Boundary behaviour of Bergman harmonic maps from strictly pseudoconvex domains

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1. Bergman metric of a bounded domain $\Omega \subset \mathbb{C}^n$

$\Omega \subset \mathbb{C}^n$ a bounded domain ($n \geq 2$).

$K(\zeta, z)$ the Bergman kernel of Ω

$\big[K : \Omega \times \Omega \to \mathbb{C}$ is a C^∞ function, holomorphic in ζ, anti-holomorphic z, and unique with

$$f(\zeta) = \int_\Omega K(\zeta, z)\, f(z)\, d\mu(z)$$

$$\forall \;\; f \in L^2(\Omega) \;\text{ with }\; \overline{\partial} f = 0 \;\text{ in }\; \Omega, \quad \forall \;\; \zeta \in \Omega\big]$$

[Discovered by S. Bergman, cf. e.g. *The kernel function and conformal mapping*, Mathematical Surveys of American Mathematical Society, No. 5, 1950]

Existence and uniqueness of $K(\zeta, z)$ follow from Riesz representation theorem applied to

$$\delta_\zeta : L^2 H(\Omega) \to \mathbb{C}, \quad \delta_\zeta(f) = f(\zeta).$$

Riesz theorem applies because

- δ_ζ linear and continuous functional;
- $L^2 H(\Omega) \subset L^2(\Omega)$ closed subspace [hence Hilbert].

Both properties follow from the basic estimate

$$\left| f(\zeta) \right| \leq C_A \, \|f\|_{L^2}$$

$$A \subset\subset \Omega, \quad \zeta \in A, \quad f \in L^2 H(\Omega).$$

Bergman kernels difficult to compute. For instance

$$\Omega = \mathbb{B}^n = \{z \in \mathbb{C}^n \ : \ |z| < 1\},$$

$$K(\zeta, z) = \frac{C_n}{\big(1 - \zeta \cdot \overline{z}\big)^{n+1}} \quad (C_n = n!\,\pi^{-n}).$$

Calculation relies on

- $L^2H(\Omega)$ separable;
- $\forall \ \{\phi_\nu\}_{\nu \geq 0} \subset L^2H(\Omega)$ complete orthonormal system

$$K(\zeta, z) = \sum_{\nu=0}^{\infty} \phi_\nu(\zeta)\,\overline{\phi_\nu(z)}$$

[uniform convergence on compact subsets $\subset \Omega \times \Omega$]

- produce an explicit $\{\phi_\nu\}_{\nu \geq 1} \subset L^2H(\mathbb{B}^n)$ and sum the series $\sum_{\nu=0}^{\infty} \phi_\nu(\zeta)\,\overline{\phi_\nu(z)}$.

Assumption $n \geq 2$ unnecessary [yet considerations to follow confined to complex analysis in several complex variables]

Assumption Ω bounded also unnecessary, yet

$$\Omega \text{ bounded } \implies \text{constants} \in L^2H(\Omega) \implies K(z,z) > 0$$

$\implies \log K(z,z)$ suitable potential to build Kählerian metric g_Ω

$$g_{j\bar{k}} = \frac{\partial^2 \log K(z,z)}{\partial z_j \, \partial \bar{z}_k}, \quad 1 \leq j,k \leq n, \tag{1}$$

[Ω bounded also enters proof that such g_Ω is Riemannian]

g_Ω is the **Bergman metric** of Ω.

Siegel domain

$$\Omega = \mathbb{S}_n = \{(z,\, w) \in \mathbb{C}^{n-1} \times \mathbb{C} \ : \ \mathrm{Im}(w) > |z|^2\}$$

unbounded, yet has a "Bergman metric"

$$g_{\mathbb{S}_n} = \left(\mathcal{C}^{-1}\right)^* \, g_{\mathbb{B}^n} \quad \left[\mathcal{C} : \mathbb{B}^n \to \mathbb{S}_n \quad \text{Cayley map}\right]$$

[and $g_{\mathbb{S}_n}$ related to Bergman kernel of $\mathbb{S}_n$ by (1)]

A few known properties of Bergman metric

- $\mathrm{Hol}(\Omega) \subset \mathrm{Isom}(\Omega\,,\, g_\Omega)$
- Ω homogeneous [i.e. $\mathrm{Hol}(\Omega)$ acts transitively on Ω]

$\Longrightarrow$ g_Ω Kähler-Einstein [$R_{j\bar{k}} = 2\, g_{j\bar{k}}$]

Historical note:

G. Fichera [in *Sulla "Kernel function"*, Bollettino dell'Unione Matematica Italiana, Serie 3, Vol. 7 (1952), n.1, p. 4-15] claims that (Bergman) kernels are useless and serve at most the purpose of writing less involved formulas.

It looks like mathematical intuition, otherwise so generous with Gaetano Fichera, had abandoned him [at the time the quoted paper was written] and that we are looking indeed at one of the major discoveries in several complex variables analysis.

2. Harmonic maps from (Ω, g_Ω)

$\Omega \subset \mathbb{C}^n$ bounded, $g = g_\Omega$ Bergman metric; (N, h) Riemannian

$$E : C^\infty(\Omega, N) \to [0, +\infty), \; E(\phi) = \frac{1}{2} \int_\Omega \|d\phi\|^2 \, d\, v_g$$

$\|d\phi\|^2 = \mathrm{trace}_g\left(\phi^* h\right)$ [Hilbert-Schmidt norm of $d\phi$]

$$d\, v_g = \sqrt{G}\, dx^1 \wedge \cdots \wedge dx^{2n}, \qquad G = \det\left[g_{AB}\right]$$

$\phi \in C^\infty(\Omega, N)$ is (**Bergman**) **harmonic** if critical for E

$$\forall \;\{\phi_t\}_{|t|<\epsilon} \subset C^\infty(\Omega, N), \quad \phi_0 = \phi, \quad \mathrm{Supp}(V) \subset \Omega,$$

$$V = (\partial\phi_t/\partial t)_{t=0}, \qquad \frac{d^2}{dt^2}\left\{E\left(\phi_t\right)\right\}_{t=0} = 0.$$

Euler-Lagrange equations for variational principle $\delta\, E(\phi) = 0$

$$\tau_g(\phi) = 0$$

$$\tau_g(\phi) \equiv \mathrm{trace}_g\, \beta(\phi) \in C^\infty\big(\phi^{-1} TN\big)$$

$$\beta(\phi) \quad \text{second fundamental form of} \ \ \phi : \Omega \to N$$

$$\tau_g(\phi)^a = -\Delta_{\mathrm{B}} \phi^a + \left(\left\{ \begin{array}{c} a \\ bc \end{array} \right\} \circ \phi \right) \frac{\partial \phi^b}{\partial x^A} \frac{\partial \phi^c}{\partial x^B}\, g^{AB}$$

$$\Delta_{\mathrm{B}} u \equiv -\frac{1}{\sqrt{G}} \frac{\partial}{\partial x^A} \left(\sqrt{G}\, g^{AB} \frac{\partial u}{\partial x^B} \right), \quad u \in C^2(\Omega),$$

[**Bergman Laplacian** i.e. Laplace-Beltrami operator of (Ω, g_Ω)]

Dirichlet problem for Bergman-harmonic map system

$$\tau_g(\phi) = 0 \quad \text{in} \;\; \Omega, \quad \phi = f \;\; \text{on} \;\; \partial\Omega, \tag{2}$$

$$f \in C(\partial\Omega\,,\,N).$$

We will not discuss existence and uniqueness of solution to Dirichlet problem for hamonic map system [dealt with in classical works by

S. Hildebrand & H. Kaul & K. Widman, *An existence theorem for harmonic maps of Riemannian manifolds*, Acta Mathematica, 138(1977), 1-16; *On the Hölder continuity of weak solutions of quasilinear elliptic systems of second order*, Ann. Scuola Norm. Sup. Pisa, 4(1977), 145-178]

Problem of interest

If $f \in C^\infty(\partial\Omega, N)$ (when) is the solution $\phi = \phi_f$ [to Dirichlet problem (2)] smooth (i.e. C^∞) up to the boundary [i.e. $\phi_f \in C^\infty(\overline{\Omega}, N)$]?

[C^∞ **regularity up to the boundary** problem]

Problem taken up first in scalar case ($N = \mathbb{R}$) by

C.R. Graham, *The Dirichlet problem for the Bergman Laplacian*, I-II, Commun. in PDEs, (5)8(1983), 433-476; ibid., (6)8(1983), 563-641.

C.R. Graham & J.M. Lee, *Smooth solutions of degenerate Laplacians on strictly pseudoconvex domains*, Duke Math. J., (3)57(1988), 697-720.

C^∞ regularity up to the boundary is a fundamental feature of the Dirichlet problem for a uniformly elliptic operator, yet may fail for operators such as Δ_{B} [merely elliptic in the interior of the domain and ellipticity degenerates at the boundary].

For instance if $\Omega = \mathbb{B}^n$

$$\Delta_{\mathrm{B}} u = -4\big(1 - |z|^2\big)\, Lu, \quad L \equiv \sum_{j,k=1}^{n} \big(\delta_{jk} - z_j \overline{z}_k\big) \frac{\partial^2}{\partial z_j\, \partial \overline{z}_k}$$

so coefficients of Δ_{B} vanish at $\partial\,\mathbb{B}^n$. Also L isn't elliptic at $\partial\,\mathbb{B}^n$ [calculation of symbol demonstrates characteristic directions at points $z \in \partial\,\mathbb{B}^n$]

By a result of C.R. Graham, despite Dirichlet problem

$$\Delta_{\mathrm{B}} u = 0 \quad \text{in } \mathbb{B}^n , \quad u = f \quad \text{on } \partial\,\mathbb{B}^n ,$$

is solvable for arbitrary continuous boundary data $f \in C(\partial\,\mathbb{B}^n)$, in order that $u \in C^\infty\left(\overline{\mathbb{B}^n}\right)$ it is necessary and sufficient that $f \in C^\infty\left(\partial\,\mathbb{B}^n\right)$ be the boundary value of a pluriharmonic function, and thus that u be pluriharmonic.

In general, for a solution $\phi \in C^\infty\left(\overline{\Omega},\ N\right)$ to exist the boundary data $f \in C^\infty\left(\partial\Omega,\ N\right)$ must satisfy necessary conditions

$$\mathcal{C}(f) = 0 \quad \text{on } \partial\Omega$$

[**compatibility conditions**, and $\mathcal{C}$ is a **compatibility operator**]

3. Cimmino-Graham-Lee method

First purpose of talk: take one step towards writing compatibility conditions [i.e. compute the normal derivatives $N(\phi^A)$ for a Bergman-harmonic map ϕ]

Method: by C.R. Graham & J.M. Lee i.e.

• $\mathcal{F}$ foliation of a one sided neighborhood $V \subset \Omega = \{\varphi < 0\}$ of $\partial\Omega$ by level sets of φ

$V/\mathcal{F} = \{M_\epsilon \ : \ 0 < \epsilon < \epsilon_0\}, \quad M_\epsilon = \{\varphi + \epsilon = 0\}$

• Write equations

$$\mathcal{C}_\epsilon(\phi) = 0 \quad \text{on} \quad M_\epsilon \tag{3}$$

induced by $\tau_g(\phi) = 0$ on each leaf M_ϵ

• Derive equations $\mathcal{C}(f) = 0 \quad \text{on} \quad \partial\Omega$

by taking the limit for $\epsilon \to 0^+$ in (3).

Method [referred by some as **invasion** of Ω from $\partial\Omega$] known classically cf. e.g.

G. Cimmino, *Nuovo tipo di condizione al contorno e nuovo metodo di trattazione per il problema generalizzato di Dirichlet*, Rend. Circ. Mat. Palermo, 61(937), 177-221.

What is **new** in Graham-Lee approach?

Ω smoothly bounded strictly pseudoconvex domain (an **assumption**) $\Longrightarrow$ $\partial\Omega$ **strictly pseudoconvex CR manifold** $\Longrightarrow$ each leaf $M_\epsilon \in V/\mathcal{F}$ strictly pseudoconvex

$\Longrightarrow$ may express $\mathcal{C}_\epsilon(\phi) = 0$ [i.e. $\tau_g(\phi) = 0$ restricted to M_ϵ] in terms of

- (Webster's) **pseudohermitian invariants** of M_ϵ
- derivatives of the **transverse curvature** r of $\mathcal{F}$

$$r = 2\,\partial\overline{\partial}\big(\xi,\,\overline{\xi}\big),$$

$$\xi \in T^{1,0}(V), \quad \partial\varphi(\xi) = 1, \quad \partial\overline{\partial}\varphi\big(\xi,\,\overline{Z}\big) = 0, \quad Z \in T_{1,0}(\mathcal{F}).$$

Main feature of (Graham-Lee) novelty:

as $\epsilon \to 0^+$ pseudohermitian invariants of M_ϵ stay bounded at $\partial\Omega$ [tend to pseudohermitian invariants of $\partial\Omega$]

Also, by a result of

J.M. Lee & R. Melrose, *Boundary behaviour of the complex Monge-Ampère equation*, Acta Math., 148(1982), 159-192

$$r \in C^{\infty}\left(\overline{\Omega}\right)$$

so derivatives of r stay bounded at $\partial\Omega$ (as well)

Why worry about boundary behaviour of $\mathcal{C}_{\epsilon}(\phi) = 0$?

Source of difficulties: choice of contact form on each $M_{\epsilon} \in V/\mathcal{F}$

Choice proposed by

A. Korányi & H.M. Reimann, *Contact transformations as limits of symplectomorphisms*, C.R. Acad. Sci. Paris, 318(1994), 1119-1124

$$\varphi(z) = -K(z,z)^{-1/(n+1)}, \quad \theta = \frac{i}{2}\big(\overline{\partial} - \partial\big)\varphi$$

is **natural** as it provides computational relation between contact geometry of M_ϵ and Kähler geometry of Ω with Bergman metric [in particular between $\partial\Omega$ and $(\Omega,\, g_\Omega)$]

$\Big[$that $\varphi(z) \to 0$ as $z \to \partial\Omega$ and $\nabla\varphi(z) \neq 0$ at $\partial\Omega$ follows from Fefferman's asymptotic expansion formula (for the Bergman kernel of a smoothly bounded strictly pseudoconvex domain $\Omega = \{\rho < 0\} \subset \mathbb{C}^n$)

$$K(z,z) = C_\Omega \big|\nabla\rho\big|^2 \cdot \det L_\rho(z) \cdot \big|\rho(z)\big|^{-(n+1)} + E(z,z)$$

$$\big|E(z,z)\big| \leq C'_\Omega \big|\rho(z)\big|^{-(n+1)+1/2} \big| \log|\rho(z)| \big|$$

$$E \in C^\infty\big(\overline{\Omega} \times \overline{\Omega} \setminus \Delta\big),$$

$$\Delta = \{(z,z) \;:\; z \in \partial\Omega\}, \quad L_\rho \;\; \text{Levi form of} \;\; \partial\Omega\big]$$

Yet if $\xi = \dfrac{1}{2}\big(N - i\,T\big)$ then

$$g(X,Y) = -\frac{n+1}{\varphi}\, g_\theta(X,Y), \;\; g(X,T) = 0, \;\; g(X,N) = 0,$$

$$g(T,\,N) = 0, \quad g(T,T) = g(N,N) = \frac{n+1}{\varphi\psi}\,,$$

$$X, Y \in H(\mathcal{F}), \quad \psi \equiv \frac{\varphi}{1 - r\varphi}$$

where g_θ **Webster metric** of the leaves [Riemannian bundle metric on $T(\mathcal{F})$ restricting to M_ϵ as the Webster metric of M_ϵ]

$\Longrightarrow$ g unbounded at $\partial\Omega$

$\Longrightarrow$ Levi-Civita connection ∇^g unbounded at $\partial\Omega$

$\Longrightarrow$ equation $-\Delta_g \phi^a + \left(\left\{ \begin{array}{c} a \\ bc \end{array} \right\} \circ \phi \right) \frac{\partial \phi^b}{\partial x^A} \frac{\partial \phi^c}{\partial x^B} g^{AB} = 0$

$$\left[\Delta_g u = - \sum_{A=1}^{2n} \left\{ E_A(E_A u) - \left(\nabla^g_{E_A} E_A \right)(u) \right\}, \;\; g(E_A \, , \, E_B) = \delta_{AB} \right]$$

makes no sense *a priori* at $\partial\Omega$ [i.e. as $\varphi \to 0$]

Solution: express ∇^g in terms of **Graham-Lee connection** ∇ of $(V,\ \mathcal{F})$ [a linear connection on V whose pointwise restriction to M_ϵ is the Tanaka-Webster connection of M_ϵ] and derivatives of r e.g. $\forall\ X, Y \in H(\mathcal{F})$

$$\nabla^g_X Y = \nabla_X Y + \left\{\psi\, g_\theta\big(\tau X\,,\ Y\big) + g_\theta\big(X\,,\ JY\big)\right\} T -$$

$$-\left\{g_\theta\big(X, Y\big) + \psi\, g_\theta\big(X,\ J\tau Y\big)\right\} N$$

$$\nabla^g_X T = \tau X - \frac{1}{\psi}\, JX - \frac{\psi}{2}\left\{X(r)\, T + (JX)(r)\, N\right\}$$

$$\vdots$$

$$\nabla^g_N N = -\frac{1}{2}\,\nabla^H r + \frac{\psi}{2}\left\{T(r)\, T - \left(N(r) + \frac{4}{\varphi^2} - \frac{2r}{\varphi}\right)\, N\right\}$$

[Given formulas: illustrative (haven't written $\nabla^g_X N$, $\nabla^g_N T$ and $\nabla^g_T T$)]

$\Big[$Precise description of ∇^g because of decompositions

$T(V) = T(\mathcal{F}) \oplus \mathbb{R}N$, $T(\mathcal{F}) = H(\mathcal{F}) \oplus \mathbb{R}T\Big]$

If $\{W_\alpha \ : \ 1 \le \alpha \le n-1\} \subset T_{1,0}(\mathcal{F})$, $g_\theta(W_\alpha\,,\ W_{\overline{\beta}}) = \delta_{\alpha\beta}$

$$E_\alpha \equiv \sqrt{-\frac{\varphi}{n+1}}\, W_\alpha\,, \quad E_n \equiv \sqrt{\frac{2\psi\varphi}{n+1}}\,\xi \Longrightarrow g\big(E_j\,,\ \overline{E}_k\big) = \delta_{jk}$$

For technical details on CR and pseudohermitian geometry may see

S. Dragomir & G. Tomassini, *Differential geometry and analysis on CR manifolds*, Progress in Mathematics, Vol. 246, Birkhäuser, Boston-Basel-Berlin, 2006.

Substitution into

$$\Delta_g = -\sum_{j=1}^{n} \left\{ E_j \overline{E}_j + \overline{E}_j E_j - \nabla^g_{E_j} E_{\bar{j}} - \nabla^g_{\overline{E}_j} E_j \right\}$$

$$(\Gamma^a_{bc} \circ \phi) \frac{\partial \phi^b}{\partial x^A} \frac{\partial \phi^c}{\partial x^B} g^{AB}$$

gives

- $$\Delta_g = -\frac{\varphi}{n+1} \Delta_b + \frac{2\varphi(n-1)}{n+1} N +$$

$$-\frac{\psi\varphi}{n+1} \{ N^2 + T^2 + \nabla^H r + 2rN \}$$

$\big[\Delta_b \equiv -\sum_{\alpha=1}^{n-1} \big(W_\alpha \overline{W}_\alpha + \overline{W}_\alpha W_\alpha - \nabla_{W_\alpha} \overline{W}_\alpha - \nabla_{\overline{W}_\alpha} W_\alpha\big)$

pointwise restriction of Δ_b to a leaf of $\mathcal{F}$: sublaplacian of that leaf$\big]$

- $$(\Gamma^a_{bc} \circ \phi) \frac{\partial \phi^b}{\partial x^A} \frac{\partial \phi^c}{\partial x^B} g^{AB} = 2 \sum_{j=1}^{n} (\Gamma^a_{bc} \circ \phi)\ E_j(\phi^b) \overline{E}_j(\phi^c)$$

Substitution into $\tau_g(\phi) = 0$ and simplification of a φ factor gives

$$-\Delta_b \phi^a + 2\sum_{\alpha=1}^{n-1} (\Gamma^a_{bc} \circ \phi)\ W_\alpha(\phi^b)\,\overline{W}_\alpha(\phi^c) + 2(n-1)(\phi^a) +$$

$$-\psi\big\{N^2 + T^2 + \nabla^H r + 2rN\big\}(\phi^a) +$$

$$-4\psi\,(\Gamma^a_{bc} \circ \phi)\ \xi(\phi^b)\,\xi(\phi^c) = 0$$

hence for $\varphi \to 0$

$$\boxed{N(\phi^a) = -\frac{1}{2(n-1)}\,\tau_b(f)^a}$$

4. Subelliptic harmonic maps

$$\tau_b(f) \in C^\infty\left(f^{-1}T(N)\right)$$

$$\tau_b(f)^a \equiv -\Delta_b f^a + 2\sum_{\alpha=1}^{n-1} (\Gamma^a_{bc} \circ f)\ W_\alpha(f^b)\,\overline{W}_\alpha(f^c)$$

J. Jost & C-J. Xu, *Subelliptic harmonic maps*, Trans. Amer. Math. Soc., 350(1998), 4633-4649.

E. Barletta & S. Dragomir & H. Urakawa, *Pseudoharmonic maps from a nondegenerate CR manifold into a Riemannian manifold*, Indiana Univ. Math. J., 50(2001), 719-746.

$$E_b(f) = \frac{1}{2} \int_{\partial\Omega} \mathrm{trace}_{G_\theta} \left(\Pi_H f^* h \right) \theta \wedge (d\theta)^{n-1}$$

$$\frac{d}{dt} \left\{ E_b\left(f_t\right) \right\}_{t=0} = - \int_{\partial\Omega} h^f \left(V, \tau_b(f) \right) \theta \wedge (d\theta)^{n-1}$$

[first variation formula]

$$\{f_t\}_{|t|<\epsilon} \subset C^\infty\left(\partial\Omega, N\right), \quad f_0 = f,$$

$$V = \left(\frac{\partial f_t}{\partial t} \right)_{t=0} \in C^\infty\left(f^{-1} TN\right)$$

f **subelliptic harmonic** $\iff \tau_b(f) = 0$

Boundary values $f : \partial\Omega \to N$ of Bergman harmonic map $\phi : \Omega \to N$ are subelliptic harmonic, provided normal derivatives vanish.

Deriving conditions for local C^∞ regularity (**compatibility operators** and **compatibility equations**) for the Siegel domain:

Confined to case $\Omega = \mathbb{S}_n$, $N = (\mathbb{R},\ dt^2)$, $u = \phi^1$, where transverse curvature is $r = 0$:

$$-\Delta_b u + 2(n-1)N(u) - \varphi\big(N^2 + T^2\big)u = 0 \quad (1_\varphi)$$

$$\varphi \to 0 \Longrightarrow N(u) = \frac{1}{2(n-1)}\,\Delta_b f \quad (1_0)$$

Apply N to (1_φ) and use $N(\varphi) = 2$ and $[\Delta_b\,,\,N] = 0$, $[N, T] = 0$

$$\mathcal{L}_0 N(u) + (n-2)N^2(u) - T^2(u) - \frac{\varphi}{2}\big(N^3 + T^2 N\big)(u) = 0 \quad (2_\varphi)$$

$$\mathcal{L}_0 N(u) + (n-2)N^2(u) - T^2(f) = 0 \quad (2_0)$$

where $\mathcal{L}_0 = -(1/2)\,\Delta_b$

Iteration:

$$\mathcal{L}_0 N^k(u) + (n-k-1)N^{k+1}(u) - k\,T^2 N^{k-1}(u) +$$

$$-\frac{\varphi}{2}\big(N^{k+2} + T^2 N^k\big)(u) = 0 \quad \big((k+1)_\varphi\big)$$

If $u \in C^\infty\big(\mathcal{D} \cap \overline{\mathbb{S}}_n\big)$ and $\Delta_g u = 0$ in $\mathbb{S}_n$ and $u = f$ on $\partial\mathbb{S}_n$ then

$$N^k(u) = (-1)^k \,\frac{(n-k-1)!}{(n-1)!}\, Q_k(f), \quad 0 \le k \le n-1$$

$Q_n(f) = 0$ on $\mathcal{D} \cap \partial\mathbb{S}_n$ (compatibility condition)

Tangential operators Q_k determined by recursive relation

$$Q_{k+1} = \mathcal{L}_0 Q_k + k(n-k)T^2 Q_{k-1}$$

$$Q_0 = I, \quad Q_1 = \mathcal{L}_0 \,.$$

Compatibility operator Q_n identified as a certain composition of **Folland-Stein operators** $\mathcal{L}_\alpha = \mathcal{L}_0 + \alpha T$, $\alpha \in \mathbb{C}$.

Aside from that: explicit formula for Q_k, $2 \le k \le n-2$, **unknown**

Extension of the result to Bergman harmonic maps $\phi : \mathbb{S}_n \to N$ **unknown**

Complete characterization of boundary data $f : \partial\Omega \to N$ for arbitrary strictly pseudoconvex domains $\Omega \subset \mathbb{C}^n$ **unknown** (even in the scalar case $N = \mathbb{R}$)

Essential assumption in previous considerations:

$$\lim_{\Omega \ni z \to z_0} \Gamma^a_{bc}\big(\phi(z)\big) = \Gamma^a_{bc}(f(z_0)), \quad z_0 \in \partial\Omega.$$

Next considered case: $\phi : \mathbb{B}^n \to \mathbb{B}^N$ proper $\Longrightarrow$

If $|z| \to 1$ then $|\phi(z)| \to 1$

[and Christoffel symbols of Bergman metric (on $\mathbb{B}^N$) are unbounded at the boundary]

5. Proper holomorphic maps of balls

Proper holomorphic maps of balls $\phi : \mathbb{B}^n \to \mathbb{B}^{n+1}$ $(n \geq 3)$ which are C^3 up to the boundary are linear fractional.

[S.M. Webster, *The rigidity of C-R hypersurfaces in a sphere*, Indiana Univ. Math. J., (3)28(1979), 405-416]

Same statement **false** for $n = 2$ e.g. $\phi(z, w) = \left(z^2, \sqrt{2}\,zw, w^2\right)$

[H. Alexander, *Proper holomorphic maps in* $\mathbb{C}^n$, Indiana Univ. Math. J., 26(1977), 137-146]

Proper holomorphic maps $\phi : \mathbb{B}^2 \to \mathbb{B}^3$ which are C^3 up to the boundary where classified by J. Faran

[J. Faran, *Maps from the two-ball to the three-ball*, Invent. Math., 68(1982), 441-475]

$\phi,\, \psi : \mathbb{B}^2 \to \mathbb{B}^3$ are **spherically equivalent** if

$$\exists\ \xi \in \mathrm{Hol}(\mathbb{B}^2), \quad \exists\ \zeta \in \mathrm{Hol}(\mathbb{B}^3)\ :\ \psi = \zeta \circ \phi \circ \xi^{-1}$$

$O(2,3)$ set of proper holomorphic maps $\mathbb{B}^2 \to \mathbb{B}^3$

$P(2,3) = \{\phi \in O(2,3)$ that extend holomorphically past $\partial\, \mathbb{B}^2\}$

$P^*(2,3) =$ quotient $P(2,3)/$(spherical equivalence)

Faran's classification

$$P^*(2,3) = \{\mathbb{F},\ \mathbb{A}_0\,,\ \mathbb{A}_1\,,\ \mathbb{I}\}$$

$$\phi_{\mathbb{F}} \in \mathbb{F}, \quad \phi_{\mathbb{A}_t} \in \mathbb{A}_t\,, \quad \phi_{\mathbb{I}} \in \mathbb{I}, \quad t \in \{0,1\}$$

$\phi_{\mathbb{F}}(z,w) = \left(z^3\,,\ \sqrt{3}\,zw\,,\ w^3\right)$, $\phi_{\mathbb{A}_0}(z,w) = \left(z^2\,,\ \sqrt{2}\,zw\,,\ w^2\right)$

$\phi_{\mathbb{A}_1}(z,w) = \left(z,\ zw,\ w^2\right)$, $\phi_{\mathbb{I}}(z,w) = \left(0,\ z,\ w\right)$

$\phi_{\mathbb{F}}$ **Faran's map**, $\phi_{\mathbb{A}_0}$ **Alexander's map**

THEOREM 1 *The maps* $U \circ \phi_{\mathbb{A}_0} \circ u^{-1}$ *and* $U \circ \phi_{\mathbb{I}} \circ u^{-1}$, $u \in \mathrm{U}(2)$, $U \in \mathrm{U}(3)$, *are all proper holomorphic maps* $\mathbb{B}^2 \to \mathbb{B}^3$ *admitting a* C^3 *extension to* $\partial\,\mathbb{B}^2 = S^3$ *whose boundary values* $S^3 \to S^5$ *are subelliptic harmonic maps of the pseudohermitian manifold* $(S^3\,,\,\theta_0)$ *into the Riemannian manifold* $\left(S^5\,,\,g_{\Theta_0}\right)$ *with*

$$\theta_0 = \frac{i}{2}\left(\overline{\partial} - \partial\right)\left(|z|^2 + |w|^2\right),\ \Theta_0 = \frac{i}{2}\left(\overline{\partial} - \partial\right)|Z|^2.$$

Every subelliptic harmonic map in the above list is unstable.

6. CR degree

$$f : S^{2n+1} \to S^{2N+1} \text{ CR map } (N = n + k,\ k \geq 1)$$

$$\deg(f) = \frac{1}{\omega_n} \int_{S^{2n+1}} \lambda\big(f;\, \theta,\, \Theta_0\big)^{n+1} \theta \wedge (d\theta)^n$$

(**CR degree** of f)

θ positively oriented contact form on S^{2n+1}

$$\omega_n = \mathrm{Vol}\big(S^{2n+1},\, \theta_0 \wedge d\theta_0\big)$$

$$f^*\Theta_0 = \lambda\big(f;\, \theta,\, \Theta_0\big)\, \theta$$

[$\lambda\big(f;\, \theta,\, \Theta_0\big)$ **dilation** of f as a map of $(S^3,\, \theta)$ into $(S^5,\, \Theta_0)$]

CR degree is a (numerical) CR invariant of S^{2n+1}.

$\|u\|_p = \left(\int_{S^{2n+1}} |u|^p \, \theta \wedge (d\theta)^n\right)^{1/p}$

$u \in L^p(S^{2n+1})$, $1 \leq p < \infty$

$\|\lambda\big(f;\, \theta,\, \Theta_0\big)\|_p$ CR invariant of $S^{2n+1} \Longleftrightarrow p = n+1$

CR degree of CR map $\equiv$ topological degree

in equidimensional case ($k = 0$)

$\phi = \big(\phi_1\,,\,\cdots\,,\,\phi_N\big) : \mathbb{B}^n \to \mathbb{B}^N$ polynomial holomorphic map

$p(\phi) =$ maximum of degrees of ϕ_j ($1 \leq j \leq N$)

THEOREM 2 i) *For every* $\mathcal{C} \in P^*(2,3)$ *there is a constant* $0 < \Lambda_{\mathcal{C}} \leq 1$ *such that*

$$\deg\big(f_{\mathcal{C}}\big) = \Lambda_{\mathcal{C}}\, p\big(\phi_{\mathcal{C}}\big)^2$$

with $\Lambda_{\mathcal{C}} = 1$ *when* $f_{\mathcal{C}}$ $[f_{\mathcal{C}}$ *boundary values of* $\phi_{\mathcal{C}}]$ *is subelliptic harmonic* and $\Lambda_{\mathcal{C}} < 1$ otherwise.

ii) *If* $\mathcal{C} \in P^*(2,3)$ and $\phi \in \mathcal{C}$ *then*

$$\deg(f) \approx \deg\big(f_{\mathcal{C}}\big)$$

$[f$ *boundary values of* $\phi]$. *Precisely*

$$\frac{1}{C_{a,A}} \deg\big(f_{\mathcal{C}}\big) \leq \deg(f) \leq C_{a,A} \deg\big(f_{\mathcal{C}}\big)$$

$$C_{a,A} = \left(\frac{1+|a|}{1-|a|}\right)^4 \left(\frac{1+|A|}{1-|A|}\right)^2$$

$$\phi = \zeta \circ \phi_{\mathcal{C}} \circ \xi^{-1}, \quad \zeta = U \circ \varphi_A\,, \quad \xi = u \circ \varphi_a\,,$$

$$U \in \mathrm{U}(3), \quad u \in \mathrm{U}(2), \quad A \in \mathbb{B}^3\,, \quad a \in \mathbb{B}^2\,.$$

Here

$$\mathrm{Hol}(\mathbb{B}^N) = \left\{U \circ \varphi_A \;:\; U \in \mathrm{U}(N),\;\; A \in \mathbb{B}^N\right\}$$

$$\varphi_A(Z) = \frac{A - L_A(Z)}{1 - \langle Z, A\rangle}\,, \quad L_A(Z) = \frac{\langle Z, A\rangle}{s_A + 1}\,A + s_A\,Z,$$

$$s_A = \left(1 - |A|^2\right)^{1/2}, \quad Z \in \mathbb{B}^N\,.$$

Previous result: tribute to J. d'Angelo

J. d'Angelo, *Polynomial proper maps between balls*, Duke Math. J., 57(1988), 211-219.

J. d'Angelo, *Proper holomorphic mappings between balls of different dimensions*, Michigan Math. J., 35(1988), 83-90; *Polynomial proper holomorphic mappings between balls*, II, Michigan Math. J., 38(1991), 53-65.

7. A Solomon type result

For any harmonic map $f : M \to S^m$ of a compact Riemannian manifold M into a sphere S^m, which omits a codimension 2 totally geodesic submanifold $\Sigma \subset S^m$, the map $\phi : M \to S^m \setminus \Sigma$ is not null-homotopic.

[B. Solomon, *Harmonic maps to spheres*, J. Differential Geometry, 21(1985), 151-162]

THEOREM 3 *Boundary values*
$f_{\mathbb{A}_0} : M = \{(z, w) \in S^3 \ : \ \mathrm{Re}(w) > 0\} \to S^5$ *of Alexander's map* $\phi_{\mathbb{A}_0}$ *is a subelliptic harmonic map omitting* S^3 *and* $f_{\mathbb{A}_0} : M \to S^5 \setminus S^3$ *links* S^3.

THANK YOU!