Mean curvature flow of graphs in $M \times \mathbb{R}$

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Main topics

- Mean curvature flow of graphs (joint work with J. Faustino)
- Translating solitons (joint works with F. Martín, E. S. Gama and E. Heinonen)

Mean curvature flow

Let Σ^n and $\bar{M}^{n+1}$ be complete oriented Riemannian manifolds. Given $T > 0$, we consider a differentiable map

$$\Psi : [0, T) \times \Sigma \to \bar{M} \tag{1}$$

such that for each $t \in [0, T)$, the immersion $\Psi_t = \Psi(t, \cdot)$ is isometric. We denote the initial immersion by $\psi = \Psi_0$.

Mean curvature flow

We say that the submanifolds $\Psi_t(\Sigma)$, $t \in [0, T)$, are evolving by their mean curvature vector field if

$$\left(\frac{\partial \Psi}{\partial t}\right)^{\perp} = n\mathbf{H}, \tag{2}$$

where $\mathbf{H}$ is the mean curvature vector of Ψ_t. Here, $\perp$ indicates the projection onto the normal bundle.

In our convention

$$\mathbf{H} = \frac{1}{n}\sum_{i=1}^{n}\left(\bar{\nabla}_{\psi_* \mathrm{e}_i}\psi_* \mathrm{e}_\mathrm{i}\right)^{\perp}.$$

Mean curvature flow

The MCF is the flow by the negative gradient of the *area* (n-dimensional volume) functional.

It is a sort of “heat equation” since in an Euclidean approximation (2) becomes

$$\frac{\partial \Psi}{\partial t} = \Delta \Psi. \tag{3}$$

MCF also describes the limiting flow (as $\varepsilon \to 0$) of the interfaces between level sets of solutions of an Allen-Cahn equation of the form

$$\frac{\partial u_\varepsilon}{\partial t} = \Delta u_\varepsilon - \frac{1}{\varepsilon} W'(u_\varepsilon) \tag{4}$$

Geometric setting

Suppose that $(\bar{M}^{n+1}, \bar{g})$ is endowed with a Killing vector field X. This means that

$$L_X \bar{g} = 0.$$

We assume for the sake of simplicity that the orthogonal distribution $\mathcal{D}$ defined by X is integrable.

Let

$$\Phi : M^n \times \mathbb{R} \to \bar{M}^{n+1}$$

be the flow generated by X where M^n is an arbitrarily fixed complete integral leaf of $\mathcal{D}$.

Geometric setting

The Killing graph of a function u defined on the closure of a regular domain Ω in M^n is the hypersurface

$$\Sigma = \{\Phi(x, u(x)) : x \in \bar{\Omega}\}. \tag{5}$$

Geometric setting

The mean curvature equation in this context is

$$\operatorname{div} \frac{\nabla^M u}{W} + \left\langle \frac{\nabla^M |X|}{|X|}, \frac{\nabla^M u}{W} \right\rangle = nH$$

or

$$\operatorname{div}_{-\log \varrho} \frac{\nabla^M u}{W} = nH,$$

where

$$\varrho = |X| \quad \text{and} \quad W = \sqrt{|X|^{-2} + |\nabla^M u|^2}.$$

Recall that

$$\operatorname{div}_f Z = e^f \operatorname{div}\left(e^{-f} Z\right).$$

Geometric setting

This is the Euler-Lagrange equation of the functional

$$\mathcal{A}_H[u] = \int_\Omega \varrho W \,\mathrm{d}M + \int_\Omega \int_0^{\varrho u} nH \,\mathrm{d}s \,\mathrm{d}M$$

In other terms, the height function $u = s|_\Sigma$ satisfies the intrinsic equation

$$\Delta u + 2\langle \nabla \log \varrho, \nabla u \rangle = nH \langle X, N \rangle. \tag{6}$$

Geometric setting

A function $u = u(x,t)$ with $(x,t) \in M \times [0,T)$ satisfying

$$\partial_t u = W \left(\operatorname{div} \frac{\nabla^M u}{W} + \left\langle \frac{\nabla^M |X|}{|X|}, \frac{\nabla^M u}{W} \right\rangle \right)$$

defines a mean curvature flow of Killing graphs.

This non-parametric formulation of the mean curvature flow (2) "fixes the gauge" of tangential diffeomorphisms.

Geometric setting

We observe that the ambient *static* Riemannian metric is locally expressed as a *warped product* metric

$$\varrho^2(x)ds^2 + \sigma_{ij}(x)dx^i dx^j, \quad x \in M, \tag{7}$$

that is, $\bar{M} = M \times_\varrho \mathbb{R}$.

For instance, *constant sectional curvature metrics* may be expressed as

$$\varrho^2(r)\, ds^2 + dr^2 + \xi^2(r)\, d\theta^2, \tag{8}$$

where

$$\varrho(r) = \mathrm{cs}_\kappa(r), \qquad \xi(r) = \mathrm{sn}_\kappa(r). \tag{9}$$

Some examples

Fix a pole $o \in M$ and denote $r(x) = \mathrm{dist}_M(o, x)$. For rotationally symmetric warped metrics

$$\varrho^2(r)\, ds^2 + dr^2 + \xi^2(r)\, d\theta^2,$$

we may consider the mean curvature flow of the graphs of

$$u(x, t) = v_{R(t)}(x),$$

where

$$v_R(x) = \int_R^{r(x)} \frac{nH(R)A(\varsigma)}{\varrho(\varsigma)(A^2(\varsigma) - n^2H^2(R)V^2(\varsigma))^{\frac{1}{2}}}\, \mathrm{d}\varsigma$$

with

$$H(R) = -\frac{1}{n}\frac{A(r)}{V(r)}, \quad A(r) = \varrho(r)\xi^{n-1}(r), \quad V(r) = \int_0^r \varrho(\varsigma)\xi^{n-1}(\varsigma)\, \mathrm{d}\varsigma.$$

Some examples

The time parameter is implicitly given by

$$\begin{cases} R'(t) = -nH(R(t)), \\ R(0) = R_0, \end{cases} \tag{10}$$

for some fixed $R_0 > 0$.

Some examples

In Euclidean space $\bar{M} = \mathbb{R}^{n+1}$ (with $\varrho(r) = 1$ and $\xi(r) = r$) we have

$$H(R) = -\frac{1}{R}$$

with

$$v_R(x) = (R^2 - r^2(x))^{\frac{1}{2}}$$

and

$$R(t) = (R_0^2 + 2nt)^{\frac{1}{2}}$$

In hyperbolic space (foliated by geodesic spheres centered at some point $o \in \mathbb{H}^{n+1}$, that is, for $\varrho(r) = \cosh r$ and $\xi(r) = \sinh r$) we have

$$R'(t) = n \coth R(t)$$

and

$$e^{u(r,t)} = \left(e^{2nt} \frac{\cosh^2 R_0}{\cosh^2 r} - 1 \right)^{\frac{1}{2}} + e^{nt} \frac{\cosh R_0}{\cosh r}.$$

Some references

- G. Huisken, 1989: non-parametric MCF in Euclidean space.
- K. Ecker and G. Huisken, 1989: evolution of entire graphs.
- K. Ecker and G. Huisken, 1991: interior estimates.
- P. Untenberger, 2003: MCF of radial (Killing) graphs in hyperbolic space.
- T. Colding and W. Minicozzi, 2004: sharp estimates for evolving graphs.
- A. Borisenko and V. Miquel, 2012: MCF of graphs in warped products (under stronger assumptions).
- Longzhi Lin, 2012: modified MCF in hyperbolic space.

Geometric assumptions

Suppose from now on that M has a pole, that we will denote by o. Let $r(x) = \mathrm{dist}_M(o, x)$.

We suppose that the radial sectional curvatures along geodesics rays issuing from o satisfy

$$K(\partial_r \wedge \mathrm{v}) \geq -\frac{\xi''(r)}{\xi(r)} \tag{11}$$

for all $r > 0$, $\mathrm{v} \in TM, \mathrm{v} \perp \partial_r$.

Here $\xi \in C^\infty([0, \infty))$ with $\xi(r) > 0$ for $r > 0$, $\xi'(0) = 1$ and $\xi^{(2k)}(0) = 0$.

In this case, the Hessian comparison theorem implies that

$$\nabla^M \nabla^M r \leq \frac{\xi'(r)}{\xi(r)} \left(g - \mathrm{d}r \otimes \mathrm{d}r\right). \tag{12}$$

Geometric assumptions

We also suppose that

$$\left| \frac{\partial_r \varrho}{\varrho} \right| \leq \frac{\xi'(r)}{\xi(r)} \tag{13}$$

For further reference, we denote

$$\bar{\zeta}(r) = \int_0^r \xi(\varsigma)\, \mathrm{d}\varsigma. \tag{14}$$

Height estimates

Proposition

Suppose that (11) *holds and that* $\varrho(x) = \varrho(r(x))$. *Let* u *be a solution of the non-parametric mean curvature flow in* $B_{R_0}(o) \times [0, T']$ *with Dirichlet boundary condition* $u(x,t) = u(x,0) \in \partial B_{R_0}(o) \times [0, T']$. *Then we have the following height estimate*

$$|u(x,t)| \leq \sup_{B_{R_0}(o)} |u| + v_{R(T)}(o) - v_{R_0}(r(x)). \tag{15}$$

Evolution equations

Proposition

Suppose that (11) holds. The restrictions of the functions r and s to the hypersurfaces $\Sigma_t = \Psi_t(\Sigma)$, $t \in [0, T)$, satisfy

$$(\partial_t - \Delta) r \geq -\frac{\xi'(r)}{\xi(r)} \left(n - |\nabla r|^2\right) - \varrho^2 |\nabla s|^2 \left(\langle \bar{\nabla} \log \varrho, \nabla r \rangle - \frac{\xi'(r)}{\xi(r)} \right) \tag{16}$$

and

$$(\partial_t - \Delta) s = -2 \langle \bar{\nabla} \log \varrho, N \rangle \langle \bar{\nabla} s, N \rangle. \tag{17}$$

In both expressions, ∇ and Δ are the intrinsic Riemannian connection and Laplacian in Σ_t, respectively, whereas $\bar{\nabla}$ denotes the Riemannian connection in $\bar{M}$.

Evolution equations

Proposition

Suppose that (11) *holds. Given the function*

$$\zeta(\Psi(x,t)) = \int_0^{r(\Psi(t,x))} \xi(\varsigma)\,\mathrm{d}\varsigma, \tag{18}$$

that is, $\zeta = \bar{\zeta}(r \circ \Psi)$*, we have*

$$(\partial_t - \Delta)\zeta \geq -n\xi'(r) - \varrho^2 |\nabla s|^2 \xi(r) \left(\langle \bar{\nabla} \log \varrho, \nabla r \rangle - \frac{\xi'(r)}{\xi(r)} \right). \tag{19}$$

Evolution equations

Proposition

If the hypersurfaces Σ_t, $t \in [0, T)$, evolve by the non-parametric mean curvature flow, then

$$(\partial_t - \Delta) W = -W(|A|^2 + \overline{\mathrm{Ric}}(N, N)) - 2W^{-1}|\nabla W|^2, \qquad (20)$$

where $W = \langle X, N\rangle^{-1}$ (note that $W = (\varrho^{-2} + |\nabla^M u)|^2)^{1/2})$ and A is the Weingarten map of Σ_t. If the ambient Ricci tensor satisfies $\overline{\mathrm{Ric}} \geq -L$ for some non-negative constant L then

$$(\partial_t - \Delta)(e^{-Lt} W) \leq 0.$$

In particular, the parabolic maximum principle implies in this case that

$$\sup_{B_R(o)\times[0,T']} W(x,t) \leq e^{LT'}\big(\sup_{B_R(o)} W(\cdot, 0) + \sup_{\partial B_R(o)\times(0,T]} W\big). \qquad (21)$$

Interior gradient estimates

For obtaining interior gradient estimates, we use a method due to N. Korevaar and L. Simon and adapted to the parabolic context by Ecker and Huisken (and by Colding and Minicozzi in a different presentation).

For that, we apply the parabolic maximum principle to a function of the form ηW, where

$$\eta(x,t) = \phi\big((\bar{\zeta}(R) - \zeta(x,t) + \chi(s(x,t)))_+\big)$$

with

$$\phi(\varsigma) = e^{\lambda\varsigma} - 1$$

and

$$\chi(s) = \frac{1}{2\beta}(s - \sup_{B_R(o)\times[0,T']}|u|).$$

Interior gradient estimates

The constants $\lambda > 0$ and $\beta > 0$ are given in terms of geometric data as follows

$$\lambda = \max\left\{1, 2\beta^2 C_R \sup^2_{B_R(o)} \varrho + \left((2\beta^2 C_R \sup^2_{B_R(o)} \varrho)^2 + 4\beta^2 (L+\delta) \sup^2_{B_R(o)} \varrho\right)^{1/2}\right\}$$

with

$$\beta := \frac{1}{\zeta(R)}\left(\sup_{B_R(o)\times[0,T']}|u| - \sup_{[0,T']} u(o,t)\right).$$

$$C_R = n \sup_{B_R(o)} \xi'$$

and

$$\delta = 2\sup^2_{B_R(o)}(2|\bar{\nabla}\log\varrho| + 2\xi).$$

Interior gradient estimates

Proposition

Let $o \in M$. *Given* $R > 0$ *and* $T' \in (0, T)$, *suppose that (11) and (13) hold in* $B_R(o) \subset M$ *and that* $\overline{\mathrm{Ric}} \geq -L$ *for some constant* $L \geq 0$ *in the Killing cylinder over* $B_R(o)$. *Then*

$$W(o,t) \leq \frac{\exp(\lambda\zeta(R))}{\exp(\lambda\zeta(R)/2) - 1} \max\left\{\sup_{B_R(o)} W(\cdot,0), \frac{\beta C_R}{2\sup_{B_R(o)}\xi} \frac{\sup_{B_R(o)}\varrho}{\inf_{B_R(o)}\varrho}\right\} \tag{22}$$

for $0 \leq t \leq T'$.

Interior gradient estimates

In the particular case of Riemannian products (when $\varrho = 1$ and X is a parallel vector field) we can also obtain a gradient estimate of the form

$$W(o,t) \le C_R \exp\left(32\mu^2 \sup_{B_R(o)} \left(n\frac{\xi'(r)}{\zeta(R)} + \frac{2}{\mu}\frac{\xi(r)}{\zeta(R)} + \frac{L}{\zeta(R)}\right)\right)$$

for $t \in [0, T')$, where $\mathrm{Ric}_M \ge -Lg$, $\zeta(r) = \int_0^r \xi$ and

$$\mu = \sup_{B_R(o)\times[0,T]} |u|, \quad C_R = n \sup_{B_R(o)} \xi'.$$

Curvature estimates

In order to obtain second order bounds we need to use a parabolic counterpart of the classical Simons' formula.

Proposition

The squared norm $|A|^2$ of the second fundamental form of Σ_t, $t \in [0, T)$, evolve as

$$\begin{aligned}\frac{1}{2}(\partial_t - \Delta)|A|^2 + |\nabla A|^2 = |A|^2\overline{\mathrm{Ric}}(N, N) + |A|^4 + \\ + g^{k\ell}(\bar{\nabla}_i \bar{R}_{kj0\ell} + \bar{\nabla}_k \bar{R}_{\ell i0j})a^{ij} + 2g^{k\ell}(a_{is}\bar{R}^s_{kj\ell} + a_{sk}\bar{R}^s_{\ell ij})a^{ij}.\end{aligned} \quad (23)$$

Curvature estimates

Following closely the papers by Ecker & Huisken and Borisenko & Miquel one applies the parabolic maximum principle to the function

$$\phi(x,t)\psi(W)|A|^2$$

where

$$\phi(x,t) = (\bar{\zeta}(R) - \zeta(x,t) - C_R t)^2$$

and

$$\psi(W) = \frac{W^2}{\gamma - \delta W^2}$$

with

$$\gamma = \frac{1}{\sup_{B_R(o)} \varrho^2}$$

and

$$\delta = \frac{1}{2} \frac{\gamma}{\sup_{B_R(o)\times[0,T']} W^2}.$$

Curvature estimates

Now, given $R' \in (0, R)$ such that

$$\bar{\zeta}(R') + C_R T' \leq \frac{1}{2}\bar{\zeta}(R),$$

we conclude that

$$\sup_{B_{R'}(o)\times[0,t]}|A| \leq \left(C \sup_{B_R(o)\times[0,T']} W^2 + \widetilde{C} \sup_{B_R(o)\times[0,T]} W^3\right)\frac{\xi(R)}{\zeta(R)},$$

for $t < T'$, where the non-negative constants C and $\widetilde{C}$ depend on $\sup_{B_R(o)}\varrho$, $\xi(R), \zeta(R)$, L, $|\nabla^M \log \varrho|$, $|\nabla^M \nabla^M \log \varrho|$, R^M and $\nabla^M R^M$.

An inductive reasoning as in the classical references permits to obtain (uniform in compacts) estimates for successive covariant derivatives of A.

Main results

Theorem (J. Faustino, –)

Suppose that (M, ϱ, ξ) satisfy (11) and (13). Then there exists a modified mean curvature flow of entire graphs $\Sigma_t \subset M \times_\varrho \mathbb{R}$, $t \in [0, +\infty)$, with initial condition given by an entire smooth graph Σ_0. By modified we mean that the speed of the flow is $nH - \sigma$ for a given $\sigma \in [0, nH_\infty)$ where

$$nH_\infty = \inf_{r>0}\left(\frac{\varrho'(r)}{\varrho(r)} + (n-1)\frac{\xi'(r)}{\xi(r)}\right).$$

Sketch of the proof for the case $\varrho = \varrho(r)$.

We consider some exhaustion of M by geodesic balls $B_{R_k}(o)$, $k \in \mathbb{N}$. Let $u_k(x,t)$ be the solution of the non-parametric mean curvature flow with Dirichlet boundary condition $u_0|_{\partial B_{R_k}(o)}$ and initial value u_0.

We can fix R_k and $s_k > 0$ large enough so that the graph $S_{R_k}(o)$ of v_{R_k} satisfies

$$\mathrm{Graph}(u_k(\cdot,0)|_{B_{R'_k(o)}}) \cap \Phi(-s_k, S_{R_k}(o)) = \emptyset.$$

and

$$\mathrm{Graph}(u_k(\cdot,t)|_{\partial B_{R'_k(o)}}) \cap \Phi(-s_k, S_{R_k}(o)) = \emptyset.$$

for $t < T'$ and some R'_k such that $\bar{\zeta}(R'_k) + C_R T' < \frac{1}{2}\bar{\zeta}(R_k)$.

Hence the parabolic maximum principle yields a height bound:

$$\mathrm{Graph}(u_k(\cdot,t)|_{B_{R'_k(o)}}) \cap \Phi(-s_k, S_{R_k(T')}(o)) = \emptyset.$$

for $t < T'$.

Evolution of entire graphs

Sketch of the proof for the case $\varrho = \varrho(r)$.

These uniform height estimates for $u_k(\cdot, t)$, restricted to some parabolic cylinder $B_{R'_k}(o) \times [0, T']$, provide uniform gradient and curvature (and higher order derivatives) in a possibly smaller cylinder $B_{R''_k}(o) \times [0, T']$.

One extracts converging subsequences obtaining smooth (due to higher order estimates) limit graphs as $R_k \to \infty$ for an arbitrary $T' > 0$.

Self-similar mean curvature flow

Definition

Let $\Phi : (\Omega_*, \Omega^*) \times \bar{M} \to \bar{M}$ be the flow generated by X. The MCF $\Psi : (\omega_*, \omega^*) \times \Sigma \to \bar{M}$ is *self-similar* if there exists an isometric immersion $\psi : \Sigma \to \bar{M}$ and a reparametrization $f : (\omega_*, \omega^*) \to (\Omega_*, \Omega^*)$ of the flow lines of X such that

$$\Psi_t(\Sigma) = \Phi_{f(t)}(\psi(\Sigma)), \tag{24}$$

for all $t \in (\omega_*, \omega^*)$.

Suppose that X is a parallel vector field ($\varrho = 1$).

We easily prove that for each $t \in (\omega_*, \omega^*)$, there exists a constant c_t such that

$$c_t X = c_t \Psi_{t*} T + n\mathbf{H} \tag{25}$$

for some vector field $T \in \Gamma(TM)$. Moreover

$$n\bar{\nabla}^{\perp}\mathbf{H} + c_t II(T, \cdot) = 0. \tag{26}$$

We also have from the fact that X is parallel that

$$II_{-n\mathbf{H}} + \frac{c_t}{2} L_T g = 0, \tag{27}$$

where g is the metric induced in M by Ψ_t and the tensor $II_{-n\mathbf{H}}$ is its second fundamental form with respect to $-n\mathbf{H}$.

Mean curvature flow solitons

Definition

An isometric immersion $\psi : M^n \to \bar{M}^{n+1}$ is a *mean curvature flow soliton* if

$$c\,X^{\perp} = n\mathbf{H} \tag{28}$$

along ψ, for some constant $c \in \mathbb{R}$. With a slight abuse of notation, we also say that the submanifold $\psi(M)$ itself is the mean curvature flow soliton. This equation can be also written as

$$nH = c\langle X, N\rangle. \tag{29}$$

The soliton equation

Proposition

Let $\psi : M \to \bar{M}$ be a mean curvature flow soliton and let X be a parallel vector field on $\bar{M}$. Then we have along ψ

$$II_{-n\mathbf{H}} + \frac{c}{2} L_T g = 0 \tag{30}$$

where g is the metric induced in M by ψ and $II_{-n\mathbf{H}}$ is its second fundamental form in the direction of $-n\mathbf{H}$. The vector field T is defined by $\psi_* T = X^T$. Furthermore

$$n\bar{\nabla}^{\perp}\mathbf{H} + c\, II(T, \cdot) = 0, \tag{31}$$

where II is the second fundamental form of ψ and $\bar{\nabla}^{\perp}$ is its normal connection.

Translating solitons

Suppose that the graph Σ of a function $u : M \to \mathbb{R}$ embedded into the Riemannian product $M \times \mathbb{R}$ is a *translating* soliton. Then the soliton equation

$$nH = c\langle X, N\rangle$$

may be written as

$$\operatorname{div}_M \left(\frac{\nabla^M u}{W}\right) = \frac{c}{W}. \tag{32}$$

Variational setting

We consider the weighted area functional

$$\mathcal{F}[\Sigma] = \int_M e^{c\tau|_\Sigma}\,\mathrm{d}\Sigma. \tag{33}$$

Critical immersions are characterized by the soliton equation

$$nH - c\langle X, N\rangle = 0.$$

Moreover the second variation formula for normal variations reads as

$$\delta^2\mathcal{F}[\Sigma](v,v) = -\int_\Sigma e^{c\tau|_\Sigma}\, v\, Lv\,\mathrm{d}\Sigma, \tag{34}$$

where the *stability* operator L is defined by

$$Lv = \Delta_{-c\tau} v + (|A|^2 + \mathrm{Ric}_{\bar{M},-c\tau}(N,N))v. \tag{35}$$

Variational setting

Here we are denoting

$$\Delta_{-c\tau} = \Delta + c\langle\nabla\tau, \cdot\rangle = e^{-c\tau}\,\mathrm{div}_\Sigma(e^{c\tau}\nabla\,\cdot), \tag{36}$$

and

$$\mathrm{Ric}_{\bar{M},-c\tau} = \mathrm{Ric}_{\bar{M}} - c\bar{\nabla}\bar{\nabla}\tau = \mathrm{Ric}_{\bar{M}} - c\,\bar{g}$$

is the Bakry-Emery-Ricci tensor associated to $-c\tau$.

Translating solitons

We focus from now on in the particular case when the metric in M is rotationally symmetric as

$$\mathrm{d}r^2 + \xi^2(r)\,\mathrm{d}\theta^2$$

for $r(x) = \mathrm{div}(o, x)$ for some pole $o \in M$.

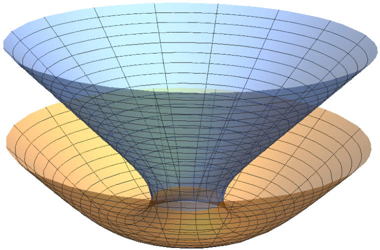
In this case the PDE for rotationally symmetric translating solitons reduces to the ODE

$$\frac{u''}{1+u'^2} + (n-1)\frac{\xi'}{\xi}u' = c.$$

In the case of $\mathbb{H}^n \times \mathbb{R}$ we have $\xi(r) = \sinh r$.

Translating solitons

In joint work with Francisco Martín, we have proved the existence of one-parameter families of translating solitons in $\mathbb{H}^n \times \mathbb{R}$.



Their asymptotic behavior as $r \to +\infty$ is

$$u'(r) = \frac{c}{n-1}\frac{\xi(r)}{\xi'(r)} + o\left(\frac{\xi'(r)}{\xi(r)}\right).$$

Translating solitons

Theorem (–, F. Martín)

Let Σ be a translating soliton in $\mathbb{H}^n \times \mathbb{R}$ and $\mathcal{G}$ a continuous subgroup of the isometries of $\mathbb{H}^n \times \mathbb{R}$ satisfying $g(\mathbb{H}^n) = \mathbb{H}^n$ and $g(\Sigma) = \Sigma$, for all $g \in \mathcal{G}$. Then we have:

i. *If $\mathcal{G}$ consists of rotations around a vertical axis, then Σ is part of either a bowl soliton or a wing-like soliton.*
ii. *If $\mathcal{G}$ consists of hyperbolic translations along a fixed geodesic γ in $\mathbb{H}^n$, then Σ is an open region of the grim reaper hyperplane.*
iii. *If $\mathcal{G}$ consists of parabolic translations around a point $p_0 \in \partial_\infty \mathbb{H}^n$, then Σ is a piece of either an ideal bowl soliton or an ideal translating catenoid.*

Translating solitons

We recall the fact that translating solitons in $M \times \mathbb{R}$ are indeed minimal surfaces for the so-called Ilmanen metric $g_c = e^{c\tau}(\mathrm{d}\tau^2 + g)$ where g is the Riemannian metric in M.

For instance, if $M = \mathbb{R}^2$ with $g = \mathrm{d}x^2 + \mathrm{d}s^2$, the coordinate vector field $X = \partial_s$ is Killing with respect to g_c.

In this case the Killing graph of a function u defined on $\Omega \subset P = X^\perp$ such that

$$\frac{1}{e^{c\tau}} \operatorname{div} \left(e^{c\tau} \frac{\nabla^P u}{W} \right) = 0$$

is a translating soliton.

Translating solitons

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In this case the Killing graph of a function u defined on $\Omega \subset P = X^\perp$ such that

$$\frac{1}{e^{c\tau}} \operatorname{div}\left(e^{c\tau}\frac{\nabla^P u}{W}\right) = 0$$

is a translating soliton. In particular, in this formulation one can explore the invariance of $\mathcal{F}$ by translations in the X direction (a sort of flux formula).

Translating solitons

This is the framework to study Dirichlet problems over *admissible* domains with (possibly infinite) boundary data.

For admissible we mean a precompact domain bounded by $e^{c\tau}$-geodesic arcs and $e^{c\tau}$-convex arcs.

Theorem (E. Gama, E. Heinonen, – , F. Martín)

Let $\Omega \subset P$ be an admissible domain such that for any admissible polygon $\mathcal{P} \subset \bar{\Omega}$ we have

$$2\alpha_c[\mathcal{P}] < L_c[\partial\mathcal{P}] \quad \text{and} \quad 2\beta_c[\mathcal{P}] < L_c[\partial\mathcal{P}].$$

If there are no $e^{c\tau}$-convex arcs in $\partial\Omega$ and $\alpha_c[\partial\Omega] = \beta_c[\partial\Omega]$, then there exists a Jenkins-Serrin Killing graph over $\bar{\Omega}$ which is a translating soliton.

Translating solitons

Key steps:

- Local existence theorem (following A. L. Pinheiro (2009) and M. H. Nguyen (2014)).
- Interior gradient estimates and compactness theorem.
- A Perron's method.
- Construction of barriers (Scherk-like translating solitons).

Thanks for your attention!