Equilibria and regularity in Mean Field Games with density penalization or constraints

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Outline

Let's be short about MFG

Mean Field Games (introduced by Lasry and Lions, and at the same time by Huang, Malhamé and Caines) describe the evolution of a population, where each agent has to choose the strategy (i.e., a path) which best fits his preferences, but is affected by the others through a global *mean field* effect.

It is a differential game, with a continuum of players, all indistinguishable and all negligible. It is a typical congestion game (agents try to avoid the regions with high concentrations) and we look for a *Nash equilibrium*, which can be translated into a system of PDEs.

J.-M. Lasry, P.-L. Lions, Jeux à champ moyen. (I & II) *C. R. Math. Acad. Sci. Paris,* 2006 + Mean-Field Games, *Japan. J. Math.* 2007

M. Huang, R.P. Malhamé, P.E. Caines, Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle, *Comm. Info. Syst.* 2006

P.-L. Lions, courses at Collège de France, 2006/12, videos available on the web

P. Cardaliaguet, lecture notes, www.ceremade.dauphine.fr/~cardalia/

MFG with density penalization- 1

In a population of agents everybody chooses its own trajectory, solving

$$\min \int_0^T \left(\frac{|x'(t)|^2}{2} + g(x(t), \rho_t(x(t))) \right) dt + \Psi(x(T)),$$

with given initial point $x(0)$; here $g(x, \cdot)$ is a given increasing function of the density ρ_t at time t. The agent hence tries to avoid overcrowded regions.

Input: the evolution of the density ρ_t.

A crucial tool is the value function φ for this problem, defined as

$$\varphi(t_0, x_0) := \min \left\{ \int_{t_0}^T \left(\frac{|x'(t)|^2}{2} + g(x(t), \rho_t(x(t))) \right) dt + \Psi(x(T)),\ x(t_0) = x_0 \right\}.$$

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MFG with density penalization- 2

Optimal control theory tells us that φ solves

$$(HJ) \qquad -\partial_t\varphi(t,x) + \frac{1}{2}|\nabla\varphi(t,x)|^2 = g(x,\rho_t(x)), \quad \varphi(T,x) = \Psi(x).$$

Moreover, the optimal trajectories $x(t)$ follow $x'(t) = -\nabla\varphi(t,x(t))$.

Hence, given the initial ρ_0, we can find the density at time t by solving

$$(CE) \qquad \partial_t\rho - \nabla\cdot(\rho\nabla\varphi) = 0,$$

which give as **Output:** the evolution of the density ρ_t.
We have an equilibrium if **Input = Output**.
This requires to solve a coupled system (HJ)+(CE):

$$\begin{cases} -\partial_t\varphi + \frac{|\nabla\varphi|^2}{2} = g(x,\rho), \\ \partial_t\rho - \nabla\cdot(\rho\nabla\varphi) = 0, \\ \varphi(T,x) = \Psi(x), \quad \rho(0,x) = \rho_0(x). \end{cases}$$

Stochastic case : we can also insert random effects $dX = \alpha dt + dB$, obtaining $-\partial_t\varphi - \Delta\varphi + \frac{|\nabla\varphi|^2}{2} - g(x,\rho) = 0$; $\quad \partial_t\rho - \Delta\rho - \nabla\cdot(\rho\nabla\varphi) = 0$.

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Variational principle

It happens that an equilibrium is found by minimizing the (global) energy

$$\mathcal{A}(\rho, v) := \int_0^T \int_\Omega \left(\frac{1}{2} \rho_t |v_t|^2 + G(x, \rho_t) \right) + \int_\Omega \Psi \rho_T$$

among pairs (ρ, v) such that $\partial_t \rho + \nabla \cdot (\rho v) = 0$, with given ρ_0, where $G(x, \cdot)$ is the anti-derivative of $g(x, \cdot)$, i.e. $G(x, \cdot)' = g(x, \cdot)$. This problem is convex in the variables $(\rho, w := \rho v)$ and admits a dual problem:

$$\sup \left\{ -\mathcal{B}(\phi, h) := \int_\Omega \phi_0 \rho_0 - \int_0^T \int_\Omega G^*(x, h) \ : \ \phi_T \leq \Psi, \ -\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 = h \right\},$$

where G^* is the Legendre transform of G (w.r.t. h).

Formally, if (ρ, v) solves the primal problem and (φ, h) the dual, then we have $v = -\nabla \varphi$ and $h = g(x, \rho)$, i.e. we solve the MFG system.

Warning: the existence of a dual solution (in a suitable weak functional space) is not always guaranteed. Also, for non-smooth functions, this is not the same as having optimal trajectories...

P. Cardaliaguet, P.J. Graber. Mean field games systems of first order. *ESAIM: COCV*, 2015.
J.-D. Benamou, G. Carlier Augmented Lagrangian methods for transport optimization, Mean-Field Games and degenerate PDEs, *JOTA*, 2015.
J.-D. Benamou, G. Carlier, F. Santambrogio, Variational Mean Field Games, *Active Particles I*, 2016

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Not all MFG are variational

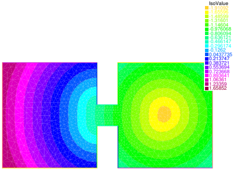
If the previous examples of MFG have indeed a variational structure, not all MFG share this feature (i.e. they are not all *potential games*). For instance

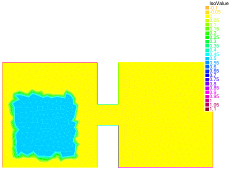
- MFG with multiplicative congestion, where the cost for the agents involves $\int_0^T \left(\frac{|x'(t)|^2}{2}\rho_t^\alpha(x(t))\right)dt$;
- Minimal-time MFG, where agents solve $\min\{T \,:\, x(T) \in \Gamma,\, g(\rho_t(x(t)))|x'(t)| \leq 1\}$;
- MFG of controls, including an old proposal about MFG with density constraints, where agents solve $\min \int_0^T \frac{|v(t)|^2}{2}dt + \Psi(x(T))$ with $x'(t) = v(t) - \nabla p(t, x(t))$, and $p \geq 0$ is a pressure generated by the incompressibility constraint $\rho \leq 1$;

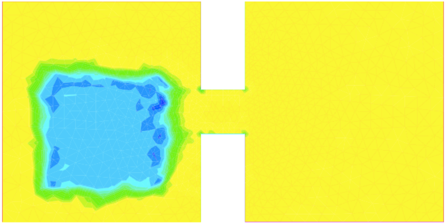
However, in this talk, we will only concentrate on the variational case.

Y. Achdou, A. Porretta Mean field games with congestion. *Ann. IHP*, 2018.
P. Cardaliaguet, C.-A. Lehalle Mean field game of controls and an application to trade crowding, *Math. Fin. Econ.*, 2018
G. Mazanti, F. Santambrogio Minimal-Time Mean Field Games, *M3AS*, to appear.
F. Santambrogio A modest proposal for MFG with density constraints, *Net. Het. Media*, 2012

An example - simulations by convex optimization







Left top: final potential Ψ,
left bottom: initial density ρ_0,
right: evolution

J.-D. Benamou, G. Carlier, F. Santambrogio, Variational Mean Field Games, *Active Particles Vol. I*, 2016

Measures on the space of trajectories

The same variational problem can also be written in the following way: let $\mathcal{C} = H^1([0,T];\Omega)$ be the space of curves valued in Ω and $e_t : \mathcal{C} \to \Omega$ the evaluation map, $e_t(\gamma) = \gamma(t)$. Solve

$$\min \left\{ \int_{\mathcal{C}} K dQ + \int_0^T \mathcal{G}((e_t)_\# Q) + \int_\Omega \Psi d(e_T)_\# Q, \; Q \in \mathcal{P}(\mathcal{C}), (e_0)_\# Q = \rho_0 \right\},$$

where $K : \mathcal{C} \to \mathbb{R}$ and $\mathcal{G} : \mathcal{P}(\Omega) \to \overline{\mathbb{R}}$ are given by $K(\gamma) = \frac{1}{2}\int_0^T |\gamma'|^2$ and $\mathcal{G}(\rho) = \int G(x, \rho(x))dx$. ($\#$ denotes image measure, or push-forward).

Existence: by semicontinuity in the space $\mathcal{P}(\mathcal{C})$.

Optimality conditions: take $\overline{Q}$ optimal, $\widetilde{Q}$ another competitor, and $Q_\varepsilon = (1-\varepsilon)\overline{Q} + \varepsilon\widetilde{Q}$. Setting $\rho_t = (e_t)_\# \overline{Q}$ and $h(t,x) = g(x, \rho_t(x))$, differentiating w.r.t. ε gives

$$J_h(\widetilde{Q}) \geq J_h(\overline{Q}),$$

where J_h is the linear functional

$$J_h(Q) = \int K dQ + \int_0^T \int_\Omega h(t,x) d(e_t)_\# Q \, dt + \int_\Omega \Psi d(e_T)_\# Q.$$

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Back to an equilibrium

Look at J_h. It is well-defined for $h \geq 0$ measurable. Yet, if $h \in C^0$ we can also write $\int_0^T \int_\Omega h(t,x) d(e_t)_\# Q \, dt = \int_C dQ \int_0^T h(t,\gamma(t))dt$ (in general we have problems in the definition a.e.) and hence we get that

$$Q \mapsto \int_C dQ(\gamma) \left(K(\gamma) + \int_0^T h(t,\gamma(t))dt + \Psi(\gamma(T)) \right)$$

is minimal for $Q = \overline{Q}$. Hence $\overline{Q}$ is concentrated on curves minimizing $\mathcal{L}_{h,\Psi}(\gamma) := K(\gamma) + \int_0^T h(t,\gamma(t))dt + \Psi(\gamma(T))$. This means **Input=Output**.

A rigorous proof can also be done even for $h \notin C^0$ but one has to choose a precise representative. Techniques from incompressible fluid mechanics **(incompressible Euler** *à la* **Brenier)** allow to handle some interesting cases using $\hat{h}(x) := \limsup_{r \to 0} \fint_{B(x,r)} h(t,y)dy$ (maximal function Mh needed to justify some convergences. . .).

L. Ambrosio, A. Figalli, On the regularity of the pressure field of Brenier's weak solutions to incompressible Euler equations, *Calc. Var. PDE*, 2008.

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Precise equilibrium statements and need for summability

An adaptation of Ambrosio-Figalli's statement is

Theorem

If $\overline{Q}$ is optimal, then $\overline{Q}$-a.e. curve γ is an optimal trajectory in the following sense:

$$\mathcal{L}_{\hat{h},\Psi}(\gamma) \leq \mathcal{L}_{\hat{h},\Psi}(\tilde{\gamma})$$

on every interval $[t_0, T]$ and for every curve $\tilde{\gamma}$ such that $\int_{t_0}^{T} M(h_+)(\tilde{\gamma}) < +\infty$.

How many curves do satisfy $\int_{t_0}^{T} M(h_+)(\tilde{\gamma}) < +\infty$? If $G(x,\rho) \approx \rho^q$, $M|h| \in L^{q'}$ then for every $\tilde{Q}$ with finite cost, $\tilde{Q}$-a.e. curves do it, since

$$\int \int_{t_0}^{T} M|h|(\tilde{\gamma}(t))dt\, d\tilde{Q}(\tilde{\gamma}) = \int_{t_0}^{T} \int_{\Omega} M|h|\, d\rho_t\, dt.$$

Need for estimates: we should prove $\rho \in L^q$ and $M|h| \in L^{q'}$ (for $q' > 1$, equivalent to $h \in L^{q'}$): this is easy for G growing as ρ^q, difficult for more exotic G ($G(\rho) = \exp(\rho), (1-\rho)^{-1}) \dots$).
Should we prove $\rho \in L^\infty$ and in particular $h_+ = (g(x,\rho)_+) \in L^\infty$, then the optimality would be among all curves, and we should not care about Mh.

Trajectories on the space of measures, time-discretization

The very same variational problem can also be written in a third way. Use the space of probabilities $\mathbb{W}_2(\Omega)$ endowed with the Wasserstein distance W_2 (enduced by optimal transport) and look for a curve $(\rho(t))_{t\in[0,T]}$ solving

$$\min\left\{\int_0^T\left(\frac{1}{2}|\rho'|(t)^2+\mathcal{G}(\rho(t))\right)dt+\int_\Omega \Psi d\rho_T \;:\; \rho(0)=\rho_0\right\},$$

(here $|\rho'|(t):=\lim_{s\to t}\frac{W_2(\rho(s),\rho(t))}{|s-t|}$ is the metric derivative of the curve ρ).

Existence is also easy by semicontinuity and by Ascoli-Arzelà applied in the space of curves from $[0,T]$ to the compact metric space $\mathbb{W}_2(\Omega)$.

A useful approximation can be obtained via time-discretization: fix $\tau = T/N$ and look for a sequence $\rho_0,\rho_1,\dots,\rho_N$ solving

$$\min\left\{\sum_{k=0}^{N-1}\left(\frac{W_2^2(\rho_k,\rho_{k+1})}{2\tau}+\tau\mathcal{G}(\rho_k)\right)+\int_\Omega \Psi d\rho_N\right\}.$$

Optimality conditions in a JKO-like scheme

If $\rho_0, \rho_1, \dots, \rho_N$ solves

$$\min \left\{ \sum_{k=0}^{N-1} \left(\frac{W_2^2(\rho_k, \rho_{k+1})}{2\tau} + \tau \mathcal{G}(\rho_k) \right) + \int_\Omega \Psi d\rho_N \right\}$$

then, for each $0 < k < N$, the measure ρ_k solves

$$\min \left\{ \frac{W_2^2(\rho, \rho_{k-1})}{2\tau} + \frac{W_2^2(\rho, \rho_{k+1})}{2\tau} + \tau \mathcal{G}(\rho) \right\},$$

i.e. it solves a minimization problem similar to what we see in the JKO scheme for gradient flows:

$$\min \left\{ \frac{W_2^2(\rho, \rho_{k-1})}{2\tau} + \mathcal{G}(\rho) \right\}.$$

For $k = N$, we have a true JKO-style problem with one only Wasserstein distance.

R. Jordan, D. Kinderlehrer, F. Otto. The variational formulation of the Fokker-Planck equation. *SIAM J. Math. An.*, 1998.

The flow-interchange estimates

Let ρ_s be the gradient flow of a functional $\mathcal{F}(\rho) := \int F(\rho(x))dx$, i.e. a solution of $\partial_s \rho - \nabla \cdot (\rho \nabla (F'(\rho))) = 0$, with initial datum at $s = 0$ equal to the optimal ρ at step k. We suppose Ω to be convex and we choose F so that $\mathcal{F}$ is geodesically convex functional on $\mathbb{W}_2(\Omega)$. This provides

$$\frac{d}{ds}\frac{W_2^2(\rho_s,\nu)}{2} \leq \mathcal{F}(\nu) - \mathcal{F}(\rho_s)$$

We also have

$$\frac{d}{ds}\mathcal{G}(\rho_s) = -\int \nabla(g(x,\rho_s)) \cdot \nabla(F'(\rho_s))\, d\rho_s.$$

the optimality of ρ_k hence gives

$$\int \nabla(g(x,\rho_k)) \cdot \nabla(F'(\rho_k))\, d\rho_k \leq \frac{\mathcal{F}(\rho_{k+1}) - 2\mathcal{F}(\rho_k) + \mathcal{F}(\rho_{k-1})}{\tau^2}.$$

R.J. McCann A convexity principle for interacting gases. *Adv. Math.* 1997.
L. Ambrosio, N. Gigli, G. Savaré *Gradient flows in metric spaces and in the space of probability measures*, 2005.
D. Matthes, R.J. McCann, G. Savaré. A family of nonlinear fourth order equations of gradient flow type. *CPDE*, 2009.

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Let ρ_s be the gradient flow of a functional $\mathcal{F}_m(\rho) := \int F_m(\rho(x))dx$, i.e. a solution of $\partial_s\rho - \nabla\cdot(\rho\nabla(F_m'(\rho))) = 0$, with initial datum at $s = 0$ equal to the optimal ρ at step k. We suppose Ω to be convex and use $F_m(\rho) = \rho^m$, so that we have a geodesically convex functional on $\mathbb{W}_2(\Omega)$. This provides

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L^m and L^∞ estimates

Suppose $g(x,\rho) = V(x) + g(\rho)$. We start from $V = 0$:

$$0 \leq \int g'(\rho_k) F''_m(\rho_k)\rho_k |\nabla \rho_k|^2 \quad \leq \quad \frac{\mathcal{F}_m(\rho_{k+1}) - 2\mathcal{F}_m(\rho_k) + \mathcal{F}_m(\rho_{k-1})}{\tau^2}$$

$k \mapsto \mathcal{F}_m(\rho_k)$ is discretely convex. If $\rho_0 \in L^m$, and we suppose $\rho_T \in L^m$, so is ρ_t, uniformly in t.
With a final penalization Ψ, if $\Psi \in C^{1,1}$, then we also obtain

$$\mathcal{F}_m(\rho_N) \leq (1 + C\tau m)\mathcal{F}_m(\rho_{N-1}),$$

hence, not only $k \mapsto \mathcal{F}_m(\rho_k)$ is convex, but we control its final derivative, which also implies boundedness of $\mathcal{F}_m$.

H. Lavenant, F. Santambrogio Optimal density evolution with congestion: L^∞ bounds via flow interchange techniques and applications to variational Mean Field Games, *CPDE*, 2018

L^m and L^∞ estimates

Suppose $g(x,\rho) = V(x) + g(\rho)$. We start from $V = 0$:

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$k \mapsto \mathcal{F}_m(\rho_k)$ is discretely convex. If $\rho_0 \in L^m$, and we suppose $\rho_T \in L^m$, so is ρ_t, uniformly in t. This also works for $m = \infty$.
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hence, not only $k \mapsto \mathcal{F}_m(\rho_k)$ is convex, but we control its final derivative, which also implies boundedness of $\mathcal{F}_m$.

H. Lavenant, F. Santambrogio Optimal density evolution with congestion: L^∞ bounds via flow interchange techniques and applications to variational Mean Field Games, *CPDE*, 2018

L^m and L^∞ estimates

Suppose $g(x,\rho) = V(x) + g(\rho)$. We start from $V = 0$:

$$? \leq \int g'(\rho_k) F''_m(\rho_k)\rho_k |\nabla \rho_k|^2 \quad \leq \quad \frac{\mathcal{F}_m(\rho_{k+1}) - 2\mathcal{F}_m(\rho_k) + \mathcal{F}_m(\rho_{k-1})}{\tau^2}$$

$k \mapsto \mathcal{F}_m(\rho_k)$ is discretely convex. Don't suppose anything on ρ_0, ρ_T and/or Ψ: we can obtain local estimates. Suppose $g'(s) \geq s^\alpha$. We use

$$\int g'(\rho_k) F''_m(\rho_k)\rho_k |\nabla \rho_k|^2 \geq c \int \rho_k^{m-1+\alpha} |\nabla \rho_k|^2 \quad = \quad c\|\nabla(\rho_k^{(m+1+\alpha)/2})\|^2_{L^2}$$
$$\geq \quad c\|(\rho_k^{(m+1+\alpha)/2})\|^2_{L^\beta},$$

for $\beta \in (2, 2^*) > 2$ and we use **Moser's iteration** on exponents $m_j \approx (\beta/2)^j$.

H. Lavenant, F. Santambrogio Optimal density evolution with congestion: L^∞ bounds via flow interchange techniques and applications to variational Mean Field Games, *CPDE*, 2018

L^m and L^∞ estimates

Suppose $g(x,\rho) = V(x) + g(\rho)$. Do not suppose anymore $V = 0$:

$$? \le \int g'(\rho_k)F''_m(\rho_k)\rho_k|\nabla\rho_k|^2 \le \frac{\mathcal{F}_m(\rho_{k+1}) - 2\mathcal{F}_m(\rho_k) + \mathcal{F}_m(\rho_{k-1})}{\tau^2} - \int(\nabla V\cdot\nabla\rho_k)F''_m(\rho_k)\rho_k.$$

The new term needs to be estimated in terms of V and $\mathcal{F}_m$. $k \mapsto \mathcal{F}_m(\rho_k)$ is no more convex (but rather it satisfies $u''+C(m)u\ge 0$). We can go on...

$$\int g'(\rho_k)F''_m(\rho_k)\rho_k|\nabla\rho_k|^2 \ge c\int \rho_k^{m-1+\alpha}|\nabla\rho_k|^2 = c\|\nabla(\rho_k^{(m+1+\alpha)/2})\|^2_{L^2} \ge c\|(\rho_k^{(m+1+\alpha)/2})\|^2_{L^\beta},$$

for $\beta\in(2,2^*)>2$ and we use **Moser's iteration** on exponents $m_j \approx (\beta/2)^j$.

H. Lavenant, F. Santambrogio Optimal density evolution with congestion: L^∞ bounds via flow interchange techniques and applications to variational Mean Field Games, *CPDE*, 2018

L^m and L^∞ estimates - summary

Theorem

Suppose $g(x,\rho) = V(x) + g(\rho)$. Suppose $g'(s) \geq s^\alpha$ for $s \geq s_0$.

- *If V is Lipschitz, $\alpha \geq -1$, and $s_0 = 0$ then $\rho \in L^\infty_{loc}((0,T)\times\overline{\Omega})$.*
- *Same result if $s_0 > 0$ but $V \in C^{1,1}$ and $\partial V/\partial n \geq 0$.*
- *These results extend to $(0,T]$ if $\Psi \in C^{1,1}$ and $\partial \Psi/\partial n \geq 0$.*
- *If $\alpha < -1$, then the same results, for $V, \Psi \in C^{1,1}$, $\partial V/\partial n \geq 0$ and $\partial \Psi/\partial n \geq 0$, are true if we already know $\rho \in L^{m_0}((0,T)\times\overline{\Omega})$ for $m_0 > d|\alpha+1|/2$. This is true in particular if $\rho_0 \in L^{m_0}$ and T is small enough.*

If g is a convex function finite on $\mathbb{R}_+$, then we also obtain upper bounds on $h = V + g(\rho)$.

Generalizations (to be done!): replace the quadratic cost in W_2^2 with the transport cost $H(x-y)$ to study agents who minimize $\int H(x') + g(x,\rho)$; add x-dependance in the Hamiltonian: geodesic convexity in the Wasserstein space on a manifold is involved (Ricci bounds. . .).

MFG with density constraints - 1

How to define a mean field game if we want to replace the penalization $+g(x,\rho)$ with the constraint $\rho \leq 1$ and a cost $V(x)$?

Naïve idea: when $(\rho_t)_t$ is given, every agent minimizes his own cost paying attention to the constraint $\rho_t(x(t)) \leq 1$. But if ρ already satisfies $\rho \leq 1$, one extra agent will not violate the constraint (it's a *non-atomic game*). Hence the constraint becomes empty.

Instead, let's look at the variational problem

$$\min\left\{\int_0^T\int_\Omega\left(\frac{1}{2}\rho_t|v_t|^2+V\rho_t\right)+\int_\Omega \Psi\rho_T\ :\ \rho\leq 1\right\}.$$

It means $G(x,\rho)=V(x)\rho$ for $\rho\in[0,1]$ and $+\infty$ otherwise. There is a dual

$$\sup\left\{\int_\Omega \phi_0\rho_0-\int_0^T\int_\Omega (h-V)_+\ :\ \phi_T\leq\Psi,\ -\partial_t\phi+\frac{1}{2}|\nabla\phi|^2=h\right\}.$$

This problem is also obtained as the limit $m\to\infty$ of $g(x,\rho)=V(x)+\rho^m$. Indeed the functional $\frac{1}{m+1}\int\rho^{m+1}$ Γ-converges to the constraint $\rho\leq 1$.

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MFG with density constraints - 2

The system we get is

$$\begin{cases} -\partial_t\varphi + \frac{|\nabla\varphi|^2}{2} = p + V, \\ \partial_t\rho - \nabla\cdot(\rho\nabla\varphi) = 0, \\ p \geq 0,\ \rho \leq 1,\ p(1-\rho) = 0, \\ \varphi(T,x) = \Psi(x), \quad \rho(0,x) = \rho_0(x). \end{cases}$$

Each agent solves min $\int_0^T \left(\frac{|x'(t)|^2}{2} + h(t,x(t))\right) dt + \Psi(x(T))$.
Here $h = p + V$ and p is a **pressure** arising from the incompressibility constraint $\rho \leq 1$ but finally acts as a **price**. In order to give a meaning to the above problem the regularity of p is crucial, in particular if we want $M|h| \in L^1$. A priori, we don't even know that p is a function!!

Results inspired by Incompressible fluid mechanics and based on convex duality gave

$$V \in C^{1,1} \Rightarrow p \in L^2_{loc}((0,T); BV_{loc}(\Omega)).$$

P. Cardaliaguet, A. Mészáros, F. Santambrogio, First order Mean Field Games with density constraints: Pressure equals Price, *SIAM J. Contr. Opt.*, 2016
Y. Brenier, Minimal geodesics on groups of volume-preserving maps and generalized solutions of the Euler equations, *Comm. Pure Appl. Math.*, 1999.

Estimates on Δp

Very informally, take the Laplacian of the HJ equation

$$-\partial_t \Delta\varphi + \nabla\varphi \cdot \nabla\Delta\varphi + |D^2\varphi|^2 = \Delta h.$$

Use material derivative $D_t := \partial_t - \nabla\varphi \cdot \nabla$. We have $D_t(\log\rho) = \Delta\varphi$ and

$$-D_t^2(\log\rho) = -D_t\Delta\varphi \leq -D_t\Delta\varphi + |D^2\varphi|^2 = \Delta h.$$

Where $p > 0$, $\log\rho$ is maximal, thus $D_t^2(\log\rho) \leq 0$. This implies

$$p > 0 \Rightarrow \Delta p \geq -\Delta V.$$

The same estimate can be justified via time-discretization. Consider

$$\min\left\{\frac{W_2^2(\rho,\rho_{k-1})}{2\tau} + \frac{W_2^2(\rho,\rho_{k+1})}{2\tau} + \tau\int V d\rho \ : \ \rho \leq 1\right\},$$

then for the optimal ρ_k we have $\frac{\phi_+}{\tau} + \frac{\phi_-}{\tau} + \tau(p+V) = c$, where $\phi_\pm$ is the Kantorovich potential from ρ_k to $\rho_{k\pm1}$ and p is the (discrete) pressure. We then use, on $\{p > 0\}$ (where $\rho_k = 1$ while $\rho_{k\pm1} \leq 1$ everywhere)

$$1 - \frac{\Delta\phi_\pm}{d} \geq \left(\det(I - D^2\phi_\pm)\right)^{1/d} = \left(\frac{\rho_k}{\rho_{k\pm1}(id - \nabla\phi_\pm)}\right)^{1/d} \geq 1.$$

H^1 and L^∞ estimates

Testing $p > 0 \Rightarrow \Delta p \geq -\Delta V$ against p we obtain

$$\int |\nabla p|^2 \leq \int \nabla V \cdot \nabla p$$

i.e. $\int |\nabla p|^2 \leq \int |\nabla V|^2$, hence $p \in L^\infty([0,T); H^1(\Omega))$, which is an improvement upon previous results. Moreover, testing against p^m allows to get

$$\frac{4}{(m+1)^2} \int |\nabla(p^{\frac{m+1}{2}})|^2 = \int p^{m-1} |\nabla p|^2 \leq \int p^{m-1} \nabla V \cdot \nabla p = \int p^{\frac{m-1}{2}} \nabla V \cdot \nabla(p^{\frac{m+1}{2}}).$$

Moser's iterations again allow to obtain $p \in L^\infty$ as soon as ∇V is summable enough, in particular if $V \in W^{1,q}$ with $q > d$.

Beware that there is anyway a final pressure at $t = T$ which adds up to Ψ (for this pressure we can prove $H^1 \cap L^\infty$ regularity in space)

H. Lavenant, F. Santambrogio New estimates on the regularity of the pressure in density-constrained Mean Field Games *J. Lon. Math. Soc.*, to appear.

Few formal applications to the value function

Besides the applications to Lagrangian equilibria, as we do not need to use Mh as soon as $h \in L^\infty$, we can also exploit some results for the solution φ of $-\partial_t\varphi + \frac{1}{2}|\nabla\varphi|^2 = h$, for $h \in L^r$, $r > 1 + \frac{d}{2}$ and h bounded from below

- φ is Hölder continuous, $\varphi \in C^{0,\alpha}$, for $\alpha = \alpha(d, r)$;
- φ has better Sobolev regularity: $\partial_t\varphi \in L^{1+\epsilon}$, $\nabla\varphi \in L^{2+\epsilon}$ $(\epsilon = \epsilon(d, r))$
- φ is differentiable a.e.
- when $r = \infty$, the function φ is the value function for the control problem with running cost $\hat{h}$ (otherwise one has to restrict to curves with integrability of Mh). Obtained by regularizing by convolution.

For penalized or contrained problems in MFG, this can be applied with $r = \infty$ (or with $r < \infty$ if we don't have $V \in W^{1,q}$, $q > d$).

P. Cardaliaguet, Weak Solutions for First Order Mean Field Games with Local Coupling, 2015.
P. Cardaliaguet, L. Silvestre Hölder continuity to Hamilton-Jacobi equations with superquadratic growth in the gradient and unbounded right-hand side, *CPDE*, 2012.
P. Cardaliaguet, A. Porretta, D. Tonon Sobolev regularity for the first order Hamilton-Jacobi equation, *Calc. Var. PDE*, 2015.

The End

Thanks for your attention

For those coming to Cetraro next week (CIME school on MFG) :
a mini-course with all the mathematical details and proofs of what presented here