Translators

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Up to a rigid motion and a homothety we can assume that $v = (0, 0, -1)$. Then the translator equation has the form

$$\vec{H} = (0, 0, -1)^{\perp}.$$

Translators as minimal surfaces

In 1994, T. Ilmanen observed that M is a translator iff M is minimal with respect the metric

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for g-minimal surfaces. Moreover, reflection in vertical planes and 180^o-rotation about vertical lines are isometries of g. Therefore, we can use **Schwarz reflection** and **Alexandrov method of moving planes** in our context.

Translators as minimal surfaces

If M is a **graphical translator** and

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its Gauss map, then

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Remark

A sequence of translating graphs will converge, subsequentially, to a translator.

Translators as minimal surfaces

Given a translator $M = \mathrm{Graph}(u)$, $u : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$, the vertical translates of M are also g-minimal and foliate $\Omega \times \mathbb{R}$.

Proposition

M is **g-area minimizing** in $\Omega \times \mathbb{R}$,

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Proposition

M is **g-area minimizing** in $\Omega \times \mathbb{R}$, and if Ω is convex $\Longrightarrow$ *M* is **g-area minimizing** in $\mathbb{R}^3$.

Corollary (local area estimates)

If U is a bounded convex open subset of $\mathbb{R}^3$ disjoint from $\Gamma := \overline{M} \setminus M$, then

$$\mathrm{area}_g(M \cap U) \leq \frac{1}{2}\,\mathrm{area}_g(\partial U).$$

Translators as minimal surfaces

Theorem (curvature estimates up to the boundary)

There is a constant $C < \infty$ with the following property. Let M be translator with velocity $-s\,\mathbf{e}_3$ in $\mathbb{R}^3$ (where $s > 0$) such that

1. M is the graph of a smooth function $F : \Omega \to \mathbb{R}$ on a convex open subset Ω of $\mathbb{R}^2$.
2. $\Gamma := \overline{M} \setminus M$ is a polygonal curve (not necessarily connected) consisting of segments, rays, and lines.
3. $\overline{M}$ is a smooth manifold-with-boundary except at the corners of Γ.

If $p \in \mathbb{R}^3$, let $r(M,p)$ be the supremum of $r > 0$ such that $\mathbf{B}(p,r) \cap \partial M$ is either empty or consists of a single line segment. Then

$$|A|(M,p)\min\{s^{-1}, r(M,p)\} \leq C,$$

where $|A|(M,p)$ is the norm of the 2^{nd} f. f. of M at p.

Translators as minimal surfaces

Compactness

Let M_i, $\Gamma_i = \overline{M_i} \setminus M_i$, and Ω_i be a sequence of examples satisfying the hypotheses of the previous theorem with $s_i \equiv 1$. Suppose that the Γ_i converge (with multiplicity 1) to a polygonal curve Γ. Thus curvature estimates imply that (after passing to a subsequence) the M_i converge smoothly in $\mathbb{R}^3 \setminus \Gamma$ to a smooth translator M.

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Translating graphs

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$$(x, y) \mapsto \frac{\log(\sin(y\cos\theta))}{(\cos\theta)^2} + x\tan\theta. \tag{2}$$

over the strip given by $\mathbb{R} \times (0, \pi/\cos\theta)$. The graph of (2), or any surface obtained from it by translation and rotation about a vertical axis, is called a **tilted grim reaper of width** $w = \pi/\cos\theta$.

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$$(x, y) \mapsto (w/\pi)^2 \log(\sin(y(\pi/w)) \pm x\sqrt{(w/\pi)^2 - 1}. \tag{3}$$

Translating graphs

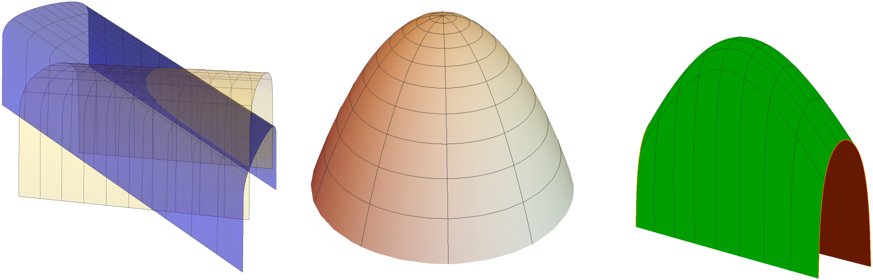

Figure: Some examples of complete graphical translators.

Theorem (Classification Theorem, Hoffman-Ilmanen-M-White)

For every $w > \pi$, there exists (up to translation) a unique complete translator $u : \mathbb{R} \times (0, w) \to \mathbb{R}$. for which the Gauss curvature is everywhere > 0. The function u is symmetric with respect to $(x, y) \mapsto (-x, y)$ and $(x, y) \mapsto (x, w - y)$ and thus attains its maximum at $(0, w/2)$. Up to isometries of $\mathbb{R}^2$ and vertical translation, the only other complete translating graphs are the tilted grim reapers and the bowl soliton, a strictly convex, rotationally symmetric graph of an entire function.

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In particular (as Spruck and Xiao had already shown), there are no complete graphical translators defined over strips of width less than π. Moreover the grim reaper surface is the only example with width π. The positively curved translator in the Classification Theorem is called a **Δ-wing**.

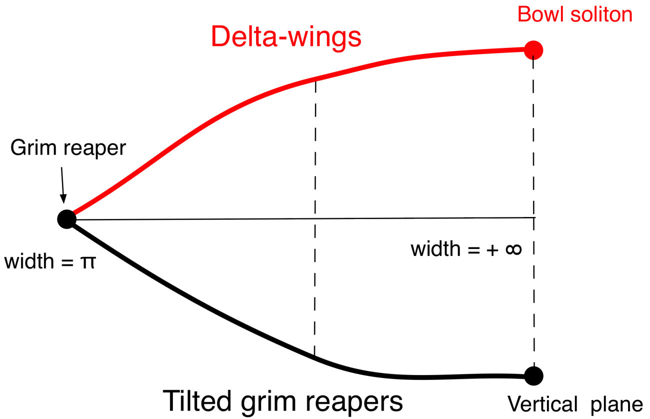


Figure: The space of complete graphical translators.

The Main Theorem

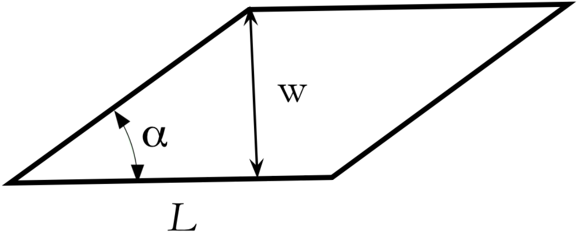
The main theorem

Definition

For $\alpha \in (0, \pi)$, $w \in (0, \infty)$, and $0 < L \leq \infty$, let $P(\alpha, w, L)$ be the set of points (x, y) in the strip $\mathbb{R} \times (0, w)$ such that

$$\frac{y}{\tan\alpha} < x < L + \frac{y}{\tan\alpha}.$$

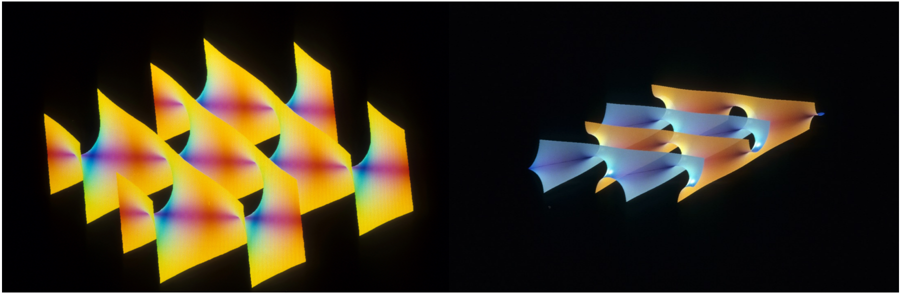
The lower-left corner of the region is at the origin and the interior angle at that corner is α.



The main theorem

Classical Scherk's surfaces are obtained by solving this boundary problem:

$$(\divideontimes)\begin{cases} u : P = P(\alpha, w, L) \to \mathbb{R}, \\ \mathrm{Div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0, \\ u = -\infty \text{ on the horizontal sides of } P, \\ u = +\infty \text{ on the nonhorizontal sides of } P \end{cases}$$



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Theorem (Classical Scherk's surfaces)

For each $\alpha \in (0, \pi)$, $w \in (0, \infty)$ and $L \in (0, \infty]$, the boundary value problem ($\divideontimes$) has a solution if and only if P is a rhombus, i.e., if and only if $L = \frac{w}{\sin \alpha}$.

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- The solution is unique up to an additive constant,
- The graph of $u_{\alpha,w}$ is bounded by the four vertical lines through the corners of P.
- It extends by repeated Schwartz reflection to a doubly periodic minimal surface $\mathcal{S}_{\alpha,w}$.
- As $\alpha \to 0$, the surface $\mathcal{S}_{\alpha,w}$ converges smoothly to the parallel vertical planes $y = nw$, $n \in \mathbb{Z}$.
- As $\alpha \to \pi$, the surface $\mathcal{S}_{\alpha,w}$ converges smoothly to the helicoid given by $z = x \cot\left(\frac{\pi}{w} y\right)$.

The main theorem

We are interested in solving this boundary problem:

$$(\divideontimes\divideontimes)\begin{cases} u : P = P(\alpha, w, L) \to \mathbb{R}, \\ \mathrm{Div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = -\frac{1}{\sqrt{1+|\nabla u|^2}}. \\ u = -\infty \text{ on the horizontal sides of } P, \\ u = +\infty \text{ on the nonhorizontal sides of } P \end{cases}$$

Theorem (Hoffman-M-White)

For each $\alpha \in (0, \pi)$ and $w \in (0, \infty)$, there is a unique $L = L(\alpha, w)$ in $(0, \infty]$ for which the boundary value problem ($\divideontimes\divideontimes$) has a solution.

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2. If $P = P(\alpha, w, L(\alpha, w))$, then the solution is unique up to an additive constant, and there is a unique solution $u_{\alpha,w}$ satisfying the additional condition
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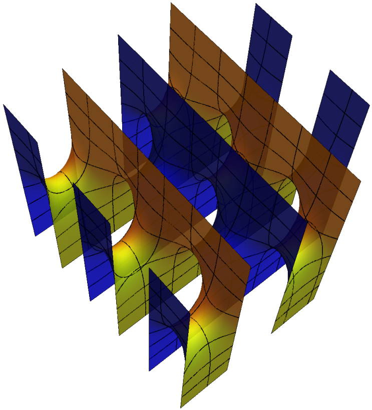
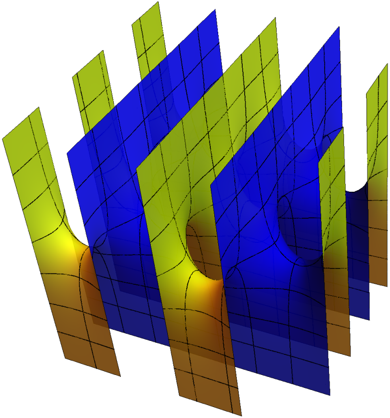
   - If $w \geq \pi$, then $\mathcal{S}_{\alpha,w}$ is singly periodic and we call it a **Scherkenoid**.

Scherk translator $\alpha = \pi/2$, $w = \pi/2$





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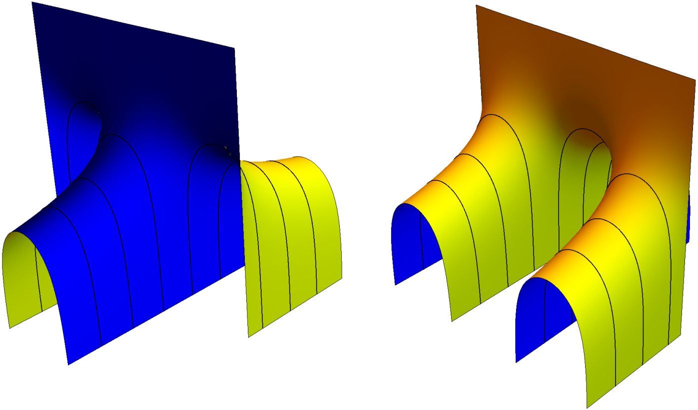

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  - If $w < \pi$, then M is helicoid-like: there is an $\hat{x} = \hat{x}_M \in \mathbb{R}$ such that M contains the vertical lines L_n through the points $n(\hat{x}, w)$, $n \in \mathbb{Z}$. Furthermore, $M \setminus \cup_n L_n$ projects diffeomorphically onto $\cup_{n \in \mathbb{Z}} \{nw < y < (n+1)w\}$.

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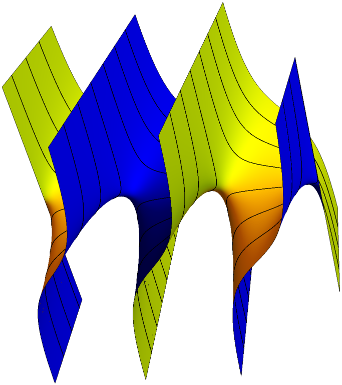
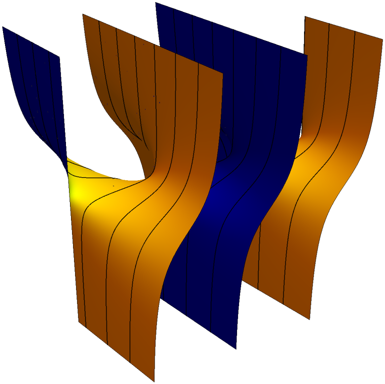
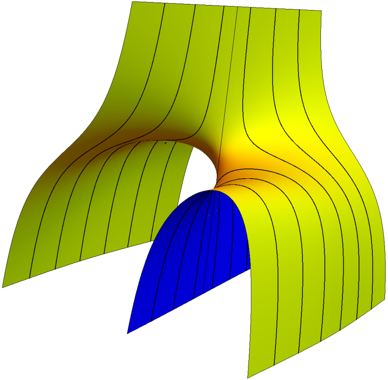
  - If $w = \pi$, then the component of M containing the origin is a pitchfork Ψ of width π, but in this case we do not know whether M is connected.

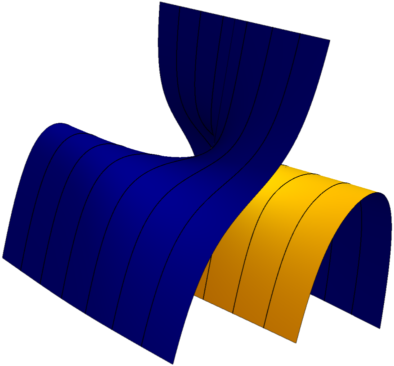
Helicoid-like translators $w = \pi/2$

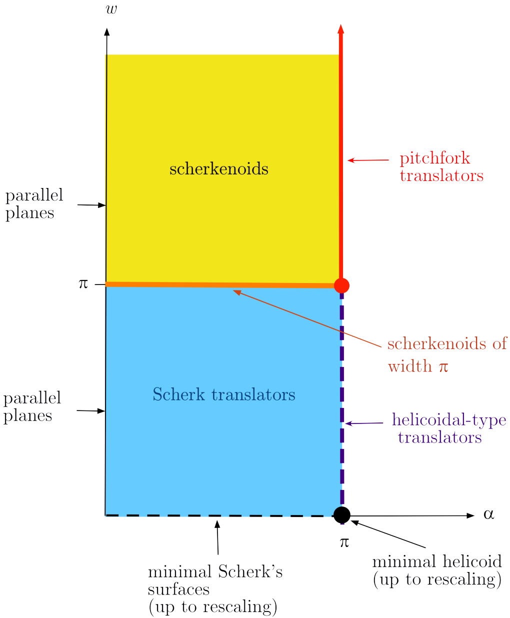




Pitchfork $w = \pi$






w
scherkenoids
pitchfork translators
parallel planes
π
scherkenoids of width π
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parallel planes
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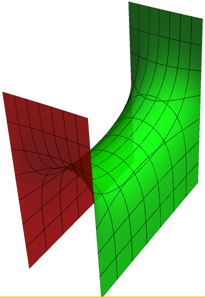
α
π
minimal Scherk's surfaces (up to rescaling)
minimal helicoid (up to rescaling)

Scherk translators

Theorem

For every $0 < \alpha < \pi$ and $0 < w < \pi$ there exists a unique $L = L(\alpha, w) > 0$ with the following property: there is a smooth surface-with-boundary $\mathcal{D} = \mathcal{D}_{\alpha,w}$ such that:

- $\mathcal{D}$ is a translator and $\mathcal{D} - \partial\mathcal{D}$ is the graph of a solution of $(\divideontimes\divideontimes)$,
- Given $(\alpha, w, L(\alpha, w))$, the surface $\mathcal{D}$ is unique up to vertical translations,
- If $w \geq \pi$ and $0 < L < \infty$, the problem $(\divideontimes\divideontimes)$ has no solutions.



Remark

Since $P(\pi - \alpha, w, L)$ is the image of $P(\alpha, w, L)$ under reflection in the line $x = L/2$, it follows that $L(\alpha, w) = L(\pi - \alpha, w)$ and that $\mathcal{D}_{\pi-\alpha,w}$ is the image of $\mathcal{D}_{\alpha,w}$ under reflection in the plane $x = L(\alpha, w)/2$, followed by a vertical translation.

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- $\mathcal{D}_{\alpha,w}$ has negative Gauss curvature everywhere,
- the Gauss map is a **diffeomorphism** from $\mathcal{D}_{\alpha,w}$ onto

$$\mathbb{S}^{2+} \setminus Q$$

where $\mathbb{S}^{2+}$ is the closed upper hemisphere and where Q is the set consisting of the four unit vectors in the equator $\partial\mathbb{S}^{2+}$ that are perpendicular to the edges of the parallelogram $P(\alpha, w, L(\alpha, w))$ over which $\mathcal{D}_{\alpha,w}$ is a graph.

Proof of Existence

From now on, we are going to fix $\boxed{0 < w < \pi \text{ and } 0 < \alpha < \pi.}$

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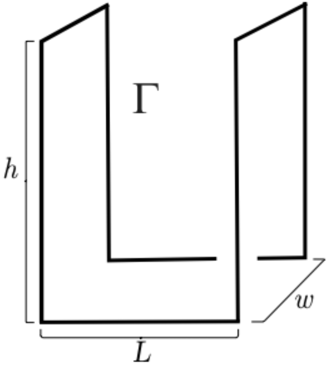
We consider the polygonal Jordan curve Γ in $\mathbb{R}^3$



We can solve the corresponding **Plateau problem associated to Γ** (in $\mathbb{R}^3$ endowed with Ilmanen's metric) and so we get a minimal disk that we are going to denote $\mathcal{D}(L, h)$.

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It has the following properties:

1. $\mathcal{D}(L, h)$ is a graph over the plane $z = 0$.
2. $\mathcal{D}(L, h)$ has the same symmetry as Γ.

For a fixed height h, we have that:

(i) As $L \to 0$, the limit of $\mathcal{D}(w, L, h)$ is the "U-shaped" curve described by Γ, as $L \to 0$.

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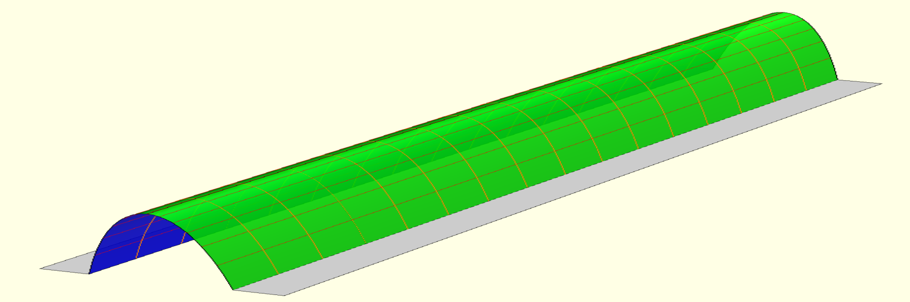
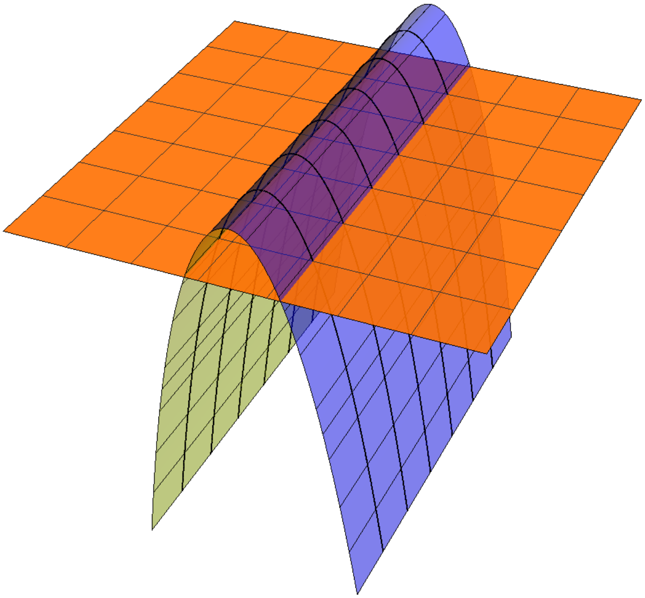
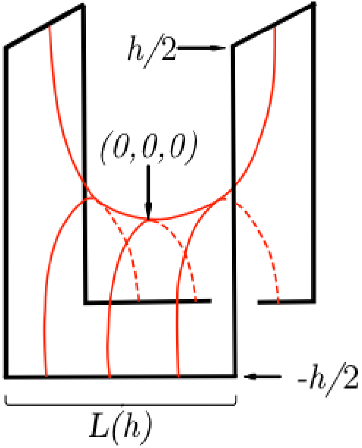
(ii) As $L \to \infty$, the limit of $\mathcal{D}(w, L, h)$ is the part of the standard grim reaper cylinder which is a graph over the strip $\mathbb{R} \times (0, w)$.





This means that (for h sufficiently large) it is possible to find $L(h) \in (0, +\infty)$, such that the height of the saddle point (which is at the intersection of the symmetry line) is precisely $h/2$. Up to a translation in space we can assume that the saddle point is placed in the origin $(0, 0, 0)$.



Now, we would like to take limit of these family of disks $\{\mathcal{D}(L(h), h) \ : \ h > 0\}$, as the height $h \to \infty$.

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This contradiction proves that $L(h)$ is bounded.

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Then, given a sequence $\{h_i\} \nearrow +\infty$ so that $\{L(h_i)\} \nearrow L$, the limit $L < +\infty$.

Now, we would like to take limit of these family of disks $\{\mathcal{D}(L(h), h) \ : \ h > 0\}$, as the height $h \to \infty$.

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We consider M the sub-sequential limit of $\mathcal{D}(L(h_i), h_i)$. M is a complete translator which is a graph over the strip $\mathbb{R} \times (-w/2, w/2)$ and $w < \pi$, which is impossible **(classification theorem)**.
This contradiction proves that $L(h)$ is bounded.
Then, given a sequence $\{h_i\} \nearrow +\infty$ so that $\{L(h_i)\} \nearrow L$, the limit $L < +\infty$. The limit of the sequence $\mathcal{D}(L(h_i), h_i)$ is the translator that we are looking for.

Proof of uniqueness

First step

Let P be the interior of a parallelogram in $\mathbb{R}^2$ with two sides parallel to the x-axis. Suppose

$$u, v : P \to \mathbb{R}$$

are translators that have boundary values $-\infty$ on the horizontal sides of P and $+\infty$ on the other sides. Then $u - v$ is constant.

If ξ is a vector in $\mathbb{R}^2$, let

$$u_\xi : P_\xi \to \mathbb{R}$$

be the result of translating $u : P \to \mathbb{R}$ by ξ, and let

$$\phi_\xi : P \cap P_\xi \to \mathbb{R},$$
$$\phi_\xi = u_\xi - v.$$

Suppose contrary to the theorem that $u - v$ is not constant. Then there is a point p where $Du(p) \neq Dv(p)$. We know that the image of the Gauss map of the graph of u is the entire upper hemisphere.

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- Equivalently, $Du(\cdot)$ takes every possible value. Thus there is a $q \in P$ such that $Du(q) = Dv(p)$. Let $\xi_0 = p - q$. Then p is a critical point of ϕ_{ξ_0}.

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- Equivalently, $Du(\cdot)$ takes every possible value. Thus there is a $q \in P$ such that $Du(q) = Dv(p)$. Let $\xi_0 = p - q$. Then p is a critical point of ϕ_{ξ_0}.
- We claim that it is an isolated critical point. For otherwise ϕ_{ξ_0} would be constant near p and therefore (by unique continuation) constant throughout $P \cap P_{\xi_0}$, which is impossible. (Note for example that ϕ_{ξ_0} is $+\infty$ on some edges of $P \cap P_{\xi_0}$ and $-\infty$ on other edges.)

- It follows (Morse theory) that for every ξ sufficiently close to ξ_0, ϕ_ξ has a critical point. In particular, there is such a ξ for which P_ξ is in general position with respect to P.

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- However, for such a ξ, ϕ_ξ cannot have a critical point, a contradiction. It cannot have a critical point because ϕ_ξ is $+\infty$ on two adjacent sides of its parallelogram domain $P \cap P_\xi$ and $-\infty$ on the other two sides.

(Note that P contains exactly one vertex of P_ξ, P_ξ contains exactly one vertex of P, and at each of the two points where an edge of P intersects an edge of P_ξ, one of the functions u and u_ξ is $+\infty$ and the other is $-\infty$.)

Second step

Suppose for $i = 1, 2$ that

$$u_i : P(\alpha, w_i, L_i) \to \mathbb{R}$$

is a graphical translator such that u_i has boundary value $-\infty$ on the horizontal edges of its domain and $+\infty$ on the nonhorizontal edges. Suppose also that $w_1 \leq w_2$ and that $L_1 \geq L_2$. Then $w_1 = w_2$, $L_1 = L_2$, and $u_1 - u_2$ is constant.

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- The proof is almost identical to the proof of Step 1.

Non-existence of Scherk translators for $w \geq \pi$.

Proposition (Non-existence)

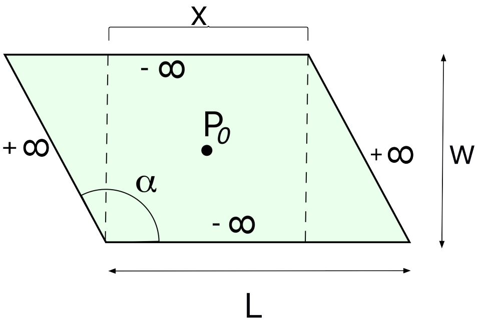
Let $w \geq \pi$ and $L < \infty$. Then there is no translator

$$u : P = P(\alpha, w, L) \to \mathbb{R}$$

with boundary values $-\infty$ on the horizontal sides and $+\infty$ on the nonhorizontal sides.

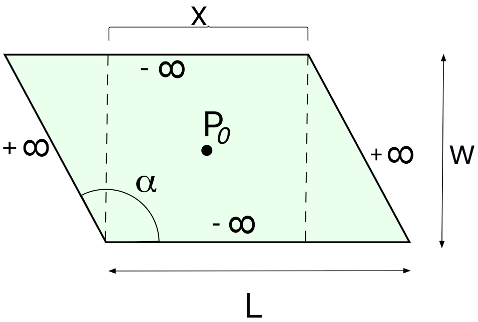
Helicoid-like translators

Translating helicoids



Using the compactness result we are able to take limit of $\mathcal{D}(\alpha, w)$, as $\alpha \to \pi$.

Translating helicoids

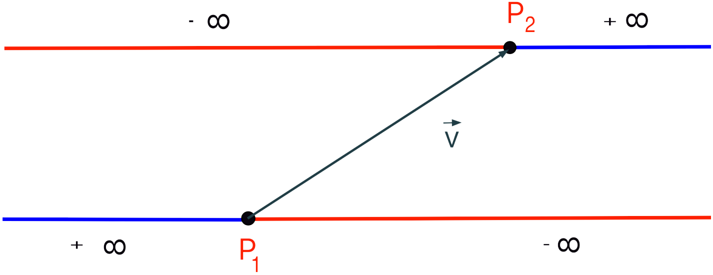


Using the compactness result we are able to take limit of $\mathcal{D}(\alpha, w)$, as $\alpha \to \pi$.

The distance x does not go to $+\infty$, as $\alpha \to \pi$.

Translating helicoid

Then, as $\alpha \to \pi$, the limit graph over the strip of width w has the shape



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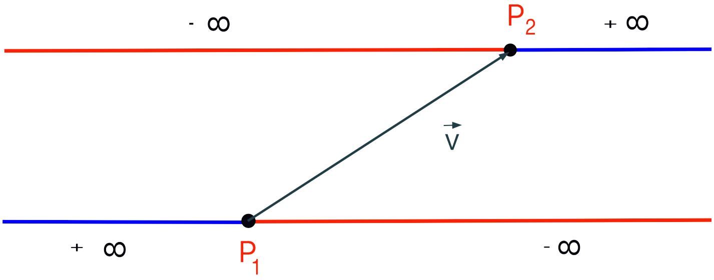
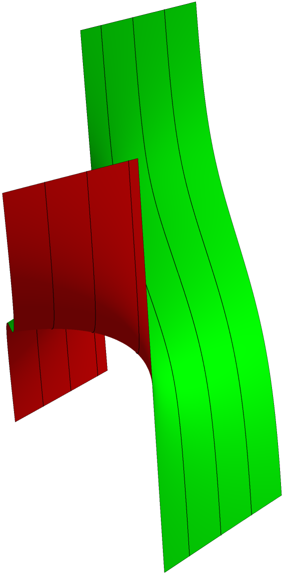
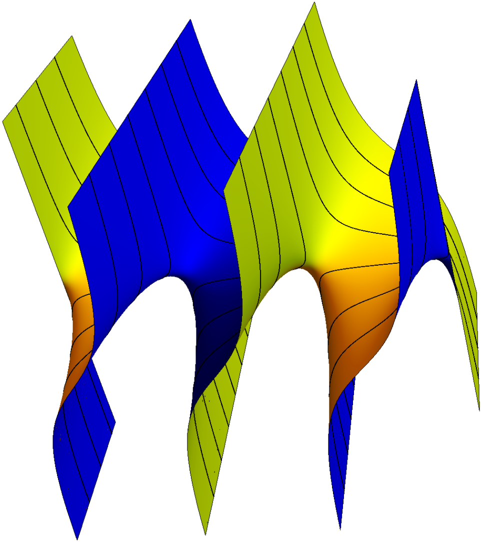


The two vertical lines $r_1 = \{P_1\} \times \mathbb{R}$ and $r_2 = \{P_2\} \times \mathbb{R}$ are contained in the surface. Then, we can reflect our graph over these lines and repeating successively this process we obtain a helicoidal-like translating soliton which is invariant under the translation by vector $2\vec{v}$ (notice that this vector is horizontal).

Translating helicoid





Scherkenoids

For $\alpha \in (0, \pi)$ and $w \in (0, \infty)$, let

$$\Omega(\alpha, w) = \{0 < y < w\} \cap \{x > y/\tan\alpha\}.$$

In the previous notation, $\Omega(\alpha, w) = P(\alpha, w, \infty)$.

Definition

For $w \geq \pi$, let $g_w : \mathbb{R} \times (0, w)$ be the unique tilted grim reaper function such that $g_w(0, w/2) = 0$ and such that $\partial g_w/\partial x \leq 0$. Let g'_w be the corresponding tilted grim reaper function with $\partial g'_w/\partial x \geq 0$. (Thus $g'_w(x, y) \equiv g_w(-x, y)$.)

Consequently

$$g_w(x, y) = (w/\pi)^2 \log(\sin(y(\pi/w)) - x\sqrt{(w/\pi)^2 - 1}$$

and therefore

$$\frac{\partial g_w}{\partial x} \equiv -\sqrt{(w/\pi)^2 - 1}.$$

Theorem

For every $\alpha \in (0, \pi)$ and $w \in [\pi, \infty)$, there exists a smooth translator $\mathcal{D}$ such that $M \setminus \partial M$ is the graph of a function

$$u : \Omega(\alpha, w) \to \mathbb{R}$$

with boundary values $-\infty$ on the horizontal edges of $\Omega(\alpha, w)$ and $+\infty$ on the non-horizontal edge. Furthermore, any such $\mathcal{D}$ has the following properties:

- $\mathcal{D}$ has negative Gauss curvature everywhere.
- As $a \to \infty$,

$$u(a + x, y) - u(a, y)$$

converges smoothly to the tilted grim reaper function $g_w(x, y)$.

Theorem

- The Gauss map takes $\mathcal{D} \setminus \partial\mathcal{D}$ diffeomorphically onto the open region $\mathbb{R} = \mathbb{R}(w)$ in the upper hemisphere bounded by $C \cup C(w)$, where C is the equatorial semicircle

$$C = \{(x, y, 0) \in \mathbb{S}^2 : x \geq 0\}$$

and where $C(w)$ is the great semicircle that is the image of graph(g_w) under its Gauss map:

$$C(w) = \left\{(x, y, z) \in \mathbb{S}^2 : z > 0 \text{ and } x = z\sqrt{(w/\pi)^2 - 1}\right\}.$$

- $\mathcal{D}$ is unique up to vertical translation. In particular, there is a unique such surface $\mathcal{D}_{\alpha,w}$ with the additional property that the vector

$$\mathbf{v}_{\alpha/2} = (\cos(\alpha/2), \sin(\alpha/2), 0)$$

is tangent to $\mathcal{D}$ at the origin.

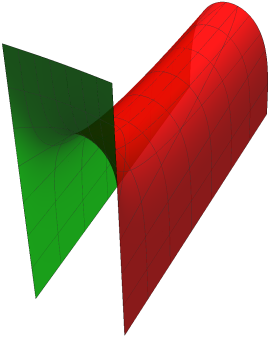


Figure: The graph of the scherkenoid function u in the theorem with $\alpha = \pi/2$ and $w = \pi$.

Remark

If $w < \pi$, there is no translator u with the indicated boundary values.

Proof of existence:

For $c >> 0$, we consider the quadrilateral

$$\Omega_{\alpha,w}(c) = \{(x,y) \in \Omega(\alpha,w) : x < c\}.$$

Let $h > 0$. Consider the translator $u_c^h : \Omega_{\alpha,w}(c) \to \mathbb{R}$ such that $u_c^h = h$ on the left edge of $\Omega_{\alpha,w}(c)$ and such that $u_c^h = 0$ on the other three edges. (Of course u_c^h also depends on α and w, but for the moment we fix α and w.)

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As $h \to \infty$, the graph of u_c^h converges (perhaps after passing to a subsequence) to a limit surface $\Sigma = \Sigma_c$ whose boundary is a polygonal curve Γ consisting of the two vertical rays with $z > 0$ over

$$(0, 0) \quad \text{and} \quad \left(\frac{w}{\tan\alpha}, w\right)$$

together with three sides of the quadrilateral $\Omega_{\alpha,w}(c)$ in the plane $z = 0$.

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- The convergence is smooth except possibly at the corners of Γ. By the compactness result, if Σ were not a graph, it would be flat and vertical.
- But Γ does not bound such a flat, vertical surface. Thus Σ is the graph of a function u_c, and u_c has boundary value $+\infty$ on the left side of the quadrilateral $\Omega_{\alpha,w}(c)$ and 0 on the other three sides.

Let

$$v_c : [-c, c] \times [0, w] \to \mathbb{R}$$

be the translator with boundary values 0. By the maximum principle, $v_c \leq u_c$ on the intersection of the two domains. In the proof of the classification theorem, it is shown that $v_c - v_c(0,0)$ converges (as $c \to \infty$) to a complete translating graph over $\mathbb{R} \times (0, w)$. Thus $u_c(x, y) \to \infty$ for every $(x, y) \in \Omega(\alpha, w)$. Consequently, Σ_c converges (as $c \to \infty$) to the pair of vertical quarterplanes

$$\{(x, 0, z) : x \geq 0,\ z \geq 0\} \tag{4}$$

and

$$\{(x, w, z) : x \geq \hat{x},\ z \geq 0\} \tag{5}$$

where $\hat{x} = w/\tan\alpha$.

By the standard curvature estimates, the convergence is smooth.
Let $(\cos\theta_c(z), \sin\theta_c(z), 0)$ be tangent to the graph of u_c at $(0, 0, z)$.
Then $\theta_c(0) = 0$ and $\theta_c(z) \to \alpha$ as $z \to \infty$. Thus there is a $z(c)$ such that $\theta_c(z(c)) = \alpha/2$.
By the smooth convergence of Σ_c to the quarterplanes (4) and (5), we see that

$$\lim_{c\to\infty} z(c) = \infty.$$

Now take a subsequential limit $\mathcal{D}$ of the graph of $u_c - z(c)$ as $c \to \infty$.
Then $\mathcal{D}$ is a smooth translator in $\overline{\Omega(\alpha, w)} \times \mathbb{R}$ whose boundary consists of the two vertical lines through the corners of $\Omega(\alpha, w)$, and
$\mathbf{v} := (\cos\alpha/2, \sin\alpha/2, 0)$ is tangent to $\mathcal{D}$ at the origin.

Let $\mathcal{D}_0$ be the component of $\mathcal{D}$ containing the origin. If $\mathcal{D}_0 \setminus \partial\mathcal{D}_0$ were not a graph, then $\mathcal{D}_0$ would be flat and vertical, and hence the halfplane $\{s\mathbf{v} + z\mathbf{e}_3 : s \geq 0, z \in \mathbb{R}\}$.

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- But this is impossible, since that halfplane is not contained in $\overline{\Omega(\alpha, w)} \times \mathbb{R}$.
- Thus $\mathcal{D}_0 \setminus \partial\mathcal{D}_0$ is the graph of a function u. The domain of u must be all of Ω, and therefore $\mathcal{D}_0$ must be all of $\mathcal{D}$. This completes the proof of existence in the theorem.

Proof of the uniqueness:

The proof of the uniqueness is quite similar to the case of Scherk translators.

Pitchforks