

Minimal Maslov number of *R*-spaces canonically embedded in Einstein-Kähler *C*-spaces

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Plan of this talk

0. What is an \mathbf{R} -space?
1. How to make \mathbf{R} -spaces and their canonical embeddings
2. The canonical embedding of \mathbf{R} -spaces into Kähler \mathbf{C} -spaces
3. The canonical embedding of \mathbf{R} -spaces into Einstein-Kähler \mathbf{C} -spaces
4. Minimal Maslov number of Lagrangian submanifolds
5. Main results
6. Some examples
8. References

What is an R -space?

Jacques Tits, *Sur les R -espaces*. (in French) C. R. Acad. Sci. Paris **239**, (1954). 850–852.

Masaru Takeuchi, *Cell decompositions and Morse equalities on certain symmetric spaces*, Fac. Sci. Univ. Tokyo, I, **12** (1965), 81–192.

Masaru Takeuchi and Shoshichi Kobayashi, *Minimal imbeddings of R -spaces*, J. Differential Geom. **2** (1969), 203–215.

Dirk Ferus, *Immersionen mit paralleler zweiter Fundamentalform: Beispiele und Nicht-Beispiele*. (in German) Manuscripta Math. **12** (1974), 153–162.

Dirk Ferus, *Symmetric submanifolds of Euclidean space*. Math. Ann. **247** (1980), no. 1, 81–93.



What is an R -space?

$$L = \mathbb{R}P^2 \quad \subset \quad \mathbb{C}P^2 = M$$



What is an R -space?

R -space

$$L = \mathbb{R}P^2$$

\subset

Einstein-Kähler C -space

$$\mathbb{C}P^2 = M$$

canonical embedding



What is an R -space?

$$\begin{array}{c}
 \mathbb{S}^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\} \subset \mathbb{R}^3: \text{2-dim. std. sphere} \\
 \downarrow \pi \quad \mathbb{Z}_2 \\
 \mathbb{R}P^2 := \{V \subset \mathbb{R}^3 \mid \text{1-dim. vect. subsp. of } \mathbb{R}^3\} \\
 \text{2-dim. real projective space ("an } R\text{-space")}
 \end{array}$$

$$\begin{array}{ccc}
 \mathbb{R}^3 & \subset & \mathbb{C}^3 = \mathbb{R}^3 + \sqrt{-1}\mathbb{R}^3 \\
 \cup & & \cup \\
 V & \mapsto & V^{\mathbb{C}} = V + \sqrt{-1}V \\
 & & \mathfrak{m}
 \end{array}$$

$$\mathbb{C}P^2 = \{\text{complex 1-dim. vect. subsp. of } \mathbb{C}^3\}$$

2-dim. complex projective space ("an Einstein-Kähler C -space")

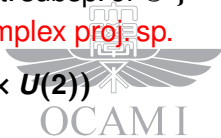


$$\begin{array}{ccc}
 \mathbb{R}^3 \supset V & \longrightarrow & V^{\mathbb{C}} \subset \mathbb{C}^3 \\
 \cup & & \cup \\
 \mathbf{S}^2 & \subset & \mathbf{S}^5 \\
 \downarrow \mathbb{Z}_2 & & \downarrow \text{Hopf fib.} \\
 \mathbb{R}P^2 \ni V & \longmapsto & V^{\mathbb{C}} \in \mathbb{C}P^2 \\
 \parallel & & \parallel \\
 \{1\text{-dim. vect. subsp. of } \mathbb{R}^3\} & \subset & \{\text{cplx. 1-dim. vect. subsp. of } \mathbb{C}^3\} \\
 \text{2-dim. real proj. sp.} & & \text{2-dim. complex proj. sp.}
 \end{array}$$

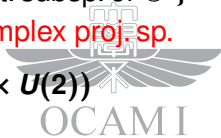


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$SO(3)$ $\mathbb{R}^3 \supset V$ \cup S^2 $\downarrow \mathbb{Z}_2$ $\mathbb{R}P^2 \ni V$ \parallel <p style="text-align: center;">{1-dim. vect. subsp. of \mathbb{R}^3}</p> <p style="text-align: center; color: red;">2-dim. real proj. sp.</p> \cong $SO(3)/S(O(1) \times O(2))$	\subset \rightarrow \subset \mapsto \subset	$SU(3)$ $V^{\mathbb{C}} \subset \mathbb{C}^3$ \cup S^5 $\downarrow \text{Hopf fib.}$ $V^{\mathbb{C}} \in \mathbb{C}P^2$ \parallel <p style="text-align: center;">{cplx. 1-dim. vect. subsp. of \mathbb{C}^3}</p> <p style="text-align: center; color: red;">2-dim. complex proj. sp.</p> \cong $SU(3)/S(U(1) \times U(2))$
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$\mathfrak{o}(3)$	\subset	$\mathfrak{su}(3)$
$SO(3)$	\subset	$SU(3)$
$\mathbb{R}^3 \supset V$	\rightarrow	$V^{\mathbb{C}} \subset \mathbb{C}^3$
\cup		\cup
S^2	\subset	S^5
$\downarrow \mathbb{Z}_2$		\downarrow Hopf fib.
$\mathbb{R}P^2 \ni V$	\mapsto	$V^{\mathbb{C}} \in \mathbb{C}P^2$
\parallel		\parallel
{1-dim. vect. subsp. of \mathbb{R}^3 }	\subset	{cplx. 1-dim. vect. subsp. of \mathbb{C}^3 }
2-dim. real proj. sp.		2-dim. complex proj. sp.
\cong		\cong
$SO(3)/S(O(1) \times O(2))$		$SU(3)/S(U(1) \times U(2))$



$$\mathfrak{su}(3) = \mathfrak{o}(3) + \sqrt{-1}S_0^2(\mathbb{R}^3)$$

$$H = \sqrt{-1} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

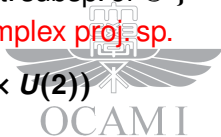
 $\mathfrak{o}(3)$
 \subset
 $\mathfrak{su}(3)$
 $SO(3)$
 \subset
 $SU(3)$
 $\mathbb{R}^3 \supset V$
 $\rightarrow V^{\mathbb{C}} \subset \mathbb{C}^3$
 \cup
 S^2
 \subset
 S^5
 $\downarrow \mathbb{Z}_2$
 \downarrow Hopf fib.

 $\mathbb{R}P^2 \ni V$
 $\mapsto V^{\mathbb{C}} \in \mathbb{C}P^2$
 \parallel
 {1-dim. vect. subsp. of \mathbb{R}^3 }

 \subset {cplx. 1-dim. vect. subsp. of \mathbb{C}^3 }

 \parallel
 2-dim. real proj. sp.

 \parallel
 2-dim. complex proj. sp.

 \parallel
 $SO(3)/S(O(1) \times O(2))$
 \parallel
 $SU(3)/S(U(1) \times U(2))$


$$\mathfrak{su}(3) = \mathfrak{o}(3) + \sqrt{-1}\mathfrak{S}_0^2(\mathbb{R}^3)$$

$$H = \sqrt{-1} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\mathfrak{o}(3) \\ K = \mathbf{SO}(3)$$

\subset
 \subset

$$\mathfrak{su}(3) \\ G = \mathbf{SU}(3)$$

$$\mathbb{R}^3 \supset V$$

\rightarrow

$$V^{\mathbb{C}} \subset \mathbb{C}^3$$

$$\cup \\ \mathbf{S}^2$$

\subset

$$\cup \\ \mathbf{S}^5$$

$$\downarrow \mathbb{Z}_2$$

$$\downarrow$$

$$\mathbb{R}P^2 \ni V$$

\mapsto

$$V^{\mathbb{C}} \in \mathbb{C}P^2$$

{1-dim. vect. subsp. of \mathbb{R}^3 }

\subset

{cplx. 1-dim. vect. subsp. of \mathbb{C}^3 }

2-dim. real proj. sp.

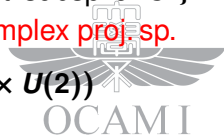
2-dim. complex proj. sp.

$$\cong \\ \mathbf{SO}(3)/\mathbf{S}(\mathbf{O}(1) \times \mathbf{O}(2))$$

$$\cong \\ \mathbf{SU}(3)/\mathbf{S}(\mathbf{U}(1) \times \mathbf{U}(2))$$

K/K_H
where $K_H := K \cap G_H$

G/G_H
 $G_H := \{a \in G \mid \text{Ad}(a)H = H\}$



$$\mathfrak{su}(3) = \mathfrak{o}(3) + \sqrt{-1}S_0^2(\mathbb{R}^3)$$

$$H = \sqrt{-1} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$K = SO(3) \subset G = SU(3)$$

$$\mathbb{R}^3 \supset V$$

$$\cup$$

$$S^2$$

$$\downarrow \mathbb{Z}_2$$

$$\mathbb{R}P^2 \ni V$$

{1-dim. vect. subsp. of \mathbb{R}^3 }

2-dim. real proj. sp.

$$\cong SO(3)/S(O(1) \times O(2))$$

$$\cong (\text{Ad}K)H \subset \sqrt{-1}S_0^2(\mathbb{R}^3)$$

$$\rightarrow V^{\mathbb{C}} \subset \mathbb{C}^3$$

$$\cup$$

$$S^5$$

$$\downarrow$$

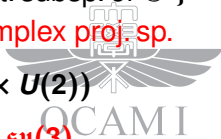
$$V^{\mathbb{C}} \in \mathbb{C}P^2$$

{cplx. 1-dim. vect. subsp. of \mathbb{C}^3 }

2-dim. complex proj. sp.

$$\cong SU(3)/S(U(1) \times U(2))$$

$$\cong (\text{Ad}G)H \subset \mathfrak{su}(3)$$



$$\mathfrak{su}(3) = \mathfrak{o}(3) + \sqrt{-1}\mathfrak{S}_0^2(\mathbb{R}^3) \quad \theta(X) := \bar{X} \quad (\forall X \in \mathfrak{su}(3))$$

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p} \quad H = \sqrt{-1} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\mathfrak{k} = \mathfrak{o}(3) \quad \subset \quad \mathfrak{g} = \mathfrak{su}(3)$$

$$K = SO(3) \quad \subset \quad G = SU(3)$$

$$\mathbb{R}^3 \supset V \quad \rightarrow \quad V^{\mathbb{C}} \subset \mathbb{C}^3$$

$$\cup \quad \cup$$

$$S^2 \quad \subset \quad S^5$$

$$\downarrow \mathbb{Z}_2 \quad \downarrow \text{Hopf fib.}$$

$$L = \mathbb{R}P^2 \ni V \quad \mapsto \quad V^{\mathbb{C}} \in \mathbb{C}P^2 = M$$

$$\parallel \quad \parallel$$

$$\{1\text{-dim. vect. subsp. of } \mathbb{R}^3\} \quad \subset \quad \{\text{cplx. 1-dim. vect. subsp. of } \mathbb{C}^3\}$$

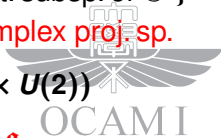
$$\text{2-dim. real proj. sp.} \quad \text{2-dim. complex proj. sp.}$$

$$\parallel \quad \parallel$$

$$SO(3)/S(O(1) \times O(2)) \quad \parallel \quad SU(3)/S(U(1) \times U(2))$$

$$\parallel \quad \parallel$$

$$(AdK)H \subset \mathfrak{p} \quad (AdG)H \subset \mathfrak{g}$$



$(\mathfrak{G}, K, \theta)$ is a Riemannian symmetric pair!

$$\mathfrak{su}(3) = \mathfrak{o}(3) + \sqrt{-1}\mathfrak{S}_0^2(\mathbb{R}^3) \quad \theta(X) := \bar{X} \quad (\forall X \in \mathfrak{su}(3))$$

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

$$H = \sqrt{-1} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\mathfrak{k} = \mathfrak{o}(3)$$

\subset

$$\mathfrak{g} = \mathfrak{su}(3)$$

$$K = SO(3)$$

\subset

$$G = SU(3)$$

$$\mathbb{R}^3 \supset V$$

\rightarrow

$$V^{\mathbb{C}} \subset \mathbb{C}^3$$

\cup

\cup

S^2

\subset

S^5

$\downarrow \mathbb{Z}_2$

\downarrow Hopf fib.

\mathbb{R} -space

Einstein-Kähler \mathbb{C} -space

$$L = \mathbb{R}P^2 \ni V$$

\mapsto

$$V^{\mathbb{C}} \in \mathbb{C}P^2 = M$$

\parallel
 {1-dim. vect. subsp. of \mathbb{R}^3 }

\subset

\parallel
 {cplx. 1-dim. vect. subsp. of \mathbb{C}^3 }

\parallel
 2-dim. real proj. sp.

\parallel
 2-dim. complex proj. sp.

$$\parallel$$

$$SO(3)/S(O(1) \times O(2))$$

$$\parallel$$

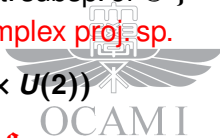
$$SU(3)/S(U(1) \times U(2))$$

$$\parallel$$

$$(\text{Ad}K)H \subset \mathfrak{p}$$

$$\parallel$$

$$(\text{Ad}G)H \subset \mathfrak{g}$$

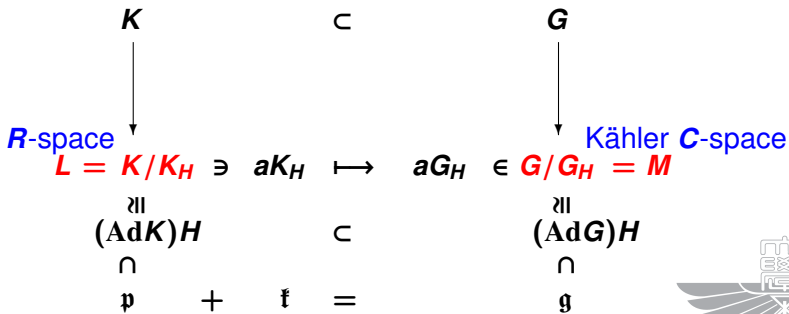


How to make R -spaces and their canonical embeddings

(G, K, θ) : a compact Riemannian symmetric pair.

$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$: canonical decomp. of sym. Lie alg. $(\mathfrak{g}, \mathfrak{k}, \theta)$

$\forall H$



where $K_H := K \cap G_H$

$G_H := \{a \in G \mid \text{Ad}(a)H = H\}$

Suppose

$(\mathbf{G}, \mathbf{K}, \theta)$: compact Riemannian symmetric pair.

Here \mathbf{G} : connected compact Lie group.

$\langle \cdot, \cdot \rangle$: $\text{Ad}(\mathbf{G})$ - and θ -inv. inner product of \mathfrak{g} .

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

: the canonical decomp. of \mathfrak{g} as a symm. Lie alg.

For each $\mathbf{H} \in \mathfrak{p}$, the compact homogeneous space

$$\mathbf{K}/\mathbf{K}_H \cong \text{Ad}(\mathbf{K})\mathbf{H} \subset \mathfrak{p}$$

is called an ***R-space***, where

$$\mathbf{K}_H := \{\mathbf{a} \in \mathbf{K} \mid \text{Ad}_{\mathfrak{p}}(\mathbf{a})(\mathbf{H}) = \mathbf{H}\} \text{ (isot. subgp. of } \mathbf{K} \text{ at } \mathbf{H}\text{)}.$$

It has the ***standard imbedding*** into the Euclidean space \mathfrak{p} :

$$\phi_H : \mathbf{K}/\mathbf{K}_H \ni \mathbf{a}\mathbf{K}_H \mapsto \text{Ad}_{\mathfrak{p}}(\mathbf{a})(\mathbf{H}) \in \mathfrak{p}.$$



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Set

$$\mathbf{G}_H := \{a \in \mathbf{G} \mid \text{Ad}_p(a)(H) = H\} \text{ (isot. subgp. of } \mathbf{G} \text{ at } H\text{).}$$

The compact homogeneous space \mathbf{G}/\mathbf{G}_H is called a *generalized flag manifold*. Since a generalized flag manifold admits \mathbf{G} -invariant Kähler structures, It is also called a *Kähler C-space*. It has the *standard imbedding* into the Euclidean space \mathfrak{g} :

$$\psi_H : \mathbf{G}/\mathbf{G}_H \ni a\mathbf{G}_H \mapsto \text{Ad}(a)(H) \in \mathfrak{g}.$$

Then it is classically well-known that

\mathbf{G}_H is connected

and by definition we have

$$K_H = K \cap \mathbf{G}_H.$$



Set

$$\begin{aligned} \mathfrak{g} &= \mathfrak{g}_H + \mathfrak{m}, & T_{eG_H}M &\cong \mathfrak{m}, \\ \mathfrak{k} &= \mathfrak{k}_H + \mathfrak{l}, & T_{eK_H}L &\cong \mathfrak{l}. \end{aligned}$$

Note that $\mathfrak{k}_H = \mathfrak{k} \cap \mathfrak{g}_H$.

Lemma

$$\theta(\mathbf{G}_H) = \mathbf{G}_H \text{ and } \theta(\mathfrak{g}_H) = \mathfrak{g}_H$$

$$\begin{aligned} \mathfrak{g} &= (\mathfrak{g}_H \cap \mathfrak{k}) + (\mathfrak{g}_H \cap \mathfrak{m}) + (\mathfrak{m} \cap \mathfrak{k}) + (\mathfrak{m} \cap \mathfrak{p}) \\ &= \mathfrak{k}_H + \mathfrak{l} + (\mathfrak{g}_H \cap \mathfrak{p}) + (\mathfrak{m} \cap \mathfrak{p}) \end{aligned}$$

We have $\mathfrak{m} = \mathfrak{m} \cap \mathfrak{k} + \mathfrak{m} \cap \mathfrak{p}$, $\mathfrak{l} = \mathfrak{m} \cap \mathfrak{k}$.

$$(\text{ad}H) : \mathfrak{m} \cap \mathfrak{k} \longrightarrow \mathfrak{m} \cap \mathfrak{p}, \quad (\text{ad}H) : \mathfrak{m} \cap \mathfrak{p} \longrightarrow \mathfrak{m} \cap \mathfrak{k}$$

are injective and thus $\dim \mathfrak{m} \cap \mathfrak{k} = \dim \mathfrak{m} \cap \mathfrak{p}$.

Lemma

$$2 \dim L = \dim M.$$



The \mathbf{G} -invariant *symplectic form* ω_H on $M = \mathbf{G}/\mathbf{G}_H$ is defined by

$$\omega_H(X, Y) := \langle [H, X], Y \rangle \quad (X, Y \in \mathfrak{m}).$$

For each $X, Y \in \mathfrak{l}$,

$$\omega_H(X, Y) = \langle [H, X], Y \rangle = 0.$$

Hence

Proposition

The canonical embedding

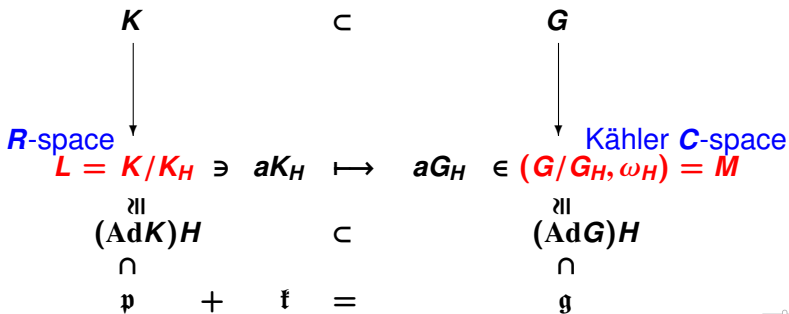
$$\iota_H : L = K/K_H \ni aK_H \mapsto aG_H \in G/G_H = M$$

is a *Lagrangian embedding* into a symplectic manifold (M, ω_H) .

(G, K, θ) : a compact Riemannian symmetric pair.

$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$: canonical decomp. of sym. Lie alg. $(\mathfrak{g}, \mathfrak{k}, \theta)$

$\forall H$



where $K_H := K \cap G_H$

$G_H := \{a \in G \mid \text{Ad}(a)H = H\}$



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The involutive automorphism θ of \mathbf{G} induces the involutive diffeomorphism $\hat{\theta}_H$ of $M = \mathbf{G}/\mathbf{G}_H$ as follows:

$$\hat{\theta}_H : M = \mathbf{G}/\mathbf{G}_H \ni a\mathbf{G}_H \mapsto \theta(a)\mathbf{G}_H \in \mathbf{G}/\mathbf{G}_H = M.$$

Then we have

$$\hat{\theta}_H^* \omega_H = -\omega_H.$$

Proposition

$\hat{\theta}_H : M = \mathbf{G}/\mathbf{G}_H \rightarrow \mathbf{G}/\mathbf{G}_H = M$ is an *anti-symplectic* involutive diffeomorphism with respect to ω_H .

Define

$$\text{Fix}(M, \hat{\theta}_H) := \{p \in M \mid \hat{\theta}_H(p) = p\}$$

: the **fixed point subset** of $\hat{\theta}_H$ on M

Here note that

$$\iota_H(L) \subset \text{Fix}(M, \hat{\theta}_H) \quad \text{as a connected component.}$$



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The group action of \mathbf{G} on $M = \mathbf{G}/\mathbf{G}_H$ is a Hamiltonian group action wrt. ω_H with the **moment map** (which is given by the standard imbedding)

$$\mu_G = \psi_H : M = \mathbf{G}/\mathbf{G}_H \longrightarrow \mathfrak{g} \cong \mathfrak{g}^*.$$

The group action of \mathbf{K} on $M = \mathbf{G}/\mathbf{G}_H$ is a Hamiltonian group action wrt. ω_H with the **moment map** (which is given by its orthogonal projection)

$$\mu_K = \pi_{\mathfrak{k}} \circ \mu_G = \pi_{\mathfrak{k}} \circ \psi_H : M = \mathbf{G}/\mathbf{G}_H \longrightarrow \mathfrak{k} \cong \mathfrak{k}^*.$$

Here

$$\pi_{\mathfrak{k}} : \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \longrightarrow \mathfrak{k} : \text{the orthogonal projection onto } \mathfrak{k}$$

Then the relationship between the moment maps with the involution diffeomorphism $\hat{\theta}_H$ is given in the following way.



Proposition

$$\mu_G \circ \hat{\theta}_H = -\theta \circ \mu_G, \quad \mu_K \circ \hat{\theta}_H = -\mu_K.$$

and it implies

Corollary

$$\text{Fix}(M, \hat{\theta}_H) = \mu_K^{-1}(0).$$

By a result of F. Kirwan (Invent. Math. 1984) we know that $\mu_K^{-1}(0)$ is connected. Then we obtain

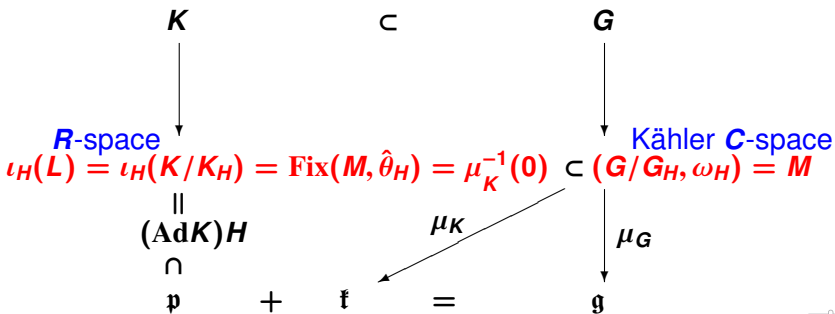
Proposition

$$\iota_H(L) = \text{Fix}(M, \hat{\theta}_H) = \mu_K^{-1}(0).$$

(G, K, θ) : a compact Riemannian symmetric pair.

$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$: canonical decomp. of sym. Lie alg. $(\mathfrak{g}, \mathfrak{k}, \theta)$

$\forall H$



where $K_H := K \cap G_H$

$G_H := \{a \in G \mid \text{Ad}(a)H = H\}$



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Invariant complex structures and Kähler metrics on M

Invariant complex structures and Kähler metrics on G/G_H

Let \mathfrak{a} : a maximal abelian subspace of \mathfrak{p} .

Choose \mathfrak{t} : a maximal abelian subalgebra of \mathfrak{g} such that

$$\begin{array}{rcccl} \mathfrak{g} & = & \mathfrak{k} & + & \mathfrak{p} \\ \cup & & & & \cup \\ \mathfrak{t} & & \supset & & \mathfrak{a} \end{array}$$



Let \mathfrak{a} : a maximal abelian subspace of \mathfrak{p} .

Choose \mathfrak{t} : a maximal abelian subalgebra of \mathfrak{g} such that

$$\begin{array}{rcc} \mathfrak{g} & = & \mathfrak{k} + \mathfrak{p} \\ \cup & & \cup \\ \mathfrak{t} & = & \mathfrak{b} + \mathfrak{a} \end{array}$$

Here $\mathfrak{b} = \mathfrak{t} \cap \mathfrak{k}$, $\mathfrak{a} = \mathfrak{t} \cap \mathfrak{p}$. In particular, \mathfrak{t} is invariant by θ .

$\Sigma(\mathfrak{g})$: the set of all roots of \mathfrak{g} relative to \mathfrak{t} .

The root space decomposition of $\mathfrak{g}^{\mathbb{C}}$ with respect to \mathfrak{t} is given by

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} + \sum_{\alpha \in \Sigma(\mathfrak{g})} \mathfrak{g}^{\alpha},$$

where

$$\mathfrak{g}^{\alpha} = \{X \in \mathfrak{g}^{\mathbb{C}} \mid \text{ad}_{\xi}(X) = \sqrt{-1}\langle \alpha, \xi \rangle X \ (\forall \xi \in \mathfrak{t})\}.$$



The invariant complex structure on G/G_H

Let \mathfrak{a} : a maximal abelian subspace of \mathfrak{p} .

Choose \mathfrak{t} : a maximal abelian subalgebra of \mathfrak{g} such that

$$\begin{array}{rcc} \mathfrak{g} & = & \mathfrak{k} + \mathfrak{p} \\ \cup & & \cup \\ \mathfrak{t} & = & \mathfrak{b} + \mathfrak{a} \end{array}$$

Here $\mathfrak{b} = \mathfrak{t} \cap \mathfrak{k}$, $\mathfrak{a} = \mathfrak{t} \cap \mathfrak{p}$. In particular, \mathfrak{t} is invariant by θ .

$\Sigma(\mathfrak{g})$: the set of all roots of \mathfrak{g} relative to \mathfrak{t} .

$\Sigma_0(\mathfrak{g}) := \{\alpha \in \Sigma(\mathfrak{g}) \mid \alpha \in \mathfrak{b}\}$



Invariant complex structures and Kähler metrics on G/G_H

Let \mathfrak{a} : a maximal abelian subspace of \mathfrak{p} .

Choose \mathfrak{t} : a maximal abelian subalgebra of \mathfrak{g} such that

$$\begin{array}{rcc} \mathfrak{g} & = & \mathfrak{k} + \mathfrak{p} \\ \cup & & \cup \cup \\ \mathfrak{t} & = & \mathfrak{b} + \mathfrak{a} \end{array}$$

Here $\mathfrak{b} = \mathfrak{t} \cap \mathfrak{k}$, $\mathfrak{a} = \mathfrak{t} \cap \mathfrak{p}$. In particular, \mathfrak{t} is invariant by θ .

$\Sigma(\mathfrak{g})$: the set of all roots of \mathfrak{g} relative to \mathfrak{t} .

$\Sigma_0(\mathfrak{g}) := \{\alpha \in \Sigma(\mathfrak{g}) \mid \alpha \in \mathfrak{b}\}$

Define $\sigma \in \mathcal{O}(\mathfrak{t})$ by

$$\sigma(H_{\mathfrak{b}} + H_{\mathfrak{a}}) := -H_{\mathfrak{b}} + H_{\mathfrak{a}} = -\theta(H_{\mathfrak{b}} + H_{\mathfrak{a}}) \quad (H_{\mathfrak{b}} \in \mathfrak{b}, H_{\mathfrak{a}} \in \mathfrak{a}).$$

$<$: a linear order of \mathfrak{t}

σ -order \iff If $\alpha \in \Sigma(\mathfrak{g}) \setminus \Sigma_0(\mathfrak{g})$ and $\alpha > \mathbf{0}$, then $\sigma(\alpha) > \mathbf{0}$.



$$\begin{array}{rcccl}
 \mathfrak{g} & = & \mathfrak{f} & + & \mathfrak{p} \\
 \cup & & \cup & & \cup \\
 \mathfrak{t} & = & \mathfrak{b} & + & \mathfrak{a}
 \end{array}$$

Fix a σ -order $<$ of \mathfrak{t} .

$$\Sigma^+(\mathfrak{g}) := \{\alpha \in \Sigma(\mathfrak{g}) \mid \alpha > \mathbf{0}\}.$$

We may assume that $H \in \mathfrak{a}$ satisfies

$$\langle \alpha, H \rangle \geq 0 \quad (\forall \alpha \in \Sigma^+(\mathfrak{g})).$$



$\Pi = \Pi(\mathfrak{g}) = \{\alpha_1, \dots, \alpha_\ell\}$: fund. root syst. (Dynkin diagram) of \mathfrak{g} .
 $\{\Lambda_1, \dots, \Lambda_\ell\}$: corresponding fund. weight syst. of \mathfrak{g} defined by

$$\frac{2(\Lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij} \quad (i, j = 1, \dots, \ell).$$

$$\Pi_H := \Pi_H(\mathfrak{g}) := \{\alpha_i \in \Pi(\mathfrak{g}) \mid (\alpha_i, H) = 0\},$$

$$\Sigma_H(\mathfrak{g}) = \{\alpha \in \Sigma(\mathfrak{g}) \mid (\alpha, H) = 0\} = \Sigma(\mathfrak{g}) \cap \left(\bigoplus_{\alpha_i \in \Pi_H} \mathbb{Z}\alpha_i \right),$$

$$\Sigma_H^+(\mathfrak{g}) := \Sigma^+(\mathfrak{g}) \cap \left(\bigoplus_{\alpha_i \in \Pi_H} \mathbb{Z}^{\geq 0} \alpha_i \right).$$

$$\mathfrak{c}_H := \bigoplus_{\alpha_i \in \Pi \setminus \Pi_H} \mathbb{R}\Lambda_{\alpha_i} = \mathfrak{c}(\mathfrak{g}_H) \subset \mathfrak{t},$$

$$\mathbb{Z}_{\mathfrak{c}_H} := \bigoplus_{\alpha_i \in \Pi \setminus \Pi_H} \mathbb{Z}\Lambda_{\alpha_i} \subset \mathfrak{c}_H.$$

Then note that $\mathfrak{c}_H = \mathfrak{c}_H \cap \mathfrak{b} + \mathfrak{c}_H \cap \mathfrak{a}$.



Invariant complex structure on G/G_H

Define a \mathbf{G} -invariant complex structure \mathbf{J}_H on \mathbf{G}/\mathbf{G}_H such that

$$T_{e\mathbf{G}_H}(\mathbf{G}/\mathbf{G}_H)^{\mathbb{C}} = T_{e\mathbf{G}_H}(\mathbf{G}/\mathbf{G}_H)^{1,0} \oplus T_{e\mathbf{G}_H}(\mathbf{G}/\mathbf{G}_H)^{0,1} \cong \mathfrak{m}^{\mathbb{C}},$$

$$T_{e\mathbf{G}_H}(\mathbf{G}/\mathbf{G}_H)^{1,0} \cong \sum_{\alpha \in \Sigma^+(\mathbf{G}) \setminus \Sigma_H^+(\mathbf{G})} \mathfrak{g}^{-\alpha},$$

$$T_{e\mathbf{G}_H}(\mathbf{G}/\mathbf{G}_H)^{0,1} \cong \sum_{\alpha \in \Sigma^+(\mathbf{G}) \setminus \Sigma_H^+(\mathbf{G})} \mathfrak{g}^{\alpha}.$$

Then

$$\theta \left(\sum_{\alpha \in \Sigma^+(\mathbf{G}) \setminus \Sigma_H^+(\mathbf{G})} \mathfrak{g}^{-\alpha} \right) = \sum_{\alpha \in \Sigma^+(\mathbf{G}) \setminus \Sigma_H^+(\mathbf{G})} \mathfrak{g}^{\alpha}.$$



Proposition

$\hat{\theta}_H : \mathbf{G}/\mathbf{G}_H \rightarrow \mathbf{G}/\mathbf{G}_H$ is anti-holomorphic with respect to \mathbf{J}_H , that is, $\mathbf{J}_H \circ d\hat{\theta}_H = -d\hat{\theta}_H \circ \mathbf{J}_H$.

For any $H' \in \mathfrak{c}_H^+$,

$$\begin{aligned}\Pi_{H'} &= \Pi_H, \Sigma_{H'} = \Sigma_H \\ \mathfrak{g}_{H'} &= \mathfrak{g}_H, \mathbf{G}_{H'} = \mathbf{G}_H, \\ \mathbf{G}/\mathbf{G}_{H'} &= \mathbf{G}/\mathbf{G}_H = M.\end{aligned}$$

We know

Lemma

The invariant complex structure $\mathbf{J}_{H'}$ coincides with the invariant complex structure \mathbf{J}_H on $M = \mathbf{G}/\mathbf{G}_{H'} = \mathbf{G}/\mathbf{G}_H$.



There is a linear isomorphism

$$\begin{array}{ccc} \mathfrak{c}_H & \ni \lambda \mapsto [\omega(\lambda)] \in & H^2(\mathbf{G}/\mathbf{G}_H, \mathbb{R}) \\ \cup & & \cup \\ \mathbf{Z}_{\mathfrak{c}_H} & \longleftrightarrow & H^2(\mathbf{G}/\mathbf{G}_H, \mathbb{Z}) \end{array}$$

Here for each $\lambda \in \mathfrak{c}_H$, define a closed \mathbf{G} -invariant 2-form on \mathbf{G}/\mathbf{G}_H by

$$\omega(\lambda)(X, Y) := -\frac{1}{2\pi} \langle [\lambda, X], Y \rangle \quad (X, Y \in \mathfrak{m})$$

For each

$$\lambda \in \mathfrak{c}_H^+ := \bigoplus_{\alpha_j \in \Pi \setminus \Pi_H} \mathbb{R}^+ \Lambda_{\alpha_j},$$

a \mathbf{G} -invariant Kähler metric $g(\lambda)$ on \mathbf{G}/\mathbf{G}_H relative to \mathbf{J}_H is defined by

$$\omega(\lambda)(X, Y) = g(\lambda)(\mathbf{J}_H(X), Y) \quad (X, Y \in \mathfrak{m}).$$



There are linear isomorphisms

$$\begin{array}{ccc}
 \mathfrak{c}_H = \bigoplus_{\alpha_j \in \Pi \setminus \Pi_H} \mathbb{R} \Lambda_{\alpha_j} & \ni \lambda \leftrightarrow [\omega(\lambda)] \in & H^2(G/G_H, \mathbb{R}) \\
 \cup & & \cup \\
 \mathfrak{z}_{\mathfrak{c}_H} = \bigoplus_{\alpha_j \in \Pi \setminus \Pi_H} \mathbb{Z} \Lambda_{\alpha_j} & \longleftrightarrow & H^2(G/G_H, \mathbb{Z})
 \end{array}$$

and

$$\begin{array}{ccc}
 \mathfrak{c}_H^* = \bigoplus_{\alpha_j \in \Pi \setminus \Pi_H} \mathbb{R} \alpha_j^* & \ni \sum_i x_i \alpha_i^* \leftrightarrow \sum_i x_i [S^2(\alpha_i^*)] \in & H_2(G/G_H, \mathbb{R}) \\
 \cup & & \cup \\
 \mathfrak{z}_{\mathfrak{c}_H}^* = \bigoplus_{\alpha_j \in \Pi \setminus \Pi_H} \mathbb{Z} \alpha_j^* & \longleftrightarrow & H_2(G/G_H, \mathbb{Z})
 \end{array}$$

Here set

$$\alpha_i^* := \frac{2\alpha_i}{(\alpha_i, \alpha_i)}.$$

$$\begin{array}{c}
 \cong \\
 \pi_2(G/G_H)
 \end{array}$$



There is a linear isomorphism

$$\begin{array}{ccc}
 \mathfrak{c}_H = \bigoplus_{\alpha_j \in \Pi \setminus \Pi_H} \mathbb{R} \Lambda_{\alpha_j} & \ni \lambda \mapsto [\omega(\lambda)] \in & H^2(G/G_H, \mathbb{R}) \\
 \cup & & \cup \\
 \mathfrak{z}_{\mathfrak{c}_H} = \bigoplus_{\alpha_j \in \Pi \setminus \Pi_H} \mathbb{Z} \Lambda_{\alpha_j} & \longleftrightarrow & H^2(G/G_H, \mathbb{Z})
 \end{array}$$

Parameter spaces of all G -inv. Kähler metrics on G/G_H

$$\begin{array}{ccc}
 \mathfrak{c}_H^+ = \bigoplus_{\alpha_j \in \Pi \setminus \Pi_H} \mathbb{R}^+ \Lambda_{\alpha_j} & \ni \lambda \leftrightarrow \omega(\lambda) \in & \{G\text{-inv. Kähler met. on } G/G_H\} \\
 \cup & & \cup \\
 \mathfrak{z}_{\mathfrak{c}_H}^+ = \bigoplus_{\alpha_j \in \Pi \setminus \Pi_H} \mathbb{Z}^+ \Lambda_{\alpha_j} & \longleftrightarrow & \{G\text{-inv. Hodge met. on } G/G_H\}
 \end{array}$$

Invariant Kähler metrics on G/G_H

For each $\lambda \in \mathfrak{c}_H$,

$$\omega(\lambda)(X, Y) := -\frac{1}{2\pi} \langle [\lambda, X], Y \rangle \quad (X, Y \in \mathfrak{m})$$

For each $\lambda \in \mathfrak{c}_H^+ := \bigoplus_{\alpha_i \in \Pi \setminus \Pi_H} \mathbb{R}^+ \Lambda_{\alpha_i}$, a \mathbf{G} -invariant Kähler metric $\mathbf{g}(\lambda)$ on \mathbf{G}/\mathbf{G}_H relative to \mathbf{J}_H is defined by

$$\omega(\lambda)(X, Y) = \mathbf{g}(\lambda)(\mathbf{J}_H(X), Y) \quad (X, Y \in \mathfrak{m}).$$

Note that $\mathbf{H} \in \mathfrak{c}_H^+ \cap \mathfrak{a}$ and $\omega(2\pi\mathbf{H}) = -\omega_H$.

Proposition

$\hat{\theta}_H : \mathbf{G}/\mathbf{G}_H \rightarrow \mathbf{G}/\mathbf{G}_H$ is an isometry with respect to $\mathbf{g}(2\pi\mathbf{H})$, that is, $\hat{\theta}_H^* \mathbf{g}(2\pi\mathbf{H}) = \mathbf{g}(2\pi\mathbf{H})$.



Define

$$\delta_m := \frac{1}{2} \sum_{\alpha \in \Sigma^+(G) \setminus \Sigma_H^+(G)} \alpha.$$

Then

Proposition (Borel-Hirzebruch 1958)

$$2\delta_m \in Z_{c_H}^+ := \bigoplus_{\alpha_j \in \Pi \setminus \Pi_H} \mathbb{Z}^+ \Lambda_{\alpha_j}$$

and

$$c_1(M) = [\omega(2\delta_m)] \in H^2(G/G_H, \mathbb{Z}).$$

Proposition (M. Takeuchi 1978)

For $\lambda \in c_H^+$, a G -invariant Kähler metric $g(\lambda)$ on G/G_H is Einstein if and only if $\lambda = \mathbf{b} \cdot 2\delta_m$ for some $\mathbf{b} > 0$.

Lemma

$$2\delta_m \in \mathfrak{a}$$

Hence

$$H^{ein} := 2\delta_m \in \mathfrak{a} \cap \mathbf{Z}_{\mathfrak{c}_H}^+.$$

Therefore,

for each $H \in \mathfrak{a}$, there exists (uniquely)

a canonical embedding of an R-space K/K_H into an
Einstein-Kähler C-space G/G_H

$$\iota_{H^{ein}} : L = K/K_H \longrightarrow G/G_H = (M, \omega_{H^{ein}}, J_H, g(2\delta_m))$$

which is a totally geodesic Lagrangian embedding.



We use expression

$$2\delta_m = \sum_{\alpha \in \Pi \setminus \Pi_H} k_\alpha \Lambda_\alpha, = \kappa(M) \sum_{\alpha \in \Pi \setminus \Pi_H} \kappa_\alpha \Lambda_\alpha.$$

Here

$\kappa(M)$: the greatest common divisor of $\{k_\alpha \mid \alpha \in \Pi \setminus \Pi_H\}$.

For each $\alpha \in \Pi \setminus \Pi_H$, set $\kappa_\alpha := \frac{k_\alpha}{\kappa(M)}$.



Lagrangian Submanifolds in Symplectic Manifolds

$$\varphi : L \longrightarrow (M^{2n}, \omega) \quad \begin{array}{l} \text{immersion} \\ \text{symplectic mfd.} \end{array}$$

Definition

“Lagrangian immersion” $\stackrel{\text{def}}{\iff}$ $\begin{array}{l} \textcircled{1} \varphi^* \omega = 0 \\ (\iff \varphi : \text{“isotropic”}) \\ \textcircled{2} \dim L = n \end{array}$

$$\begin{array}{ccc} \varphi^{-1} TM / \varphi_* TL & \cong & T^*L \\ \downarrow \psi & & \downarrow \psi \\ \mathfrak{v} & \longmapsto & \alpha_{\mathfrak{v}} := \omega(\mathfrak{v}, \cdot) \end{array} \quad \text{linear isom.}$$



Hamiltonian Deformations

$\varphi_t : L \rightarrow (M^{2n}, \omega)$ immersion with $\varphi_0 = \varphi$

$$V_t := \frac{\partial \varphi_t}{\partial t} \in C^\infty(\varphi_t^{-1} TM)$$

“Lagrangian deformation” $\stackrel{\text{def}}{\iff} \varphi_t : \text{Lagr. imm. for } \forall t$

$$\iff \alpha_{V_t} \in \mathbf{Z}^1(L) \text{ for } \forall t$$

closed

“Hamiltonian deformation” $\stackrel{\text{def}}{\iff} \alpha_{V_t} \in \mathbf{B}^1(L) \text{ for } \forall t$

exact

Hamil. deform. \implies Lagr. deform.

The difference between Lagr. deform. and Hamil. deform. is equal to $H^1(L; \mathbf{R}) \cong \mathbf{Z}^1(L)/\mathbf{B}^1(L)$.

Characterization of Hamiltonian Deformations in terms of isomonodromy deformations

$\varphi_t : L \rightarrow M$: Lagr. deform.

Suppose $\frac{1}{\gamma} [\omega]$ integral ($\exists \gamma$).

$\{\varphi_t\}$: Hamil. deform.



A family of flat connections $\{\varphi_t^{-1} \nabla\}$ has same holonomy homom.

$\rho : \pi_1(L) \rightarrow U(1)$
 (“isomonodromy deformation”)

$$\begin{array}{ccc}
 \varphi_t^{-1} E & \longrightarrow & \exists (E, \nabla) \\
 \varphi_t^{-1} \nabla \downarrow \text{flat} & & \downarrow \\
 L & \xrightarrow{\varphi_t} & (M, \omega)
 \end{array}$$

Two invariants of Lagrangian submanifolds

L : Lagr. submfd. of a sympl. mfd. (M, ω)

Define two kinds of group homomorphisms

$$I_{\mu, L} : \pi_2(M, L) \rightarrow \mathbb{Z}$$

and

$$I_{\omega, L} : \pi_2(M, L) \rightarrow \mathbb{R}.$$



The invariant $I_{\mu,L}$

For a smooth map $u : (D^2, \partial D^2) \rightarrow (M, L)$ with $A = [u] \in \pi_2(M, L)$, choose a trivialization of the pull-back bdl. as a symplectic vect bdl. (which is unique up to the homotopy).

$$u^{-1} TM \cong D^2 \times \mathbb{C}^n.$$

This gives a smooth map

$$\tilde{u} : S^1 = \partial D^2 \rightarrow \Lambda(\mathbb{C}^n).$$

Here $\Lambda(\mathbb{C}^n)$: Grassmann mfd. of Lagrangian vect. subsp. of \mathbb{C}^n .
Using the Moslov class $\mu \in H^1(\Lambda(\mathbb{C}^n), \mathbb{Z}) \cong \mathbb{Z}$, we define

$$I_{\mu,L}(A) := \mu(\tilde{u}).$$

The invariant $I_{\omega,L}$

Next, $I_{\omega,L} : \pi_2(M, L) \rightarrow \mathbb{R}$ is defined by

$$I_{\omega,L}(A) := \int_{D^2} u^* \omega.$$

Note that

- $I_{\mu,L}$ is invariant under symplectic diffeomorphisms
- $I_{\omega,L}$ is invariant under Hamiltonian diffeomorphisms but not under symplectic diffeomorphisms.



Monotonicity of Lagrangian submanifolds

A Lagr. submfd. L is *monotone*

\iff
def

$$I_{\mu,L} = \lambda I_{\omega,L} \quad (\exists \lambda > 0).$$

$\Sigma_L \in \mathbb{Z}_+$: a positive generator of $\mathbf{Im}(I_{\mu,L}) \subset \mathbb{Z}$ as an additive subgroup

Σ_L : *minimal Maslov number* of L .

Theorem (K. Cieliebak and E. Goldstein 2004, Hajime Ono 2004)

(M, ω, J, g) : Einstein-Kähler mfd. of Einstein const. $\kappa > 0$

L : compact minimal Lagr. submfd. of M

$\implies L$ is monotone.

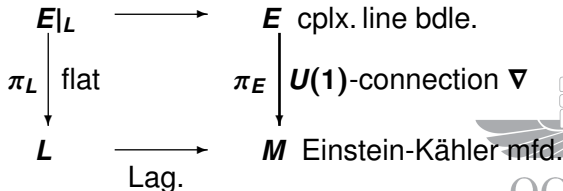
Monotone Lagrangian Submanifolds in Einstein-Kähler Manifolds

(M, ω, J, g) : simply connected Einstein-Kähler mfd. with $\kappa > 0$,
 L : compact monotone Lagrangian submfd. of M

Proposition (Hajime Ono, Japan J. Math. 2004)

$$n_L \Sigma_L = 2\gamma_{c_1},$$

where $\gamma_{c_1} := \min\{c_1(M)(A) \mid A \in H_2(M; \mathbb{Z}), c_1(M)(A) > 0\}$,
 $n_L := \min\{k \in \mathbb{Z}^+ \mid \otimes^k E \text{ trivial}\}$.



Here $\frac{1}{\gamma}\omega = c_1(P, \nabla)$ for some nonzero $\gamma > 0$.

Minimal Maslov number of the Gauss images of isoparametric hypersurfaces

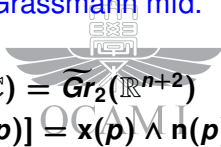
$N^n \subset S^{n+1}(1) \subset \mathbb{R}^{n+2}$: oriented hypersurface of $S^{n+1}(1)$

$\hat{N}^n := \{(x(p), n(p)) \in V_2(\mathbb{R}^{n+2}) \mid p \in N\}$: Legendrian lift of N^n .

$$\begin{array}{ccc}
 N \subset S^{n+1}(1) & & \text{unit sphere tangent bundle of } S^{n+1} \\
 \cong & \xrightarrow{\text{Leg.}} & = \text{Stiefel mfd. of o.n. 2-frames of } \mathbb{R}^{n+2} \\
 \hat{L} = \hat{N} & \xrightarrow{\text{Leg.}} & P = T^1 S^{n+1} = V_2(\mathbb{R}^{n+2}) \cong \frac{SO(n+2)}{SO(n)} \\
 \text{Gauss map } \mathcal{G} \downarrow \rho(\pi_1(L)) & & \downarrow \pi \quad SO(2) \cong U(1) \\
 L = \mathcal{G}(N) & \xrightarrow{\text{Lag.}} & M = Q_n(\mathbb{C}) = \widetilde{Gr}_2(\mathbb{R}^{n+2}) \cong \frac{SO(n+2)}{SO(2) \times SO(n)} \\
 \text{Gauss image} & & \text{complex hyperquadric} \\
 & & = \text{real ori. 2-plane Grassmann mfd.}
 \end{array}$$

The **Gauss Map** is defined by

$$\begin{array}{ccccc}
 \mathcal{G} : N & \longrightarrow & \hat{N} & \longrightarrow & \mathcal{G}(N) \subset Q_n(\mathbb{C}) = \widetilde{Gr}_2(\mathbb{R}^{n+2}) \\
 p & \longmapsto & (x(p), n(p)) & \longmapsto & [x(p) + \sqrt{-1}n(p)] \cong x(p) \wedge n(p)
 \end{array}$$



Minimal Maslov number of the Gauss images of isoparametric hypersurfaces

$N^n \subset S^{n+1}(1)$: isoparametric hypersurf. with g dist. prin. curv.

$\hat{N}^n := \{(x(p), n(p)) \in V_2(\mathbb{R}^{n+2}) \mid p \in N\}$: Legendrian lift of N^n .

$$\begin{array}{ccc}
 N \subset S^{n+1}(1) & & \text{min. Leg embed.} \\
 \hat{L} = \hat{N} & \xrightarrow{\quad} & P = V_2(\mathbb{R}^{n+2}) = \frac{SO(n+2)}{SO(n)} = T^1 S^{n+1} \\
 \downarrow \rho(\pi_1(L)) \cong \mathbb{Z}_g & & \downarrow \begin{matrix} SO(2) \\ \cong U(1) \end{matrix} \\
 \text{Gauss map } \mathcal{G} & & \pi \\
 \text{Gauss image } L = \mathcal{G}(N) & \xrightarrow{\quad} & M = Q_n(\mathbb{C}) = \frac{SO(n+2)}{SO(2) \times SO(n)} \\
 & \text{min. Lag. embed.} & \text{Einstein-Kähler HSS} \\
 & & \text{complex hyperquadric}
 \end{array}$$

simply conn.
 Einstein Sasakian homog. sp.

Theorem (Hui Ma-O. (2010), JDG (2014))

$$\Sigma_L = \frac{2n}{g}.$$

R-spaces canonically embedded in Einstein-Kähler C-spaces

Suppose \mathbf{G} : semi-simple. Take $\tilde{\mathbf{G}} \rightarrow \mathbf{G}$: universal cover of \mathbf{G}
 $(\tilde{\mathbf{G}}, \tilde{\mathbf{K}}, \theta)$: Riem sym. pair with simply conn $\tilde{\mathbf{G}}$ and conn $\tilde{\mathbf{K}}$.
 Suppose that $H = 2\delta_m$.

$$\begin{array}{ccc}
 \hat{L} = \tilde{\mathbf{K}}/\tilde{\mathbf{K}}'_H & \xrightarrow{\text{tot.geod.Leg.}} & \mathbf{P} = \tilde{\mathbf{G}}/\tilde{\mathbf{G}}'_H \\
 \hat{\pi} \downarrow \rho(\pi_1(L)) & & \pi \downarrow U(1) \cong S^1 \\
 L = \tilde{\mathbf{K}}/\tilde{\mathbf{K}}_H & \xrightarrow{\text{canon. embed.}} & M = \tilde{\mathbf{G}}/\tilde{\mathbf{G}}_H \\
 \text{R-sp.} & & \text{Einstein-Kähler C-sp.} \\
 & & \text{tot.geod.Lag.}
 \end{array}$$

simply conn.
Einstein Sasakian homog. sp.

Here $\mathfrak{g}_H = \mathbb{R} \cdot 2\delta_m \oplus \mathfrak{g}'_H$: orthog. direct sum decomp.

$\tilde{\mathbf{G}}'_H$: conn. Lie subgroup of $\tilde{\mathbf{G}}_H$ with Lie alg. \mathfrak{g}'_H .

$\tilde{\mathbf{K}}'_H := \tilde{\mathbf{K}} \cap \tilde{\mathbf{G}}'_H$.



Main result

Suppose that $H = 2\delta_m$

Theorem

The minimal Maslov number Σ_L of an \mathbf{R} -space L canonically embedded in an Einstein-Kähler \mathbf{C} -space M is given by

$$\Sigma_L = \frac{2\kappa(M)}{\#(\tilde{K}_H/\tilde{K}'_H)}.$$

Proof.

$\gamma_{c_1} = \kappa(M)$ and $n_L = \#(\tilde{K}_H/\tilde{K}'_H)$. □



Some examples for maximal flag manifolds

$$L = K/T$$

$$\iota_H : L = K/T \longrightarrow M = K/T \times \overline{K/T}$$

Then

$$n_L = 1, \quad \gamma_{c_1} = 2$$

and hence

$$\Sigma_L = 4.$$



Some examples for *regular R-spaces L* in $(G, K) = (SU(n+1), SO(n+1))$

Suppose

$$(G, K) = (SU(n+1), SO(n+1)),$$

$$H = 2\delta_m \in \mathfrak{p} = \sqrt{-1}\text{Sym}_0(\mathbb{R}^{n+1}): \text{regular.}$$

$$\begin{aligned} \iota_H : L &= \frac{SO(n+1)}{S(O(1) \times \cdots \times O(1))} = F_{1, \dots, 1}(\mathbb{R}^{n+1}) \\ &\rightarrow M = \frac{SU(n+1)}{S(U(1) \times \cdots \times U(1))} = F_{1, \dots, 1}(\mathbb{C}^{n+1}) \end{aligned}$$

Then

$$n_L = 2, \quad \gamma_{c_1} = 2$$

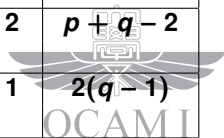
and hence

$$\Sigma_L = 2.$$



Some examples for symmetric R -spaces L

$M = G/G_H$	$L = K/K_H$	$\dim L$	γ_{C_1}	n_L	Σ_L
$G_{p,q}(\mathbb{C})$	$G_{p,q}(\mathbb{R})$	pq	$p + q$	2	$p + q$
$G_{2p,2q}(\mathbb{C})$	$G_{p,q}(\mathbb{H})$	$4pq$	$2p + 2q$	1	$4(p + q)$
$G_{m,m}(\mathbb{C})$	$U(m)$	m^2	$2m$	2	$2m$
$\frac{SO(2m)}{U(m)}$	$SO(m)$	$\frac{m(m-1)}{2}$	$2m - 2$	2	$2(m-1)$
$\frac{SO(4m)}{U(2m)}$	$\frac{U(2m)}{Sp(m)}$	$m(2m-1)$	$2(2m-1)$	2	$2(2m-1)$
$\frac{Sp(2m)}{U(2m)}$	$Sp(m)$	$m(2m+1)$	$2m+1$	1	$2(2m+1)$
$\frac{Sp(m)}{U(m)}$	$\frac{U(m)}{O(m)}$	$\frac{m(m+1)}{2}$	$m+1$	2	$m+1$
$Q_{p+q-2}(\mathbb{C})$	$Q_{p,q}(\mathbb{R})$ $p \geq 2$	$p + q - 2$	$p + q - 2$	2	$p + q - 2$
$Q_{q-1}(\mathbb{C})$	$Q_{1,q}(\mathbb{R})$ $q \geq 3$	$q - 1$	$q - 1$	1	$2(q - 1)$



$M = G/G_H$	$L = K/K_H$	$\dim L$	γ_{c_1}	n_L	Σ_L
$\frac{E_6}{T \cdot Spin(10)}$	$P_2(K)$	16	12	1	24
$\frac{E_6}{T \cdot Spin(10)}$	$G_{2,2}(H)/Z_2$	16	12	2	12
$\frac{E_7}{T \cdot E_6}$	$\frac{SU(8)}{Sp(4)Z_2}$	27	18	2	18
$\frac{E_7}{T \cdot E_6}$	$\frac{T \cdot E_6}{F_4}$	27	18	1	36



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Thank you very much for your attention !



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