On Morse index estimates for minimal surfaces

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CIRM
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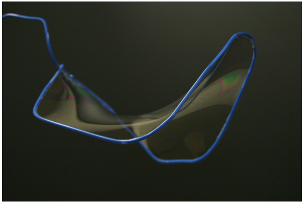

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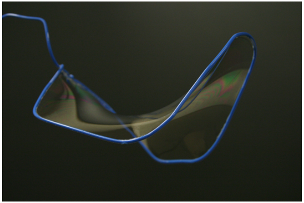

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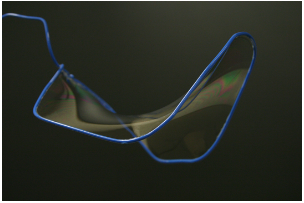

- Plateau's Problem became a central question in the field, until it was independently solved in 1930 by Douglas and Rado (as mappings of the unit disk). The search for solving the Plateau's problem in greater generality lead to the development of Geometric Measure Theory in the 1960s.

Definitions-II

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- It is also equivalent to the Gauss map

$$N : \Sigma \to \mathbb{S}^2$$

being anti-holomorphic.

Embedded Examples-I

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1. The flat two-plane

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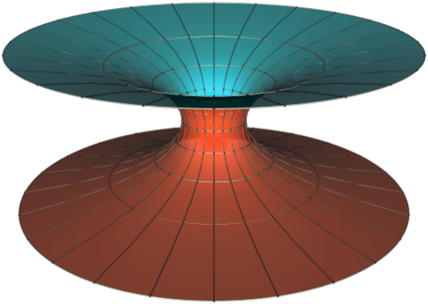
2. The catenoid (Euler 1744)



$$x_1^2 + x_2^2 = \cosh^2 x_3$$

Embedded Examples-II

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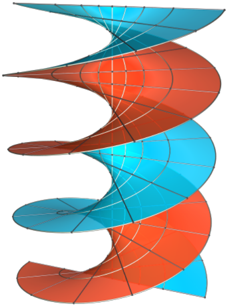

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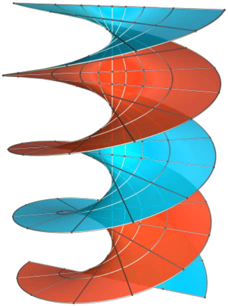

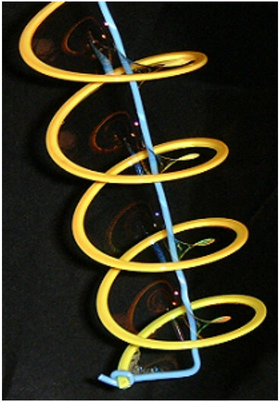

$$(x_1, x_2, x_3) = (t \cos s, t \sin s, s)$$

19th Century

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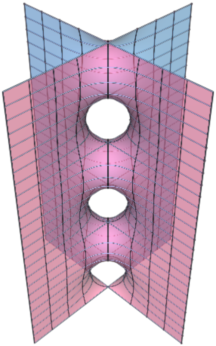

Scherk 1835

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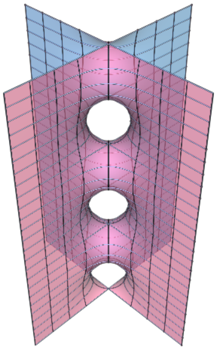

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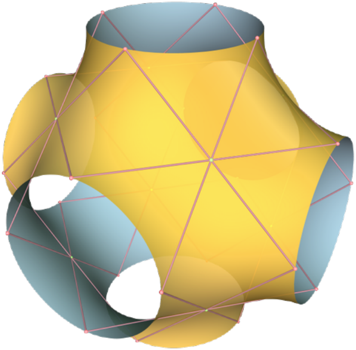

Schwarz-Neovius 1880

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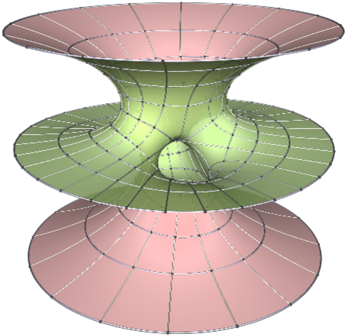

Costa 1982

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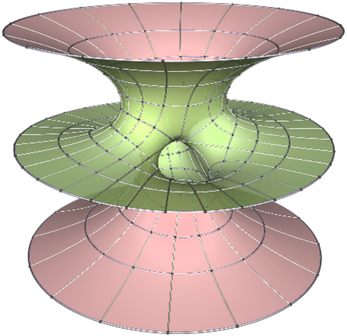

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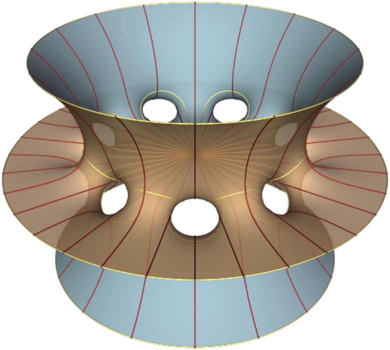

Costa-Hoffman-Meeks 1983

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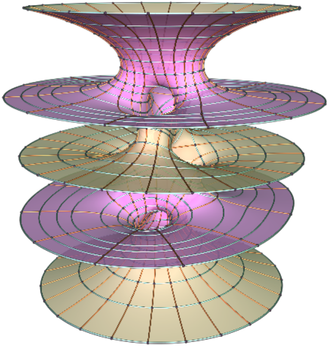

Weber-Wolf 2002

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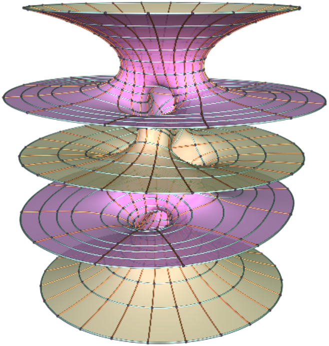

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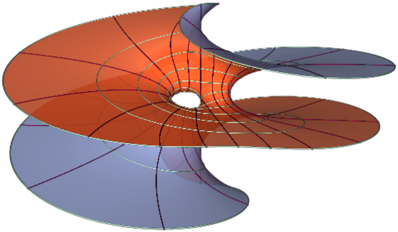

Hoffman-Weber-Wolf 2004, Hoffman-White 2006, Hoffman-Traizet-White 2015

Variational Theory-I

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Let $F : (-\epsilon, \epsilon) \times \Sigma \to \mathbb{R}^3$ be a smooth variation with $F(0, \cdot) = \mathrm{id}|_\Sigma$ and $X = \frac{\partial F}{\partial t}(0, \cdot)$ its initial velocity.

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- Assuming Σ minimal and $X = \varphi N$, where $\varphi \in C_0^\infty(\Sigma)$, we compute the second variation:

$$Q(\varphi, \varphi) = |\Sigma_t|''(0) = \int_\Sigma |\nabla\varphi|^2 - |A|^2\varphi^2 = -\int_\Sigma \varphi L\varphi$$

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  where $L = \Delta + |A|^2$, and $A(X, Y) = -\langle \nabla_X N, Y \rangle$ the 2nd fundamental form.
- The *Morse index* of Σ is the number of negative eigenvalues of L. Thus, a minimal surface of index k minimizes area up to second order in all directions orthogonal to a k-dimensional space.

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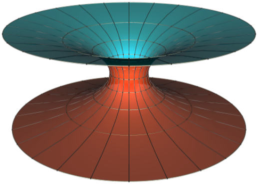

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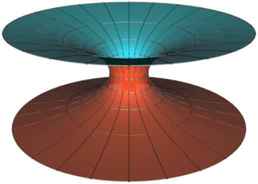

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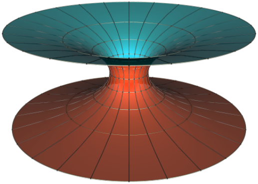

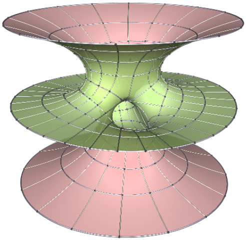

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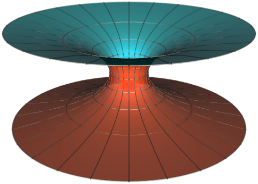

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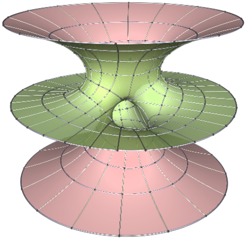

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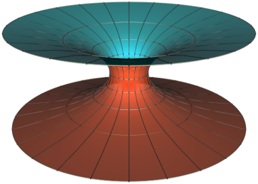

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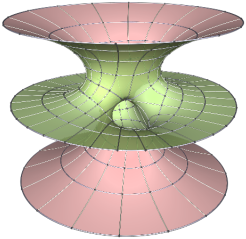

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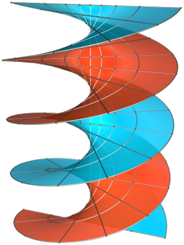

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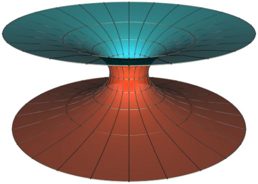

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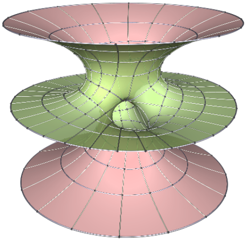

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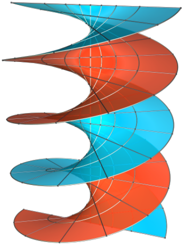

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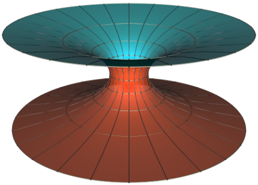

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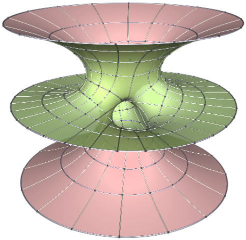

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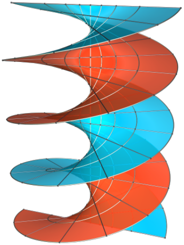

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- Thm(Fisher-Colbrie 1985) $\Sigma \subset \mathbb{R}^3$ has finite index if and only if it has finite total curvature, $\int_\Sigma |K| < \infty$.

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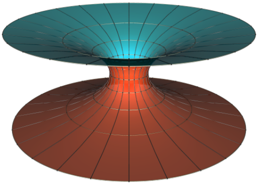

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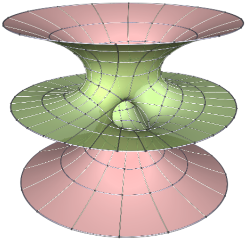

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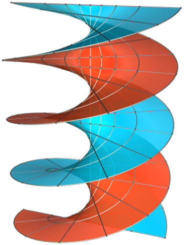

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- Detecting the topology from N is easier:

  Thm(Jorge-Meeks 1982) If Σ has finite total curvature, genus g, and r ends, then

$$\deg(N) = -\frac{1}{4\pi}\int_\Sigma K = g + r - 1.$$

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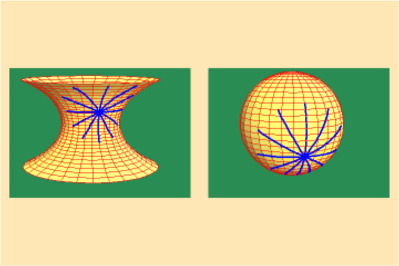

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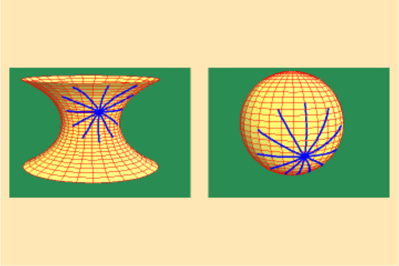

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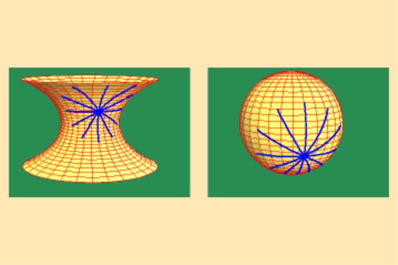

- Therefore, the second variation of Σ is a conformal invariant of the induced metric on $\langle \cdot, \cdot \rangle$ on Σ. Hence, we may calculate the index with respect to the metric $N^*(ds^2)$, where ds^2 is the standard metric of S^2. Issue: branch points!

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  First time the number of ends appeared on a lower bound for the index. Upper bounds had been known for a long time (Tysk 87).

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More recently, we showed:

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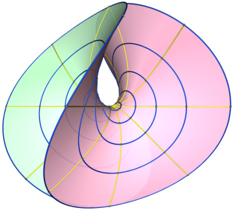

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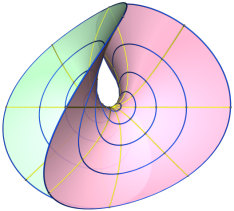

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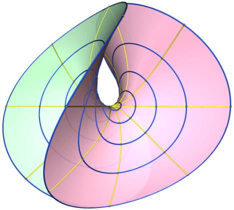

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the above index estimate proves the conjecture.

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