Constant Mean Curvature surfaces in 3-Manifolds

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CIRM

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1. Concentration of CMC in the compact
2. Asymptotic flatness, mass, center of mass
3. Huisken-Yau and Ye foliations
4. A new theorem on large CMC in asymptotically flat manifolds
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contents

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Definition

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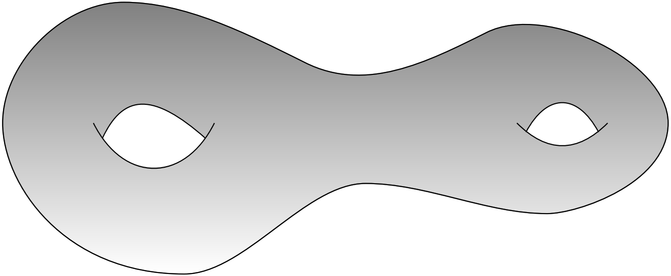
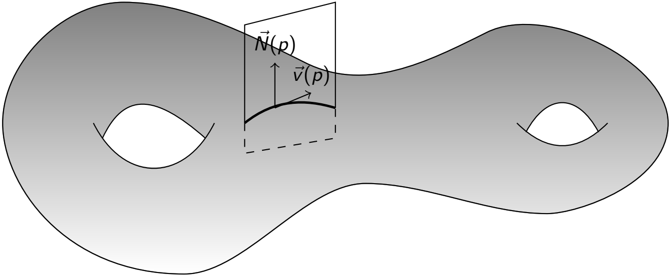
We define two maps K et H, namely the Gauss and the mean curvature, as follows

$$K(p) = det(d\vec{N}_p)$$

and

$$H(p) = \frac{1}{2} trace(d\vec{N}_p).$$




$\vec{N}(p)$
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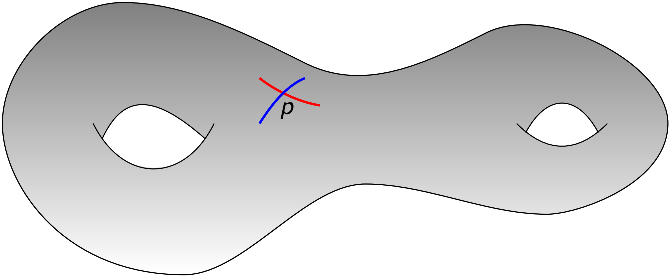
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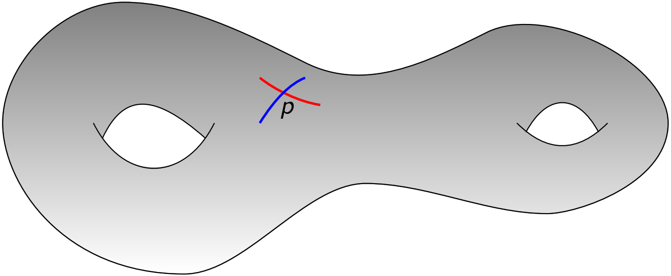

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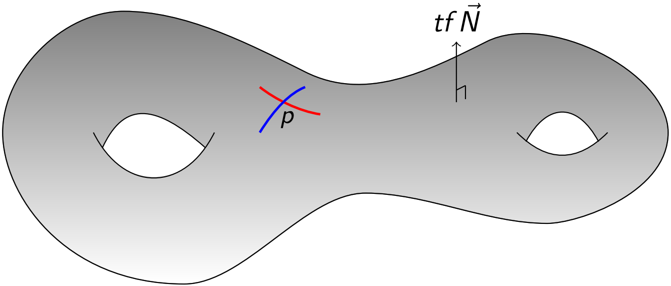

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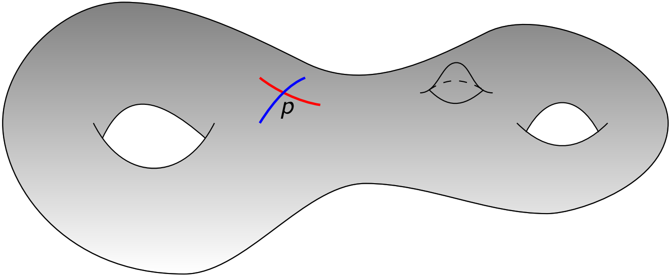

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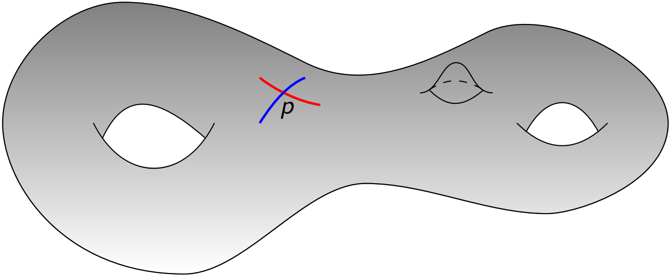
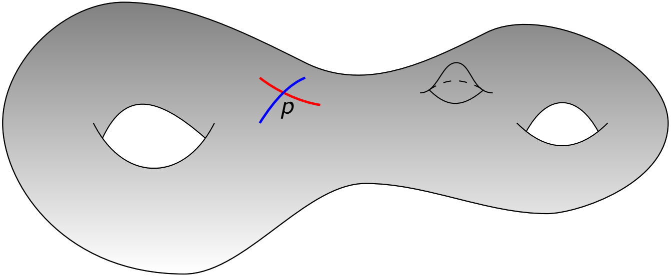


- $H = \frac{\kappa_1 + \kappa_2}{2}$ and $K = \kappa_1 \kappa_2$
- $A'(0) = -2 \int_\Sigma f \, H \, d\Sigma$ et $V'(0) = \int_\Sigma f \, d\Sigma$



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Theorem

Surfaces which minimize their area with a fixed volume have constant mean curvature.

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Kapouleas proved in the 90' that there are CMC surfaces of arbitrary genus.

Theorem (Barbosa, Do Carmo, 84)

The only (weakly) stable CMC (complete) surfaces of $\mathbb{R}^3$ are planes and round spheres.

Theorem (Ye, 1991)

Let $(\mathcal{N}, g)$ be a Riemannian manifold and $p \in \mathcal{N}$ a non-degenerate critical point of the scalar curvature. Then there exists a surface with constant mean curvature in every neighbourhood of p.

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We can relax the hypothesis of non-degeneracy for existence : Pacard et Xu 2009.

Theorem (Zoltareva, 2016)

Let $p \in \mathcal{N}$ a non- degenerate critical point of the scalar curvature such that $\nabla^2 Scal$ admits a positive eigenvalue. We can perturb the connected sum of two spheres in a surface of constant mean curvature in any neighbourhood of p.

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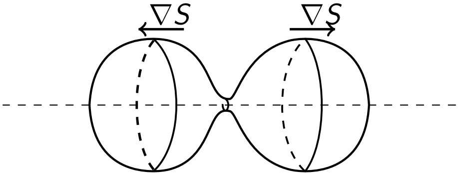

Gluing of two spheres thanks to the gradient of the scalar curvature .

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Theorem (Druet, 2002)

Let $(\mathcal{N}, g)$ be a compact Riemannian manifold and Ω_V a sequence of isoperimetric domains of volume V, then

$$\Omega_V \to p \text{ as } V \to 0,$$

where p is a point of maximum of the scalar curvature.

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Especially, it appears the classical argument for compactness of stable CMC.

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Theorem (Laurain, 2010)

Let $(\mathcal{N}, g)$ be a compact Riemannian 3-manifold and Σ_H a sequence of embedded spheres with constant mean curvature H satisfying the following hypothesis

$$\begin{cases} \delta(\Sigma_H) = o(1) \\ A(\Sigma_H) = O\left(\frac{1}{H^2}\right). \end{cases} \qquad \text{as } H \to +\infty,$$

Then Σ_H converge to critical point of the scalar curvature.

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Uniqueness of the profile of concentration.

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How to get pointwise estimate, without maximum principle ?
Study of the linearized operator.

contents

Definition

Let (M, g) be a 3-manifold, it is said to be Asymptotically Flat (AF) (with one end), if there exists $\varepsilon > 0$ and a compact K such that $M \setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus B(0, 1)$ and in those coordinates

$$g = \delta^{ij} + O_2(|x|^{-\frac{1}{2}-\varepsilon}).$$

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Theorem (Arnowitt, Deser, Misner, Bartnik, Chrusciel)

Let (M, g) be an asymptotically flat manifold whose scalar curvature is in L^1, then the following limit exists

$$\lim_{R\to+\infty} \frac{1}{16\pi} \int_{S(0,R)} (g_{ij,i} - g_{ii,j})\nu^j \, d\sigma,$$

moreover it depends only on the metric. Let denote it m for mass.

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Alternative definition using the Einstein tensor : $Ric - \frac{S}{2} g$, see Huang, Corvino & Wu, Herzlich...

Theorem (Schoen-Yau, 79)

Let (M, g) be an AF manifold with nonnegative scalar curvature. Then $m \geq 0$ with equality if and only if M is isometric to $\mathbb{R}^3$.

$\mathbb{R}^3 \setminus \{0\}, \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij}$ is an AF manifold with vanishing scalar curvature and mass m .

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The mass is unchanged by translation, considering $\mathbb{R}^3 \setminus \{p\}, \left(1+\frac{m}{2|x-p|}\right)^4 \delta_{ij}$ but can we detect the "center" of this translated version of Schwarzschild ?

Definition

Let (M, g) be an AF manifold, such that

$$|g_{ij} - \delta_{ij}| + |x||\Gamma_{ij}^k| + |x|^2|Ric_{ij}| + |x|^{\frac{5}{2}}|S| \leq \frac{C}{|x|^{\frac{1}{2}+\varepsilon}}.$$

Then it satisfies the **weak Regge-Teitelboim** *condition, if*

$$|g^{odd}(x)| + |x||\Gamma^{odd}(x)| \leq \frac{C}{|x|^{1+\varepsilon}}.$$

It satisfies the **strong Regge-Teitelboim** *condition, if*

$$|g^{odd}(x)| + |x||\Gamma^{odd}(x)| + |x|^2|Ric^{odd}(x)| + |x|^{\frac{5}{2}}|S^{odd}(x)| \leq \frac{C}{|x|^{\frac{3}{2}+\varepsilon}}.$$

Theorem (Beig, O'Murchadha, 90)

Let (M, g) *an AF manifold satisfying* **the strong RT condition with non vanishing mass**, *then the following limit exists*

$$\lim_{R \to +\infty} \frac{1}{16\pi m} \int_{S(0,R)} (g_{ij,i} - g_{ii,j})\nu^j x^\alpha - (g_{i\alpha}\nu^i - g_{ii}\nu^\alpha)\, d\sigma,$$

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The strong RT condition has been proved to be optimal by Cederbaum & Nerz (13) : Constructing metric with divergent center of mass.

contents

Theorem (Christodoulou-Yau, 88)

Let (M, g) be a 3-manifold with none-negative scalar curvature, then the Hawking quasi-local mass

$$m(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{4\pi} \int_{\Sigma} H^2 \, d\sigma\right)$$

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Improvements : L.H. Huang, J. Metzger and finally C. Nerz(14) who prove the existence into an AF manifold. .

Theorem (Qing, Tian, 07)

Let (M, g) a Schwarzschildean manifold with positive mass. Then exists a compact set K such that stable CMC spheres which separates the infinity from K coincide with the leafs of the CMC foliation.

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- **Global uniqueness** of large stable CMC surfaces in almost of Schrwarzschildean 3-manifolds, arxiv april 2017 Otis Chodosh & Michael Eichmair.

contents

Theorem (L.,Metzger 19)

Let (M, g) *an AF manifold satisfying the* **weak RT condition, with non vanishing mass** *then there exist a compact* K *and* $\varepsilon_0 > 0$ *and* $C > 0$ *such that for every CMC surfaces which separates* K *and infinity such that*

$$\int_\Sigma |\mathring{A}|^2 \, d\sigma \leq \varepsilon_0$$

satisfy

$$\sup_\Sigma |x| \leq C \inf_\Sigma |x|.$$

Moreover if $|\Sigma_n| \to +\infty$ *then* $\lim_n \frac{\sup_{\Sigma_n} |x|}{\inf_{\Sigma_n} |x|} \to 1.$

Consequently : **Uniqueness of large, not outlying, stable CMC surfaces** in this "optimal" setting.

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Then we can parametrize Σ_H by $u_H : \mathbb{S}^2 \to (\mathbb{R}^3, g)$ conformal, such that it decomposes like

$$u_H = \frac{1}{H}(a + Id) + r_H$$

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Moreover it solves

$$\Delta u_H^l - \Gamma_{ij}^l \langle \nabla u_H^i, \nabla u_H^j \rangle = -2H\sqrt{|g|}g^{lj}((u_H)_x \wedge (u_H)_y)_j.$$

where x, y are conformal coordinates given by a stereographic projection.

Let us expand the equation

$$\Delta r_H + 2((r_H)_x \wedge \omega_y + \omega_x \wedge (r_H)_y) =$$
$$\frac{1}{H^2}(\Gamma^l_{ij})\langle \nabla\omega^i, \nabla\omega^j\rangle + \frac{2}{H} h_{ij}(\omega_x \wedge \omega_y)_j - \frac{h_{ii}}{H}(\omega_x \wedge \omega_y)_l$$
$$+ O\left(\frac{|\Gamma|}{H}|\nabla\omega||\nabla r_H| + |\Gamma||\nabla r_H|^2 + H|\nabla r_H|^2\right.$$
$$\left. + |h||\nabla\omega||\nabla r_H| + H|h||\nabla r_H|^2\right)$$

where ω is the inverse of the stereographic projection and $h = g - \delta$.
The right-hand side behaves like $\left(\frac{1}{H}\right)^{\frac{1}{2}-\varepsilon}$.

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$$|\nabla r_H| = O\left(\left(\frac{1}{H}\right)^{\frac{1}{2}-\varepsilon}\right)$$

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Then we assume that the estimate is not true and we divide the equation by $\|\nabla r_H\|_\infty >> \left(\frac{1}{H}\right)^{\frac{1}{2}-\varepsilon}$. The right hand side disappear and we have a **non trivial** solution of the linearized operator

$$\begin{cases} \Delta r + 2r_x \wedge \omega_y + \omega_x \wedge r_y = 0 \\ \langle r_x, \omega_x \rangle = \langle r_y, \omega_y \rangle, \quad \langle r_x, \omega_y \rangle + \langle r_y, \omega_x \rangle = 0 \end{cases}$$

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Then, thanks to the classification of (L. 10), r must be 0, we get a contradiction and the desired estimate.

Step 3 : General flux formula

$$0 = \int_{\mathbb{R}^2} \frac{1}{H^2}\Gamma^l_{ij}\langle\nabla\omega^i, \nabla\omega^j\rangle + \frac{2}{H}h_{lj}(\omega_x \wedge \omega_y)_j - \frac{h_{ii}}{H}(\omega_x \wedge \omega_y)_l \, dz + o(1)$$

$$= \int_{S(a_H, \frac{1}{H})} \frac{1}{2}(2g_{il,j} - g_{ij,l})(\delta_{ij} - \nu^i\nu^j) - 2Hh_{lj}\nu^j \; + Hh_{ii}\nu^l d\sigma \; + o(1)$$

Thanks to the weak RT condition

$$= \frac{1}{2}\int_{S(a_H, \frac{1}{H})} (g_{ii,j} - g_{ij,i})\nu^j\nu^l + H(-g_{lj}\nu^j + g_{ii}\nu^l)\, d\sigma + o(1)$$

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But $\nu^l = H(y - a_H)^l$.

$$
\begin{aligned}
0 = & -Ha_H^l \left(\int_{S(a,\frac{1}{H})} (g_{ii,j} - g_{ij,i})\nu^j \, d\sigma \right) \\
& + H \left(\int_{S(a_H,\frac{1}{H})} ((g_{ii,j} - g_{ij,i}))\, \nu^j y^l + (-g_{lj}\nu^j + g_{ii}\nu^l) \, d\sigma \right) \\
= & -Ha_H^l \left(\int_{S(a,\frac{1}{H})} (g_{ii,j} - g_{ij,i})\nu^j \, d\sigma \right) \\
& + H \left(\int_{S(0,\frac{2}{H})} ((g_{ii,j} - g_{ij,i}))\, \nu^j y^l + (-g_{lj}\nu^j + g_{ii}\nu^l) \, d\sigma \right) \\
& - H \int_{B(0,\frac{2}{H})\setminus B(a_H,\frac{1}{H})} (g_{ij,ij} - g_{ii,jj}) y^l \, dx \\
= & \, I_H + II_H + III_H
\end{aligned}
$$

Thanks to AF,

$$III_H \to 0.$$

Thanks to some Nerz's results, when we are AF with the weak RT, we have

$$\int_{S(0,\frac{2}{H})} \left((g_{ii,j} - g_{ij,i}) \right) \nu^j y^l + \left(-g_{lj}\nu^j + g_{ii}\nu^l \right) d\sigma = O\left(\left(\frac{1}{H} \right)^{1-\varepsilon} \right)$$

Hence

$$II_H \to 0.$$

Finally

$$a_H = o(1/H).$$

contents

Theorem (L., Mondino 13)

Let (M, g) be a 3-manifold and let $\Phi_k : \mathbb{S}^2 \to M$ be a sequence of area-constrained Willmore surfaces satisfying

$$\limsup_k W(\Phi_k) < 8\pi.$$

and Hausdorff converging to a point $\bar{p} \in M$.
Then $\nabla Scal(\bar{p}) = 0$; moreover, if we rescale (M, g) around $\bar{p}$ in such a way that the rescaled immersions $\tilde{\Phi}_k$ have fixed area equal to one, then $\tilde{\Phi}_k$ converges smoothly to a round sphere.

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Such a result was also prove by Lamm-Metzger in 2012 assuming $\limsup_k W(\Phi_k) < 4\pi + \varepsilon$ for ε small enough and with a $W^{2,2}$ convergence.

Theorem (Lamm-Metzger-Schulze 11)

Let (M, g) a Schwarzschildean manifold with positive mass, then there exists $\lambda_0 > 0$ such that for every $\lambda \in (0, \lambda_0)$ there exist an area-constrained Willmore surface Σ_λ satisfying $|\Sigma_\lambda| = \frac{1}{\lambda}$. Moreover there are locally unique.

Work In Progress (L.,Metzger 19)

Let (M, g) an AF manifold satisfying the **weak** *RT condition,* **with non vanishing mass** *then there exist a compact K and $\varepsilon_0 > 0$ and $C > 0$ such that for every area constrained Willmore surfaces which separates K and infinity such that*

$$\int_\Sigma |\mathring{A}|^2 \, d\sigma \leq \varepsilon_0$$

then

$$\sup_\Sigma |x| \leq C \inf_\Sigma |x|.$$

Moreover if $|\Sigma_n| \to +\infty$ then $\lim_n \frac{\sup_{\Sigma_n} |x|}{\inf_{\Sigma_n} |x|} \to 1$.

Consequently : **Uniqueness of large, not outlying, Willmore surfaces** in this setting.

Theorem (Christodoulou-Yau, 88)

Let (M, g) be a 3-manifold with none-negative scalar curvature, then the Hawking quasi-local mass

$$m(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{4\pi} \int_\Sigma H^2 \, d\sigma\right)$$

of a closed stable CMC is non negative.

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A new isoperimetric problem : What about surfaces which maximise the quasi local-mass with fixed area ?

Let

$$m_\alpha(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{4\pi} \int_\Sigma (H^2 + \alpha|\mathring{A}|^2)\, d\sigma\right),$$

$\alpha \in (0, 1/6]$.

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Question

Let (M, g) be an AF manifold satisfying the **weak** *RT condition,* **with nonnegative scalar curvature and positive mass**, *are surfaces which maximise m_α under large area constraint are close to large centred sphere ?*

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Question

The surfaces which maximise m_α under area constraint in Schwarzschild space are the round spheres ?

Thank you for your attention.