Databases and Despriptive Complexity — Part 1:

Using Logical Formulas to Describe Computations

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Overview

Descriptive Complexity

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Datalog is highly expressive

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Seminal Results in Descriptive Complexity (1/2)

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Fagin's Theorem: NP is captured by ESO on graphs.

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Later on in this talk, we will prove a variant of the Immerman-Vardi Theorem for Datalog rather than LFP.

The Quest for a Logic Capturing PTIME (1/3)

Immerman-Vardi Theorem: PTIME is captured by LFP on ordered graphs.

Major open research question: [Chandra & Harel 1982; Gurevich 1988]
Is there a logic L (instead of LFP) such that the Immerman-Vardi theorem can be generalized to arbitrary graphs? I.e.:

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Such a logic L would be a great query language: It is guaranteed that

- all queries described by a user can be evaluated in PTIME, and
- all tractable queries can be formulated in the language.

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In order to really get this, the notions of "logic" and "capturing PTIME" have to be defined very carefully:

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- An abstract logic L consists of
  - a set of L[σ]-sentences for each signature σ, and
  - a mapping that associates a property p_φ of σ-structures with each L[σ]-sentence φ.

For every σ-structure G we write $G \models_{\mathrm{L}} \varphi \quad :\Longleftrightarrow \quad G \in p_\varphi$.

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- An abstract logic L captures PTIME on graphs if the following 3 conditions are satisfied for the signature $\sigma = \{E\}$:
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     $\mathbb{B}$ is the query optimizer, which produces the query evaluation plan $\mathbb{A}_\varphi$
  3. For every PTIME-algorithm $\mathbb{A}$ that decides a graph property, there is a sentence $\varphi \in$ L[σ] such that for every graph G we have: $G \models_L \varphi \iff$ $\mathbb{A}$ accepts G. All PTIME graph properties can be expressed in L[σ].

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- But we don't have an algorithm $\mathbb{B}$ that associates with every $n \in \mathbb{N}$ a PTIME algorithm $\mathbb{A}_n$ that decides p_n. I.e., condition 2 is not met.

Overview

Descriptive Complexity

Datalog is poorly expressive

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Datalog queries are closed under homomorphisms

For simplicity, throughout this talk Datalog queries don't contain any constants.

[1] Recall: *adom*(**I**) is the active domain, i.e., the set of all elements of **dom** occurring in **I**.

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Definition: A query Q of schema $\mathbf{S}$ is closed under homomorphisms if for all DBs $\mathbf{I}$ and $\mathbf{J}$ and all mappings $h : \mathbf{dom} \to \mathbf{dom}$, the following is true

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I.e.: Datalog cannot even count to two!

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Represent words w of alphabet Σ by databases $\mathbf{I}_w$ of schema $\mathbf{S}_\Sigma$ consisting of a binary relation SUCC and unary relations MIN, MAX and P_α for every $\alpha \in \Sigma$.

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Definition: For a word $w = w_0 \cdots w_{n-1}$ with $w_i \in \Sigma$, let $\mathbf{I}_w$ be the database of schema $\mathbf{S}_\Sigma$ with

- $\mathbf{I}_w(\text{SUCC}) = \{(i, i{+}1) : 0 \leqslant i < n{-}1\}$, $\mathbf{I}_w(\text{MIN}) = \{0\}$, $\mathbf{I}_w(\text{MAX}) = \{n{-}1\}$,
- $\mathbf{I}_w(P_\alpha) = \{i \in \{0, \ldots, n{-}1\} : w_i = \alpha\}$ for each letter $\alpha \in \Sigma$.

Example: $\Sigma = \{a, b, c\}$, $w = aaba$, $\rightsquigarrow$ $\mathbf{I}_w$:

SUCC:

0	1
1	2
2	3

MIN:

0

MAX:

3

P_a:

0
1
3

P_b:

2

P_c:

Datalog can simulate runs of Turing machines (2/2)

Simulation Lemma: For every deterministic Turing machine M with input alphabet Σ and every integer $k \geqslant 1$, there is a Datalog program $P_{M,k}$ with edb-predicates $\mathbf{S}_\Sigma$ and a 0-ary idb-predicate GOAL, such that the following is true for the Datalog query $Q_{M,k} := (P_{M,k}, \text{GOAL})$ and for every non-empty word $w \in \Sigma^*$:

$Q_{M,k}(\mathbf{I}_w) = $ "yes" $\iff$ upon input w, M stops in an accepting state after at most $|w|^k - 1$ steps.

[2] This is the variant of the Immerman-Vardi Theorem promised at the beginning of the talk.

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Furthermore, $Q_{M,k}$ and $\mathbf{I}_w$ can be constructed in time polynomial in k, i.e., polynomial in $|u|$. □

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Proof of the Simulation Lemma (1/5)

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$$\text{TAPE}_\Box(\overline{x}, \overline{y}) \leftarrow \text{MIN}(x_{k-1}), \ldots, \text{MIN}(x_0), \text{NOTMIN}(y_i)$$

Proof of the Simulation Lemma (2/5)

Start with $P_{M,k} := \emptyset$ and add rules as follows.

Step 1: Add rules to achieve the intended meaning at time 0.

- At time 0, head is at tape position 0:

$$\text{HEAD}(\overline{x},\overline{y}) \leftarrow \text{MIN}(x_{k-1}),\ldots,\text{MIN}(x_0),\ \text{MIN}(y_{k-1}),\ldots,\text{MIN}(y_0)$$

- At time 0, M is in the starting state q_0:

$$\text{STATE}_{q_0}(\overline{x}) \leftarrow \text{MIN}(x_{k-1}),\ldots,\text{MIN}(x_0)$$

- At time 0, tape positions $0,\ldots,n-1$ carry the input string w:

$$\text{TAPE}_a(\overline{x},\overline{y}) \leftarrow \text{MIN}(x_{k-1}),\ldots,\text{MIN}(x_0),\ \text{MIN}(y_{k-1}),\ldots,\text{MIN}(y_1),\text{P}_a(y_0)$$

  Add this rule for every letter $a \in \Sigma$.

- At time 0, tape positions $n,\ldots,n^k-1$ carry the blank symbol $\Box$:
  For each $i \in \{1,\ldots,k-1\}$ add the rule

$$\text{TAPE}_{\Box}(\overline{x},\overline{y}) \leftarrow \text{MIN}(x_{k-1}),\ldots,\text{MIN}(x_0),\text{NOTMIN}(y_i)$$

  And add the rule

$$\text{NOTMIN}(z) \leftarrow \text{SUCC}(z',z)$$

Proof of the Simulation Lemma (3/5)

Step 2: Add rules so that if intended meaning is achieved at time t, then also at $t+1$.

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- Auxiliary rules for reasoning about "$+1$": For each $\ell \in \{1, \ldots, k\}$, use a 2ℓ-ary predicate SUCC_ℓ to represent the "successor on ℓ-tuples".

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$$\text{SUCC}_{\ell+1}(x_\ell,x_{\ell-1},\ldots,x_0,\ y_\ell,y_{\ell-1},\ldots,y_0) \leftarrow \text{MAX}(x_{\ell-1}),\ldots,\text{MAX}(x_0),\text{SUCC}(x_\ell,y_\ell),\text{MIN}(y_{\ell-1}),\ldots,\text{MIN}(y_0)$$

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  and the rule $\text{ADOM}(z) \leftarrow \text{X}(z)$ for each unary edb-predicate X.

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And all other tape positions carry the same letter at time $t+1$ as at time t:
For every tape symbol b add the rule

$$\text{TAPE}_b(\overline{x}', \overline{y}') \leftarrow \text{TAPE}_b(\overline{x}, \overline{y}'),\ \text{SUCC}_k(\overline{x}, \overline{x}'),\ \text{STATE}_q(\overline{x}),\ \text{HEAD}(\overline{x}, \overline{y}),\ \text{NEQ}(\overline{y}, \overline{y}')$$

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Add similar rules for representing the head movement of M:
$m \in \{0, 1, -1\}$ indicates whether the head stays or moves one position to the right or the left, respectively.

Proof of the Simulation Lemma (5/5)

Recall that we consider M's transition $(q', a', m) := \delta(q, a)$.

- If $m = 0$, we add to $P_{M,k}$ the rule

$$\text{HEAD}(\overline{x}', \overline{y}) \leftarrow \text{SUCC}_k(\overline{x}, \overline{x}'),\ \text{STATE}_q(\overline{x}),\ \text{HEAD}(\overline{x}, \overline{y}),\ \text{TAPE}_a(\overline{x}, \overline{y})$$

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To ensure that the Datalog query outputs the correct result, we add the rule

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This finally completes the construction of the Datalog program $P_{M,k}$.
It is straightforward to verify that this proves the Simulation Lemma. □

Datalog can simulate runs of Turing machines (2/2)

Simulation Lemma: For every deterministic Turing machine M with input alphabet Σ and every integer $k \geqslant 1$, there is a Datalog program $P_{M,k}$ with edb-predicates $\mathbf{S}_\Sigma$ and a 0-ary idb-predicate GOAL, such that the following is true for the Datalog query $Q_{M,k} := (P_{M,k}, \text{GOAL})$ and for every non-empty word $w \in \Sigma^*$:

$Q_{M,k}(\mathbf{I}_w) = \text{“yes”} \iff$ upon input w, M stops in an accepting state after at most $|w|^k - 1$ steps.

Furthermore, upon input of M and k, the query $Q_{M,k}$ can be constructed in time polynomial in k and the size of M. Moreover, there is a log-space algorithm which, upon input of a string w, constructs the database $\mathbf{I}_w$.

Easy consequences:

(1) Datalog captures PTIME on database-respresentations of strings.[2]

(2) Datalog query evaluation is EXPTIME-complete w.r.t. combined complexity.

(3) Datalog query evaluation is PTIME-complete w.r.t. data complexity.

(4) The Boundedness Problem for Datalog is undecidable.

[2] This is the variant of the Immerman-Vardi Theorem promised at the beginning of the talk.

Databases and Despriptive Complexity — Part 2:

A Toolkit for Proving Limitations of the Expressive Power of Logics

Nicole Schweikardt

Humboldt-Universität zu Berlin

EPIT 2019 — Spring School on Theoretical Computer Science:
Databases, logic and automata

Luminy, 11 April 2019

In this talk

- Consider finite directed graphs $G = (V^G, E^G)$.

  Sometimes, nodes are additionally labeled (i.e., colored) by a symbol from a finite alphabet Σ.

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  if $G \cong H$, then $\quad G$ has property $p \iff H$ has property p

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- q is a k-ary graph query, if the following is true:

  if $\pi : G \cong H$, then for all $a_1, \ldots, a_k \in V^G$,

  $$(a_1, \ldots, a_k) \in q(G) \iff (\pi(a_1), \ldots, \pi(a_k)) \in q(H)$$

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- I.e., graph properties and queries are closed under isomorphisms.

Logics expressing graph properties and queries

Classical logics like, e.g.

- FO (first-order logic: Boolean combinations + quantification over nodes)

express graph properties and queries in a straightforward way.

Example:

- $q(G) := \{\ x \in V^G\ :\ x \text{ lies on a triangle}\ \}$ is expressed in FO via

$$\varphi(x) \ := \ \exists y\, \exists z\, \big(\ E(x,y) \ \wedge\ E(y,z)\ \wedge\ E(z,x)\ \big)$$

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- FO (first-order logic: Boolean combinations + quantification over nodes)
- EMSO (existential monadic second-order logic: FO + existential quantification over sets of nodes)

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Example:

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$$\varphi(x) \;:=\; \exists y \, \exists z \, \big(\, E(x,y) \,\wedge\, E(y,z) \,\wedge\, E(z,x) \, \big)$$

- $p \;=\; \{ \, G \; : \; G \text{ is 3-colorable} \, \}$ is expressed in EMSO via

$$\exists R \, \exists B \, \exists G \Big(\; \forall x \, \big(R(x) \vee B(x) \vee G(x)\big) \;\; \wedge \;\; \forall x \, \forall y \, \big(\, E(x,y) \rightarrow \; \neg \big((R(x) \wedge R(y)) \,\vee\, (B(x) \wedge B(y)) \,\vee\, (G(x) \wedge G(y))\big) \, \big) \Big)$$

Question

How can we prove that

certain properties or queries

are NOT expressible in a particular logic?

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Note: There are properties with $\mu(p) \in \{0, 1\}$ which cannot be expressed in FO.
Example: Connectivity.

Overview

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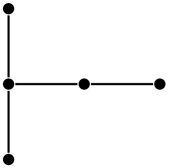

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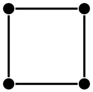

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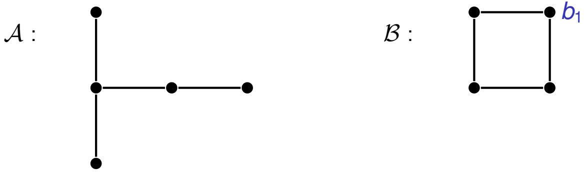

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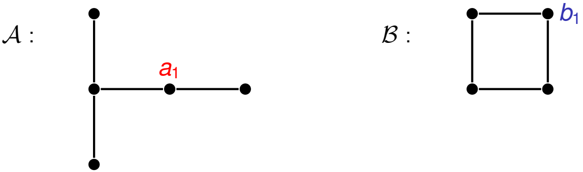

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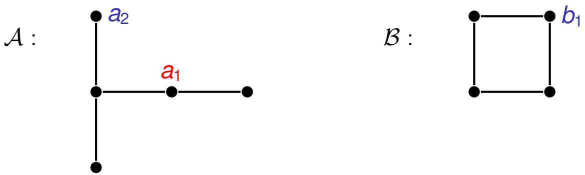

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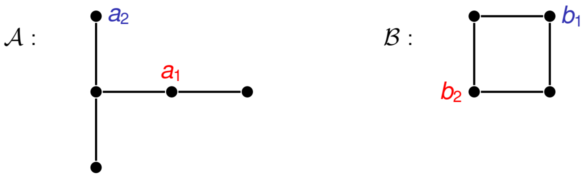

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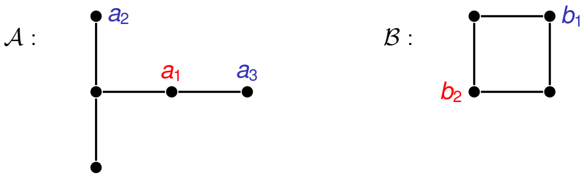

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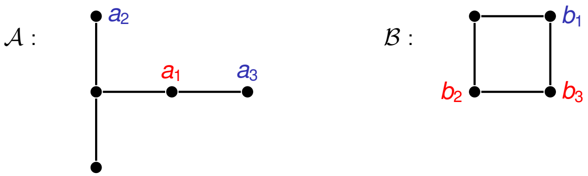

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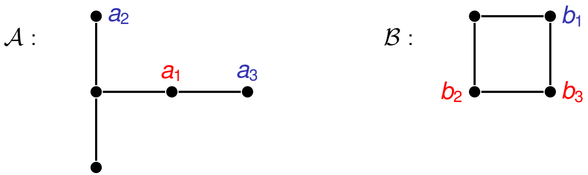

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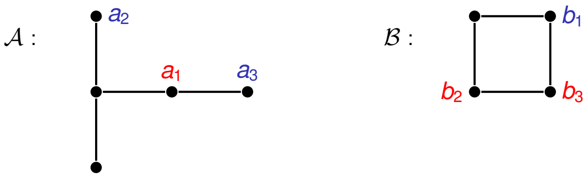

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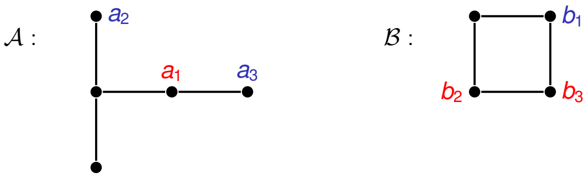

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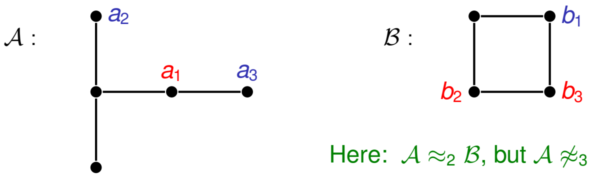

Here: $\mathcal{A} \approx_2 \mathcal{B}$, but $\mathcal{A} \not\approx_3 \mathcal{B}$.

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Note: Finding winning strategies for Duplicator often requires highly non-trivial combinatorial arguments.

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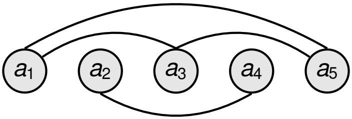

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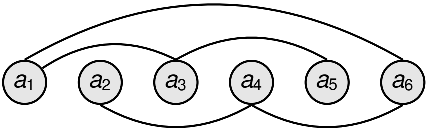


Overview

Introduction

Zero-One Laws

Ehrenfeucht-Fraïssé games

Logical Reductions

Locality Results

Reductions to known results in circuit complexity

The "Algebraic" Approach

Final Remarks

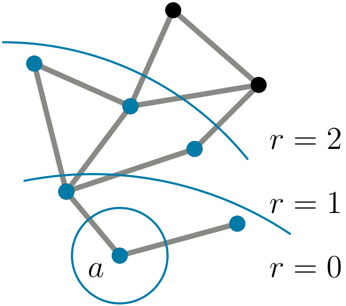
Neighborhoods

Graph $G = (V, E)$

Distance *dist*(u, v) : length of a shortest path between u, v in G.

Shell $S_r(a)$ of nodes at distance exactly r from a.

Ball $N_r(a)$ of radius r at a in G.



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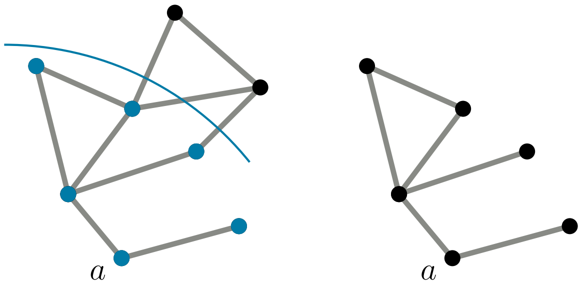

Gaifman-local queries

- For a list $a = a_1, \ldots, a_k$ of nodes, $N_r^G(a) = N_r^G(a_1) \cup \cdots \cup N_r^G(a_k)$.
- The r-neighborhood $\mathcal{N}_r^G(a)$ is the structure $(G_{|N_r^G(a)}, a)$ consisting of the induced subgraph of G on $N_r^G(a)$, together with the distinguished nodes a.

Definition: Let q be a k-ary graph query. Let $f : \mathbb{N} \to \mathbb{N}$.
q is called $f(n)$-local if there is an n_0 such that for every $n \geqslant n_0$ and every graph G with $|V^G| = n$, the following is true for all k-tuples a and b of nodes:

$$\text{if} \quad \mathcal{N}_{f(n)}^G(a) \cong \mathcal{N}_{f(n)}^G(b) \quad \text{then} \quad a \in q(G) \iff b \in q(G).$$

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Theorem:

- For every graph query q that is FO-definable, there is a constant c such that q is c-local.

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- For every graph query q that is FO-definable on graphs with arbitrary numerical predicates (for short: q is definable in Arb-invariant FO), there is a constant c such that q is $(\log n)^c$-local.
  (Anderson, van Melkebeek, S., Segoufin '11)

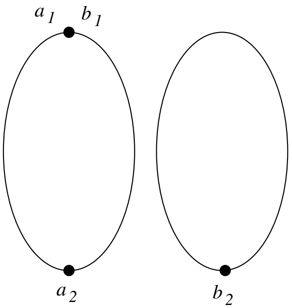
Use locality for proving non-expressibility

Example: The reachability query

$$\text{REACH}(G) := \{ (a_1, a_2) : \text{there is a directed path from } a_1 \text{ to } a_2 \text{ in } G \}$$

is not $\frac{n}{5}$-local an thus cannot be expressed in Arb-invariant FO.

Proof: Consider the graph G:



Use locality for proving non-expressibility

Similarly, one obtains that the following queries are not definable in Arb-invariant FO:

- Does node x lie on a cycle?
- Does node x belong to a connected component that is acyclic?
- Is node x reachable from a node that belongs to a triangle?
- Do nodes x and y have the same distance to node z?

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Idea: Use known lower bounds in circuit complexity!

- Let q be expressible by an Arb-invariant FO formula.
- Then, q can be computed by an AC^0 circuit family $\mathcal{C}$ (Immerman '87).

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- Assume that q is *not* $(\log n)^c$-local (for any $c \in \mathbb{N}$), and modify $\mathcal{C}$ to obtain an AC^0 circuit family computing

$$\text{PARITY} := \{w \in \{0,1\}^* : |w|_1 \text{ is even}\}.$$

- This contradicts known lower bounds in circuit complexity theory (Håstad'86).

Proof of Gaifman-locality theorem (2/5)

How to compute a graph query $q(x)$ by an AC^0 circuit family $\mathcal{C}$?

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- Let $Rep(G, a)$ be the set of all bitstrings $\beta(G)\beta(a)$, corresponding to all adjacency matrices of G (i.e., all ways of embedding V in $\{1, \ldots, |V|\}$). Thus, $Rep(G, a)$ is the set of all bitstrings representing (G, a).

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- A unary graph query $q(x)$ is computed by a circuit family $\mathcal{C} = (C_n)_{n\in\mathbb{N}}$ iff the following is true:
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- *Known:* A unary graph query $q(x)$ is definable in Arb-invariant FO $\iff$ it is computed by a circuit family of constant depth and polynomial size.
  (implicit in Immerman'87)

Proof of Gaifman-locality theorem (3/5)

Let $q(x)$ be a unary graph query expressible in Arb-invariant FO.

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Theorem: *(Håstad '86)*

There exist $\ell, m_0 > 0$ such that for all $m \geqslant m_0$, no circuit of depth d and size $2^{\ell \cdot m^{1/(d-1)}}$ computes parity on m bits.

Contradiction for $c = 2d$, since $2^{\ell \cdot m^{1/(d-1)}} > 2^{\ell \cdot (\log n)^2} = n^{\ell \log n} > p(n)$. □

Proof of Gaifman-locality theorem (4/5)

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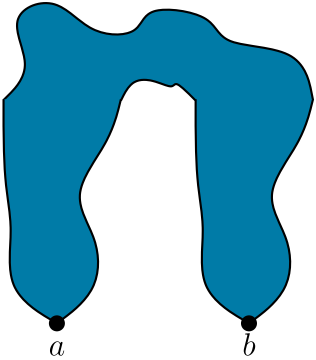

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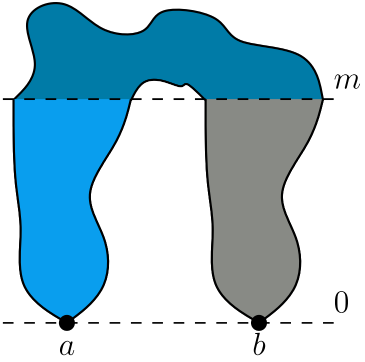

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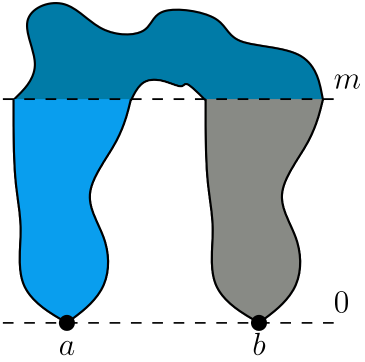

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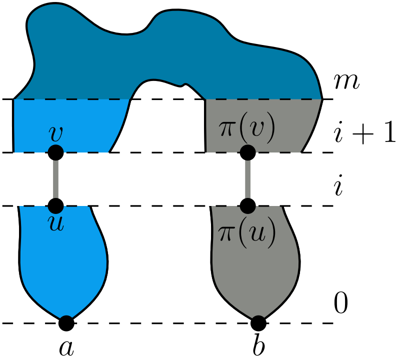

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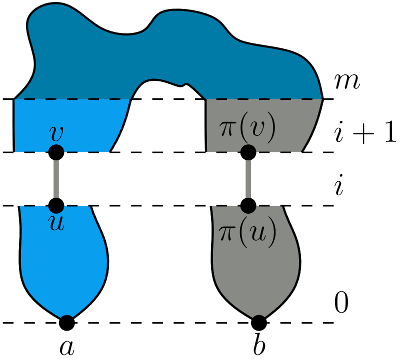

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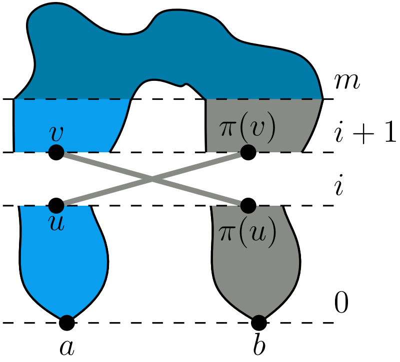

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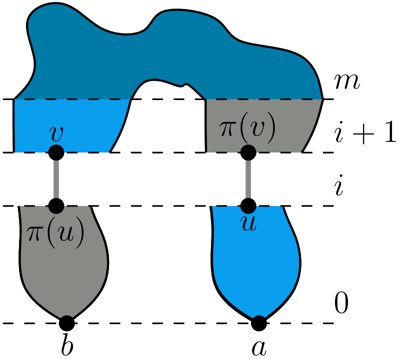

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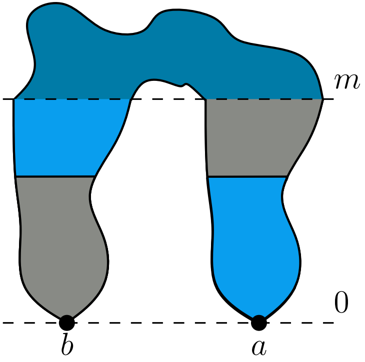

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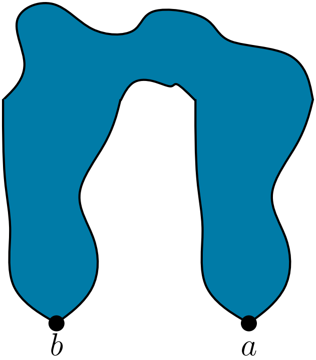

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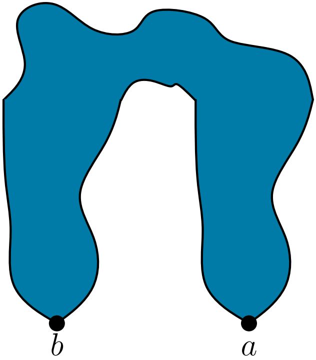

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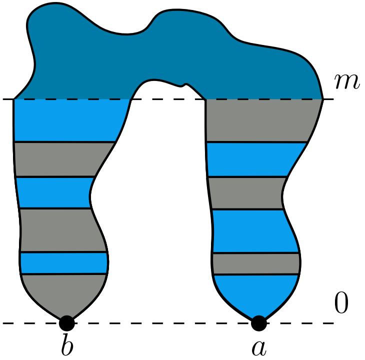

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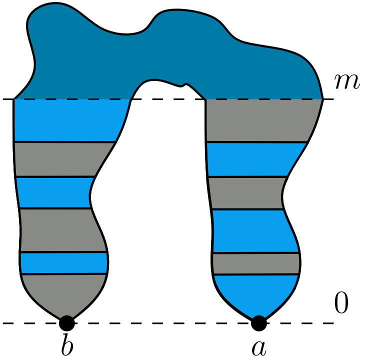

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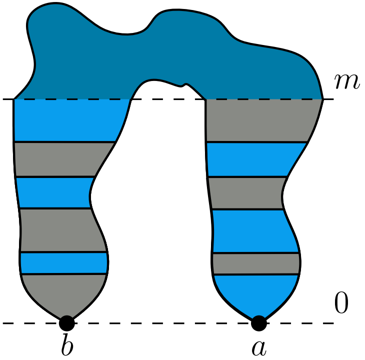
Circuit C distinguishes these cases.



Proof of Gaifman-locality theorem (5/5)

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How to obtain $\tilde{C}$ from C?

- Consider C for a fixed input string $\gamma \in Rep(G, a)$.
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- For all $i < m$ and all $u \in S_i(a)$, $v \in S_{i+1}(a)$ consider the potential edges $e = \{u, v\}$, $e' = \{\pi(u), \pi(v)\}$, $\tilde{e} = \{u, \pi(v)\}$, $\tilde{e}' = \{\pi(u), v\}$.

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- Consider C for a fixed input string $\gamma \in Rep(G, a)$.
- Fix all input bits (as in γ) that do *not* correspond to potential edges between the shells S_i and S_{i+1}, for $i < m$.
- For all $i < m$ and all $u \in S_i(a)$, $v \in S_{i+1}(a)$ consider the potential edges $e = \{u, v\}$, $e' = \{\pi(u), \pi(v)\}$, $\tilde{e} = \{u, \pi(v)\}$, $\tilde{e}' = \{\pi(u), v\}$.
- Replace input gates of C as follows:

$$e \text{ by } (e \wedge \neg w_i) \qquad e' \text{ by } (e' \wedge \neg w_i)$$
$$\tilde{e} \text{ by } (e \wedge w_i) \qquad \tilde{e}' \text{ by } (e' \wedge w_i)$$

Proof of Gaifman-locality theorem (5/5)

Key Lemma:
Let $m \in \mathbb{N}$, $G = (V, E)$, $a, b \in V$ such that $\mathcal{N}_m^G(a) \cong \mathcal{N}_m^G(b)$ and $dist(a, b) > 2m$.
Let circuit C accept all strings in $Rep(G, a)$ and reject all strings in $Rep(G, b)$.
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- This yields a circuit $\tilde{C}$ of the same size and depth as C which, on input $w \in \{0, 1\}^m$ does the same as C on input (G_w, a).
  Thus, $\tilde{C}$ accepts iff $|w|_1$ is even. □

Hanf-local graph properties

- Let $G = (V^G, E^G)$ and $H = (V^H, E^H)$ be two graphs.
- Let $r \in \mathbb{N}$.
- $G \rightleftarrows_r H \quad :\Longleftrightarrow \quad$ there is a bijection $\beta : V^G \to V^H$ such that for every $a \in V^G$

$$N_r^G(a) \;\cong\; N_r^H(\beta(a))$$

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Definition

A graph property p is Hanf $f(n)$-local if there is an n_0 such that for all graphs G and H of size $n \geqslant n_0$ the following is true:

If $G \rightleftarrows_{f(n)} H$ then G has property p iff H has property p.

Hanf-locality of FO

Theorem:

- For every graph property p that is FO-definable, there is a constant c such that p is Hanf c-local.

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Example: The class of all strings of the form $c^* a\, c^* b\, c^*$ is not definable in Arb-invariant FO($Succ$).

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Reductions to known results in circuit complexity

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Idea: Use known lower bounds in circuit complexity to show non-expressibility in certain logics.

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    (1) Note that for every k-variable Arb-invariant FO-sentence φ there exists a constant depth circuit family $(C_n)_n$ of size n^k such that C_n evaluates φ on graphs of size n.
    (2) Prove a new lower bound of $\omega(n^{k/4})$ on the size of constant-depth circuits solving the k-clique problem on n-vertex graphs.

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  - L is closed under swaps iff for all strings u, v, e, x, y, z such that e, f are idempotents (i.e., for all u, v we have $uev \in L$ iff $ue^2v \in L$), we have

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$$u\; e\,x\,f\; y\; e\,z\,f\; v \;\in\; L \iff u\; e\,z\,f\; y\; e\,x\,f\; v \;\in\; L.$$

- Show that every string-language definable in $<$-invariant FO(*Succ*) is aperiodic and closed under swaps.

  (For this, you can use Ehrenfeucht-Fraïssé games.)

Some further results proved using this method

Theorem:

- A tree-language is definable in $<$-invariant FO(*Succ*) iff it is definable in FO(*Succ*). (Benedikt, Segoufin '09)
  (They use aperiodicity and closure under guarded swaps.)

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- A regular string- or tree-language is definable in $+$-invariant FO(*Succ*) iff it is definable in $\text{FO}_{\text{card}}(\mathit{succ})$. (S., Segoufin '10 and Harwath, S. '12)
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- A regular string-language is definable in Arb-invariant FO(*Succ*) iff it is definable in $\text{FO}_{\text{card}}(\mathit{Succ})$. (Anderson, van Melkebeek, S., Segoufin '11)

Overview

Gaifman-locality

If $\mathcal{N}_r^G(a) \cong \mathcal{N}_r^G(b)$ then $\big(a \in q(G) \iff b \in q(G)\big)$.

Known:

- Queries definable in order-invariant FO are Gaifman-local with respect to a constant locality radius. (Grohe, Schwentick '98)
- Queries definable in Arb-invariant FO are Gaifman-local with respect to a poly-logarithmic locality radius. (Anderson, Melkebeek, S., Segoufin '11)

Open Question:

- How about addition-invariant FO — is it Gaifman-local with respect to a constant locality radius?

Hanf-locality

A graph property *p* is Hanf-local w.r.t. locality radius *r*, if any two graphs having the same *r*-neighbourhood types with the same multiplicities, are not distinguished by *p*.

Known:

- Properties of graphs definable in FO are Hanf-local w.r.t. a constant locality radius. (Fagin, Stockmeyer, Vardi '95)
- Properties of strings or trees definable by order-invariant FO are Hanf-local w.r.t. a constant locality radius. (Benedikt, Segoufin '09)
- Properties of strings definable by Arb-invariant FO are Hanf-local w.r.t. a poly-logarithmic locality radius. (Anderson, van Melkebeek, S., Segoufin '11)

Open Question:

- Do these results generalise from strings to arbitrary finite graphs?

Decidable Characterisations

Open Question:

Are there decidable characterisations of

- order-invariant FO?
- addition-invariant FO?
- $(+, \times)$-invariant FO?

Known:

- On finite strings and trees: order-invariant FO $\equiv$ FO. (Benedikt, Segoufin '10)
- On finite coloured sets: addition-invariant FO $\equiv$ FO enriched by "cardinality modulo" quantifiers. (S., Segoufin '10)

Thank You!