Willmore stability and conformal rigidity of minimal surfaces in $\mathbb{S}^n$

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A minimal surface M in the round sphere $\mathbb{S}^n$ is critical for area, as well as for the Willmore bending energy $W = \iint(1+H^2)da$. Willmore stability of M is equivalent to a gap between -2 and 0 in its area-Jacobi operator spectrum. We show the W-stability of M persists in all higher dimensional spheres if and only if the Laplacian of M has first eigenvalue 2. The square Clifford 2-torus in $\mathbb{S}^3$ and the equilateral minimal 2-torus in $\mathbb{S}^5$ have this spectral gap, and each is embedded by first eigenfunctions, so both are "persistently" W-stable. On the other hand, we discovered the equilateral torus has nontrivial third variation (with vanishing second variation) of W, and thus is not a W-minimizer (though it is the W-minimizer if we fix the conformal type!) This is evidence the Willmore Conjecture holds in every codimension. Another result concerns higher genus minimal surfaces (such as those constructed by Lawson and those by Karcher-Pinkall-Sterling) in $\mathbb{S}^3$ which Choe-Soret showed are embedded by first eigenfunctions: we show their first eigenspaces are always 4-dimensional, and that this implies each is (up to Möbius transformations of $\mathbb{S}^n$) the unique W-minimizer in its conformal class. (Some analogous results hold for free boundary minimal surfaces in the unit ball $\mathbb{B}^n$....). This is joint work with Peng Wang.

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