on $M = G/G_H$ is a Hamiltonian group action wrt. ω_H with the moment map (which is given by its orthogonal projection)

$$\mu_K = \pi_{\mathfrak{k}} \circ \mu_G = \pi_{\mathfrak{k}} \circ \psi_H : M = G/G_H \longrightarrow \mathfrak{k} \cong \mathfrak{k}^*.$$

Here

$\pi_{\mathfrak{k}} : \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \longrightarrow \mathfrak{k}$: the orthogonal projection onto $\mathfrak{k}$

Then the relationship between the moment maps with the involution diffeomorphism $\hat{\theta}_H$ is given in the following way.

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Proposition

$$\mu_G \circ \hat{\theta}_H = -\theta \circ \mu_G, \quad \mu_K \circ \hat{\theta}_H = -\mu_K.$$

and it implies

Corollary

$$\mathrm{Fix}(M, \hat{\theta}_H) = \mu_K^{-1}(0).$$

By a result of F. Kirwan (Invent. Math. 1984) we know that $\mu_K^{-1}(0)$ is connected. Then we obtain

Proposition

$$\iota_H(L) = \mathrm{Fix}(M, \hat{\theta}_H) = \mu_K^{-1}(0).$$

(G, K, θ): a compact Riemannian symmetric pair.

$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$: canonical decomp. of sym. Lie alg. $(\mathfrak{g}, \mathfrak{k}, \theta)$

$$\mathfrak{p} \ni \forall H$$

R-space and **Kähler C-space**

$$\begin{tikzcd}
K \arrow[r, phantom, "\subset"] \arrow[d] & G \arrow[d] \\
\iota_H(L) = \iota_H(K/K_H) = \mathrm{Fix}(M, \hat{\theta}_H) = \mu_K^{-1}(0) \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\parallel"] & (G/G_H, \omega_H) = M \arrow[d, "\mu_G"] \arrow[dl, "\mu_K"'] \\
(\mathrm{Ad}K)H \subset \mathfrak{p} + \mathfrak{k} \arrow[r, phantom, "="] & \mathfrak{g}
\end{tikzcd}$$

where $K_H := K \cap G_H$ $\qquad G_H := \{a \in G \mid \mathrm{Ad}(a)H = H\}$

Invariant complex structures and Kähler metrics on M

Invariant complex structures and Kähler metrics on G/G_H

Let $\mathfrak{a}$: a maximal abelian subspace of $\mathfrak{p}$.
Choose $\mathfrak{t}$: a maximal abelian subalgebra of $\mathfrak{g}$ such that

$$\begin{array}{ccccc} \mathfrak{g} & = & \mathfrak{k} & + & \mathfrak{p} \\ \cup & & & & \cup \\ \mathfrak{t} & & \supset & & \mathfrak{a} \end{array}$$

Let $\mathfrak{a}$: a maximal abelian subspace of $\mathfrak{p}$.
Choose $\mathfrak{t}$: a maximal abelian subalgebra of $\mathfrak{g}$ such that

$$\begin{array}{ccccc} \mathfrak{g} & = & \mathfrak{k} & + & \mathfrak{p} \\ \cup & & \cup & & \cup \\ \mathfrak{t} & = & \mathfrak{b} & + & \mathfrak{a} \end{array}$$

Here $\mathfrak{b} = \mathfrak{t} \cap \mathfrak{k}$, $\mathfrak{a} = \mathfrak{t} \cap \mathfrak{p}$. In particular, $\mathfrak{t}$ is invariant by θ.
$\Sigma(\mathfrak{g})$: the set of all roots of $\mathfrak{g}$ relative to $\mathfrak{t}$.
The root space decomposition of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{t}$ is given by

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} + \sum_{\alpha \in \Sigma(\mathfrak{g})} \mathfrak{g}^{\alpha},$$

where

$$\mathfrak{g}^{\alpha} = \{X \in \mathfrak{g}^{\mathbb{C}} \mid \mathrm{ad}\xi(X) = \sqrt{-1}\langle \alpha, \xi \rangle X \ (\forall \xi \in \mathfrak{t})\}.$$

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The invariant complex structure on G/G_H

Let $\mathfrak{a}$: a maximal abelian subspace of $\mathfrak{p}$.
Choose $\mathfrak{t}$: a maximal abelian subalgebra of $\mathfrak{g}$ such that

$$\begin{array}{ccccc} \mathfrak{g} & = & \mathfrak{k} & + & \mathfrak{p} \\ \cup & & \cup & & \cup \\ \mathfrak{t} & = & \mathfrak{b} & + & \mathfrak{a} \end{array}$$

Here $\mathfrak{b} = \mathfrak{t} \cap \mathfrak{k}$, $\mathfrak{a} = \mathfrak{t} \cap \mathfrak{p}$. In particular, $\mathfrak{t}$ is invariant by θ.
$\boldsymbol{\Sigma}(\mathfrak{g})$: the set of all roots of $\mathfrak{g}$ relative to $\mathfrak{t}$.
$\boldsymbol{\Sigma}_0(\mathfrak{g}) := \{\alpha \in \boldsymbol{\Sigma}(\mathfrak{g}) \mid \alpha \in \mathfrak{b}\}$

Invariant complex structures and Kähler metrics on G/G_H

Let $\mathfrak{a}$: a maximal abelian subspace of $\mathfrak{p}$.
Choose $\mathfrak{t}$: a maximal abelian subalgebra of $\mathfrak{g}$ such that

$$\begin{array}{ccccc} \mathfrak{g} & = & \mathfrak{k} & + & \mathfrak{p} \\ \cup & & \cup & & \cup \\ \mathfrak{t} & = & \mathfrak{b} & + & \mathfrak{a} \end{array}$$

Here $\mathfrak{b} = \mathfrak{t} \cap \mathfrak{k}$, $\mathfrak{a} = \mathfrak{t} \cap \mathfrak{p}$. In particular, $\mathfrak{t}$ is invariant by θ.
$\boldsymbol{\Sigma}(\mathfrak{g})$: the set of all roots of $\mathfrak{g}$ relative to $\mathfrak{t}$.
$\boldsymbol{\Sigma}_0(\mathfrak{g}) := \{\alpha \in \boldsymbol{\Sigma}(\mathfrak{g}) \mid \alpha \in \mathfrak{b}\}$

Define $\sigma \in O(\mathfrak{t})$ by

$$\sigma(H_\mathfrak{b} + H_\mathfrak{a}) := -H_\mathfrak{b} + H_\mathfrak{a} = -\theta(H_\mathfrak{b} + H_\mathfrak{a}) \ (H_\mathfrak{b} \in \mathfrak{b}, H_\mathfrak{a} \in \mathfrak{a}).$$

$<$: a linear order of $\mathfrak{t}$
σ-order $\iff$ If $\alpha \in \boldsymbol{\Sigma}(\mathfrak{g}) \setminus \boldsymbol{\Sigma}_0(\mathfrak{g})$ and $\alpha > 0$, then $\sigma(\alpha) > 0$.

$$\begin{array}{ccccc} \mathfrak{g} & = & \mathfrak{k} & + & \mathfrak{p} \\ \cup & & \cup & & \cup \\ \mathfrak{t} & = & \mathfrak{b} & + & \mathfrak{a} \end{array}$$

Fix a σ-order $<$ of $\mathfrak{t}$.

$$\Sigma^+(\mathfrak{g}) := \{\alpha \in \Sigma(\mathfrak{g}) \mid \alpha > 0\}.$$

We may assume that $H \in \mathfrak{a}$ satisfies

$$\langle \alpha, H \rangle \geq 0 \quad (\forall \alpha \in \Sigma^+(\mathfrak{g})).$$

$\Pi = \Pi(\mathfrak{g}) = \{\alpha_1, \cdots, \alpha_\ell\}$: fund. root syst. (Dynkin diagram) of $\mathfrak{g}$.
$\{\Lambda_1, \cdots, \Lambda_\ell\}$: corresponding fund. weight syst. of $\mathfrak{g}$ defined by

$$\frac{2(\Lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij} \quad (i, j = 1, \cdots, \ell).$$

$$\Pi_H := \Pi_H(\mathfrak{g}) := \{\alpha_i \in \Pi(\mathfrak{g}) \mid (\alpha_i, H) = 0\},$$

$$\Sigma_H(\mathfrak{g}) = \{\alpha \in \Sigma(\mathfrak{g}) \mid (\alpha, H) = 0\} = \Sigma(\mathfrak{g}) \cap \left(\bigoplus_{\alpha_i \in \Pi_H} \mathbb{Z}\alpha_i\right),$$

$$\Sigma_H^+(\mathfrak{g}) := \Sigma^+(\mathfrak{g}) \cap \left(\bigoplus_{\alpha_i \in \Pi_H} \mathbb{Z}^{\geq 0}\alpha_i\right).$$

$$\mathfrak{c}_H := \bigoplus_{\alpha_i \in \Pi \setminus \Pi_H} \mathbb{R}\Lambda_{\alpha_i} = \mathfrak{c}(\mathfrak{g}_H) \subset \mathfrak{t},$$

$$Z_{\mathfrak{c}_H} := \bigoplus_{\alpha_i \in \Pi \setminus \Pi_H} \mathbb{Z}\Lambda_{\alpha_i} \subset \mathfrak{c}_H.$$

Then note that $\quad \mathfrak{c}_H = \mathfrak{c}_H \cap \mathfrak{b} + \mathfrak{c}_H \cap \mathfrak{a}.$

Invariant complex structure on G/G_H

Define a G-invariant complex structure J_H on G/G_H such that

$$T_{eG_H}(G/G_H)^{\mathbb{C}} = T_{eG_H}(G/G_H)^{1,0} \oplus T_{eG_H}(G/G_H)^{0,1} \cong \mathfrak{m}^{\mathbb{C}},$$

$$T_{eG_H}(G/G_H)^{1,0} \cong \sum_{\alpha \in \Sigma^+(G) \setminus \Sigma_H^+(G)} \mathfrak{g}^{-\alpha},$$

$$T_{eG_H}(G/G_H)^{0,1} \cong \sum_{\alpha \in \Sigma^+(G) \setminus \Sigma_H^+(G)} \mathfrak{g}^{\alpha}.$$

Then

$$\theta\left(\sum_{\alpha \in \Sigma^+(G) \setminus \Sigma_H^+(G)} \mathfrak{g}^{-\alpha} \right) = \sum_{\alpha \in \Sigma^+(G) \setminus \Sigma_H^+(G)} \mathfrak{g}^{\alpha}.$$

Proposition

$\hat{\theta}_H : G/G_H \to G/G_H$ is anti-holomorphic with respect to J_H, that is, $J_H \circ d\hat{\theta}_H = -d\hat{\theta}_H \circ J_H$.

For any $H' \in \mathfrak{c}_H^+$,

$$\Pi_{H'} = \Pi_H, \Sigma_{H'} = \Sigma_H$$
$$\mathfrak{g}_{H'} = \mathfrak{g}_H, G_{H'} = G_H,$$
$$G/G_{H'} = G/G_H = M.$$

We know

Lemma

The invariant complex structure $J_{H'}$ coincides with the invariant complex structure J_H on $M = G/G_{H'} = G/G_H$.

There is a linear isomorphism

$$\begin{array}{ccc} \mathfrak{c}_H & \ni \lambda \longmapsto [\omega(\lambda)] \in & H^2(G/G_H, \mathbb{R}) \\ \cup & & \cup \\ Z_{\mathfrak{c}_H} & \longleftrightarrow & H^2(G/G_H, \mathbb{Z}) \end{array}$$

Here for each $\lambda \in \mathfrak{c}_H$, define a closed G-invariant 2-form on G/G_H by

$$\omega(\lambda)(X, Y) := -\frac{1}{2\pi}\langle[\lambda, X], Y\rangle \quad (X, Y \in \mathfrak{m})$$

For each

$$\lambda \in \mathfrak{c}_H^+ := \bigoplus_{\alpha_i \in \Pi \setminus \Pi_H} \mathbb{R}^+ \Lambda_{\alpha_i},$$

a G-invariant Kähler metric $g(\lambda)$ on G/G_H relative to J_H is defined by

$$\omega(\lambda)(X, Y) = g(\lambda)(J_H(X), Y) \quad (X, Y \in \mathfrak{m}).$$

There are linear isomorphisms

$$\begin{array}{ccccc}
\mathfrak{c}_H = \bigoplus_{\alpha_i \in \Pi \setminus \Pi_H} \mathbb{R}\Lambda_{\alpha_i} & \ni \lambda \leftrightarrow [\omega(\lambda)] \in & H^2(G/G_H, \mathbb{R}) \\
\cup & & \cup \\
Z_{\mathfrak{c}_H} = \bigoplus_{\alpha_i \in \Pi \setminus \Pi_H} \mathbb{Z}\Lambda_{\alpha_i} & \longleftrightarrow & H^2(G/G_H, \mathbb{Z})
\end{array}$$

and

$$\begin{array}{ccccc}
\mathfrak{c}_H^* = \bigoplus_{\alpha_i \in \Pi \setminus \Pi_H} \mathbb{R}\alpha_i^* & \ni \sum_i x_i \alpha_i^* \leftrightarrow \sum_i x_i [S^2(\alpha_i^*)] \in & H_2(G/G_H, \mathbb{R}) \\
\cup & & \cup \\
Z_{\mathfrak{c}_H}^* = \bigoplus_{\alpha_i \in \Pi \setminus \Pi_H} \mathbb{Z}\alpha_i^* & \longleftrightarrow & H_2(G/G_H, \mathbb{Z}) \\
& & \wr\Vert \\
& & \pi_2(G/G_H)
\end{array}$$

Here set

$$\alpha_i^* := \frac{2\alpha_i}{(\alpha_i, \alpha_i)}.$$

There is a linear isomorphism

$$\mathfrak{c}_H = \bigoplus_{\alpha_i \in \Pi \backslash \Pi_H} \mathbb{R}\Lambda_{\alpha_i} \ni \lambda \longmapsto [\omega(\lambda)] \in H^2(G/G_H, \mathbb{R})$$

$$\begin{array}{ccc} \cup & & \cup \end{array}$$

$$Z_{\mathfrak{c}_H} = \bigoplus_{\alpha_i \in \Pi \backslash \Pi_H} \mathbb{Z}\Lambda_{\alpha_i} \longleftrightarrow H^2(G/G_H, \mathbb{Z})$$

Parameter spaces of all G-inv. Kähler metrics on G/G_H

$$\mathfrak{c}_H^+ = \bigoplus_{\alpha_i \in \Pi \backslash \Pi_H} \mathbb{R}^+\Lambda_{\alpha_i} \ni \lambda \leftrightarrow \omega(\lambda) \in \{G\text{-inv. Kähler met. on } G/G_H\}$$

$$\begin{array}{ccc} \cup & & \cup \end{array}$$

$$Z_{\mathfrak{c}_H}^+ = \bigoplus_{\alpha_i \in \Pi \backslash \Pi_H} \mathbb{Z}^+\Lambda_{\alpha_i} \longleftrightarrow \{G\text{-inv. Hodge met. on } G/G_H\}$$

Invariant Kähler metrics on G/G_H

For each $\lambda \in \mathfrak{c}_H$,

$$\omega(\lambda)(X, Y) := -\frac{1}{2\pi}\langle[\lambda, X], Y\rangle \quad (X, Y \in \mathfrak{m})$$

For each $\lambda \in \mathfrak{c}_H^+ := \bigoplus_{\alpha_i \in \Pi \setminus \Pi_H} \mathbb{R}^+ \Lambda_{\alpha_i}$, a G-invariant Kähler metric $g(\lambda)$ on G/G_H relative to J_H is defined by

$$\omega(\lambda)(X, Y) = g(\lambda)(J_H(X), Y) \quad (X, Y \in \mathfrak{m}).$$

Note that $H \in \mathfrak{c}_H^+ \cap \mathfrak{a}$ and $\omega(2\pi H) = -\omega_H$.

Proposition

$\hat{\theta}_H : G/G_H \to G/G_H$ is an isometry with respect to $g(2\pi H)$, that is, $\hat{\theta}_H^* g(2\pi H) = g(2\pi H)$.

Define

$$\delta_{\mathfrak{m}} := \frac{1}{2} \sum_{\alpha \in \Sigma^+(G) \setminus \Sigma^+_H(G)} \alpha.$$

Then

Proposition (Borel-Hirzebruch 1958)

$$2\delta_{\mathfrak{m}} \in Z^+_{\mathfrak{c}_H} := \bigoplus_{\alpha_i \in \Pi \setminus \Pi_H} \mathbb{Z}^+ \Lambda_{\alpha_i}$$

and

$$c_1(M) = [\omega(2\delta_{\mathfrak{m}})] \in H^2(G/G_H, \mathbb{Z}).$$

Proposition (M. Takeuchi 1978)

For $\lambda \in \mathfrak{c}^+_H$, a G-invariant Kähler metric $g(\lambda)$ on G/G_H is Einstein if and only if $\lambda = b \cdot 2\delta_{\mathfrak{m}}$ for some $b > 0$.

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Lemma

$$2\delta_{\mathfrak{m}} \in \mathfrak{a}$$

Hence

$$H^{ein} := 2\delta_{\mathfrak{m}} \in \mathfrak{a} \cap Z^{+}_{\mathfrak{c}_H}.$$

Therefore,

for each $H \in \mathfrak{a}$, there exists (uniquely)

a canonical embedding of an R-space K/K_H into an *Einstein*-Kähler C-space G/G_H

$$\iota_{H^{ein}} : L = K/K_H \longrightarrow G/G_H = (M, \omega_{H^{ein}}, J_H, g(2\delta_{\mathfrak{m}}))$$

which is a totally geodesic Lagrangian embedding.

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We use expression

$$\mathbf{2}\delta_{\mathfrak{m}} = \sum_{\alpha \in \boldsymbol{\Pi} \setminus \boldsymbol{\Pi}_{\boldsymbol{H}}} \boldsymbol{k}_\alpha \boldsymbol{\Lambda}_\alpha, = \boldsymbol{\kappa}(\boldsymbol{M}) \sum_{\alpha \in \boldsymbol{\Pi} \setminus \boldsymbol{\Pi}_{\boldsymbol{H}}} \boldsymbol{\kappa}_\alpha \boldsymbol{\Lambda}_\alpha.$$

Here
$\boldsymbol{\kappa}(\boldsymbol{M})$: the greatest common divisor of $\{\boldsymbol{k}_\alpha \mid \alpha \in \boldsymbol{\Pi} \setminus \boldsymbol{\Pi}_{\boldsymbol{H}}\}$.
For each $\alpha \in \boldsymbol{\Pi} \setminus \boldsymbol{\Pi}_{\boldsymbol{H}}$, set $\boldsymbol{\kappa}_\alpha := \dfrac{\boldsymbol{k}_\alpha}{\boldsymbol{\kappa}(\boldsymbol{M})}$.

Lagrangian Submanifolds in Symplectic Manifolds

$$\varphi : L \longrightarrow \underset{\text{symplectic mfd.}}{(M^{2n}, \omega)} \quad \text{immersion}$$

Definition

"Lagrangian immersion" $\underset{\text{def}}{\Longleftrightarrow}$

1. $\varphi^*\omega = 0$

   ($\Leftrightarrow$ φ : "isotropic ")

2. $\dim L = n$

$$\begin{array}{ccc} \varphi^{-1}TM/\varphi_*TL & \cong & T^*L \\ \Psi & & \Psi \\ v & \longmapsto & \alpha_v := \omega(v, \cdot\,) \end{array} \quad \text{linear isom.}$$

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Hamiltonian Deformations

$\varphi_t : L \longrightarrow (M^{2n}, \omega)$ immersion with $\varphi_0 = \varphi$

$V_t := \dfrac{\partial \varphi_t}{\partial t} \in C^\infty(\varphi_t^{-1} TM)$

$$\text{“Lagrangian deformation”} \underset{\text{def}}{\Longleftrightarrow} \varphi_t : \text{Lagr. imm. for}^\forall t$$

$$\Longleftrightarrow \alpha_{V_t} \in \underset{\text{closed}}{Z^1(L)} \text{ for } {}^\forall t$$

$$\text{“Hamiltonian deformation”} \underset{\text{def}}{\Longleftrightarrow} \alpha_{V_t} \in \underset{\text{exact}}{B^1(L)} \text{ for } {}^\forall t$$

Hamil. deform. $\Longrightarrow$ Lagr. deform.
The difference between Lagr. deform. and Hamil. deform. is equal to $H^1(L; \mathbf{R}) \cong Z^1(L)/B^1(L)$.

Characterization of Hamiltonian Deformations in terms of isomonodromy deformations

$\varphi_t : L \longrightarrow M$: Lagr. deform.

Suppose $\dfrac{1}{\gamma}[\omega]$ integral ($\exists\,\gamma$).

$\{\varphi_t\}$: Hamil. deform.

$$\Updownarrow$$

A family of flat connections $\left\{\varphi_t^{-1}\nabla\right\}$ has same holonomy homom.

$\rho : \pi_1(L) \longrightarrow U(1)$

("isomonodromy deformation")

$$\begin{tikzcd}
\varphi_t^{-1}E \arrow[r] \arrow[d, "\substack{\varphi_t^{-1}\nabla \\ \text{flat}}"'] & {}^{\exists}(E,\nabla) \arrow[d] \\
L \arrow[r, "\varphi_t"] & (M,\omega)
\end{tikzcd}$$

Two invariants of Lagrangian submanifolds

L: Lagr. submfd. of a sympl. mfd. (M, ω)
Define two kinds of group homomorphisms

$$I_{\mu,L} : \pi_2(M, L) \to \mathbb{Z}$$

and

$$I_{\omega,L} : \pi_2(M, L) \to \mathbb{R}.$$

The invariant $I_{\mu,L}$

For a smooth map $u : (D^2, \partial D^2) \to (M, L)$ with $A = [u] \in \pi_2(M, L)$, choose a trivialization of the pull-back bdl. as a symplectic vect bdl. (which is unique up to the homotopy).

$$u^{-1}TM \cong D^2 \times \mathbb{C}^n.$$

This gives a smooth map

$$\tilde{u} : S^1 = \partial D^2 \to \Lambda(\mathbb{C}^n).$$

Here $\Lambda(\mathbb{C}^n)$: Grassmann mfd. of Lagrangian vect. subsp. of $\mathbb{C}^n$. Using the Moslov class $\mu \in H^1(\Lambda(\mathbb{C}^n), \mathbb{Z}) \cong \mathbb{Z}$, we define

$$I_{\mu,L}(A) := \mu(\tilde{u}).$$

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The invariant $I_{\omega,L}$

Next, $I_{\omega,L} : \pi_2(M, L) \to \mathbb{R}$ is defined by

$$I_{\omega,L}(A) := \int_{D^2} u^*\omega.$$

Note that

- $I_{\mu,L}$ is invariant under symplectic diffeomorphisms
- $I_{\omega,L}$ is invariant under Hamiltonian diffeomorphisms but not under symplectic diffeomorphisms.

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Monotonicity of Lagrangian submanifolds

A Lagr. submfd. L is *monotone*

$\underset{\text{def}}{\Longleftrightarrow}$

$$I_{\mu,L} = \lambda\, I_{\omega,L} \quad (\exists \lambda > 0).$$

$\Sigma_L \in \mathbb{Z}_+$: a positive generator of $\mathrm{Im}(I_{\mu,L}) \subset \mathbb{Z}$ as an additive subgroup

Σ_L: *minimal Maslov number* of L.

Theorem (K. Cieliebak and E. Goldstein 2004, Hajime Ono 2004)

(M, ω, J, g)*: Einstein-Kähler mfd. of Einstein const.* $\kappa > 0$

L*: compact minimal Lagr. submfd. of* M

$\Longrightarrow$ L *is monotone.*

Monotone Lagrangian Submanifolds in Einstein-Kähler Manifolds

(M, ω, J, g): *simply connected* Einstein-Kähler mfd. with $\kappa > 0$,
L: compact *monotone* Lagrangian submfd. of M

Proposition (Hajime Ono, Japan J. Math. 2004)

$$n_L \Sigma_L = 2\gamma_{c_1},$$

where $\gamma_{c_1} := \min\{c_1(M)(A) \mid A \in H_2(M;\mathbb{Z}), c_1(M)(A) > 0\}$,
$n_L := \min\{k \in \mathbb{Z}^+ \mid \otimes^k E \text{ trivial}\}$.

$$\begin{array}{ccccl} E|_L & \longrightarrow & E & & \text{cplx. line bdle.} \\ \pi_L \downarrow \text{ flat} & & \pi_E \downarrow & & U(1)\text{-connection } \nabla \\ L & \xrightarrow[\text{Lag.}]{} & M & & \text{Einstein-Kähler mfd.} \end{array}$$

Here $\frac{1}{\gamma}\omega = c_1(P, \nabla)$ for some nonzero $\gamma > 0$.

Minimal Maslov number of the Gauss images of isoparametric hypersurfaces

$N^n \subset S^{n+1}(1) \subset \mathbb{R}^{n+2}$: oriented hypersurface of $S^{n+1}(1)$
$\hat{N}^n := \{(\mathrm{x}(p), \mathrm{n}(p)) \in V_2(\mathbb{R}^{n+2}) \mid p \in N\}$: Legendrian lift of N^n.

$$\begin{tikzcd}
N \subset S^{n+1}(1) \arrow[d, phantom, "\wr\|"] & \\
\hat{L} = \hat{N} \arrow[r, "\text{Leg.}"] \arrow[d, "\mathcal{G}"', "\rho(\pi_1(L))"] & P = T^1 S^{n+1} = V_2(\mathbb{R}^{n+2}) \cong \frac{SO(n+2)}{SO(n)} \arrow[d, "\pi"', "SO(2) \cong U(1)"] \\
L = \mathcal{G}(N) \arrow[r, "\text{Lag.}"'] & M = Q_n(\mathbb{C}) = \widetilde{Gr}_2(\mathbb{R}^{n+2}) \cong \frac{SO(n+2)}{SO(2)\times SO(n)}
\end{tikzcd}$$

unit sphere tangent bundle of S^{n+1}
$=$ Stiefel mfd. of o.n. 2-frames of $\mathbb{R}^{n+}$

Gauss map

Gauss image

complex hyperquadric
$=$real ori. 2-plane Grassmann mfd.

The Gauss Map is defined by

$$\begin{array}{cccccc}
\mathcal{G}: & N & \longrightarrow & \hat{N} & \longrightarrow & \mathcal{G}(N) \subset Q_n(\mathbb{C}) = \widetilde{Gr}_2(\mathbb{R}^{n+2}) \\
 & p & \longmapsto & (\mathrm{x}(p), \mathrm{n}(p)) & \longmapsto & [\mathrm{x}(p) + \sqrt{-1}\,\mathrm{n}(p)] = \mathrm{x}(p) \wedge \mathrm{n}(p)
\end{array}$$

Minimal Maslov number of the Gauss images of isoparametric hypersurfaces

$N^n \subset S^{n+1}(1)$: isoparametric hypersurf. with g dist. prin. curv.
$\hat{N}^n := \{(\mathrm{x}(p), \mathrm{n}(p)) \in V_2(\mathbb{R}^{n+2}) \mid p \in N\}$: Legendrian lift of N^n.

simply conn.
Einstein Sasakian homog. sp.

Gauss map $\mathcal{G}$

Gauss image

$$\begin{tikzcd}
N \subset S^{n+1}(1) \arrow[d, phantom, sloped, "\cong"] & \\
\hat{L} = \hat{N} \arrow[r, "\text{min. Leg embed.}"] \arrow[d, "\mathcal{G}"', "\rho(\pi_1(L)) \cong \mathbb{Z}_g"] & P = V_2(\mathbb{R}^{n+2}) = \frac{SO(n+2)}{SO(n)} = T^1 S^{n+1} \arrow[d, "\pi"', "SO(2) \cong U(1)"] \\
L = \mathcal{G}(N) \arrow[r, "\text{min. Lag. embed.}"'] & M = Q_n(\mathbb{C}) = \frac{SO(n+2)}{SO(2)\times SO(n)}
\end{tikzcd}$$

Einstein-Kähler HSS

complex hyperquadric

Theorem (Hui Ma-O. (2010), JDG (2014))

$$\Sigma_L = \frac{2n}{g}.$$

R-spaces canonically embedded in Einstein-Kähler C-spaces

Suppose G: semi-simple. Take $\widetilde{G} \to G$: universal cover of G
$(\widetilde{G}, \widetilde{K}, \theta)$: Riem sym. pair with simply conn $\widetilde{G}$ and conn $\widetilde{K}$.
Suppose that $H = 2\delta_{\mathfrak{m}}$.

simply conn.
Einstein Sasakian homog. sp.

$$
\begin{tikzcd}
\hat{L} = \widetilde{K}/\widetilde{K}'_H \arrow[r, "\text{tot.geod.Leg.}"] \arrow[d, "\hat{\pi}"', "\rho(\pi_1(L))"] & P = \widetilde{G}/\widetilde{G}'_H \arrow[d, "\pi"', "U(1) \cong S^1"] \\
L = \widetilde{K}/\widetilde{K}_H \arrow[r, "\text{canon. embed.}"', "\text{tot.geod.Lag.}"'] & M = \widetilde{G}/\widetilde{G}_H
\end{tikzcd}
$$

R-sp. canon. embed. Einstein-Kähler C-sp.

tot.geod.Lag.

Here $\mathfrak{g}_H = \mathbb{R} \cdot 2\delta_{\mathfrak{m}} \oplus \mathfrak{g}'_H$: orthog. direct sum decomp.
$\widetilde{G}'_H$: conn. Lie subgroup of $\widetilde{G}_H$ with Lie alg. $\mathfrak{g}'_H$.
$\widetilde{K}'_H := \widetilde{K} \cap \widetilde{G}'_H$.

Main result

Suppose that $H = 2\delta_{\mathfrak{m}}$

Theorem

The minimal Maslov number Σ_L of an R-space L canonically embedded in an Einstein-Kähler C-space M is given by

$$\Sigma_L = \frac{2\,\kappa(M)}{\sharp(\widetilde{K}_H/\widetilde{K}'_H)}.$$

Proof.

$\gamma_{c_1} = \kappa(M)$ and $n_L = \sharp(\widetilde{K}_H/\widetilde{K}'_H)$. $\square$

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Some examples for maximal flag manifolds $L = K/T$

$$\iota_H : L = K/T \longrightarrow M = K/T \times \overline{K/T}$$

Then

$$n_L = 1, \quad \gamma_{c_1} = 2$$

and hence

$$\Sigma_L = 4.$$

Some examples for *regular R-spaces L* in $(G,K)=(SU(n+1),SO(n+1))$

Suppose
$(G,K)=(SU(n+1),SO(n+1))$,
$H=2\delta_{\mathfrak{m}}\in\mathfrak{p}=\sqrt{-1}\mathrm{Sym}_0(\mathbb{R}^{n+1})$: regular.

$$\iota_H : L = \frac{SO(n+1)}{S(O(1)\times\cdots\times O(1))} = F_{1,\cdots,1}(\mathbb{R}^{n+1})$$
$$\longrightarrow M = \frac{SU(n+1)}{S(U(1)\times\cdots\times U(1))} = F_{1,\cdots,1}(\mathbb{C}^{n+1})$$

Then

$$n_L = 2, \quad \gamma_{c_1} = 2$$

and hence

$$\Sigma_L = 2.$$

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Some examples for *symmetric R-spaces L*

$M = G/G_H$	$L = K/K_H$	$\dim L$	γ_{c_1}	n_L	Σ_L
$G_{p,q}(\mathbb{C})$	$G_{p,q}(\mathbb{R})$	pq	$p+q$	2	$p+q$
$G_{2p,2q}(\mathbb{C})$	$G_{p,q}(\mathbb{H})$	$4pq$	$2p+2q$	1	$4(p+q)$
$G_{m,m}(\mathbb{C})$	$U(m)$	m^2	$2m$	2	$2m$
$\frac{SO(2m)}{U(m)}$	$SO(m)$	$\frac{m(m-1)}{2}$	$2m-2$	2	$2(m-1)$
$\frac{SO(4m)}{U(2m)}$	$\frac{U(2m)}{Sp(m)}$	$m(2m-1)$	$2(2m-1)$	2	$2(2m-1)$
$\frac{Sp(2m)}{U(2m)}$	$Sp(m)$	$m(2m+1)$	$2m+1$	1	$2(2m+1)$
$\frac{Sp(m)}{U(m)}$	$\frac{U(m)}{O(m)}$	$\frac{m(m+1)}{2}$	$m+1$	2	$m+1$
$Q_{p+q-2}(\mathbb{C})$	$Q_{p,q}(\mathbb{R})$ $p \geq 2$	$p+q-2$	$p+q-2$	2	$p+q-2$
$Q_{q-1}(\mathbb{C})$	$Q_{1,q}(\mathbb{R})$ $q \geq 3$	$q-1$	$q-1$	1	$2(q-1)$

$M = G/G_H$	$L = K/K_H$	$\dim L$	γ_{c_1}	n_L	Σ_L
$\frac{E_6}{T \cdot Spin(10)}$	$P_2(\mathrm{K})$	16	12	1	24
$\frac{E_6}{T \cdot Spin(10)}$	$G_{2,2}(\mathrm{H})/\mathrm{Z}_2$	16	12	2	12
$\frac{E_7}{T \cdot E_6}$	$\frac{SU(8)}{Sp(4)\mathrm{Z}_2}$	27	18	2	18
$\frac{E_7}{T \cdot E_6}$	$\frac{T \cdot E_6}{F_4}$	27	18	1	36

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Thank you very much for your attention !

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